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Hyperbolic Functions that Linearize Inverted Manifold Surfaces

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Abstract

In this paper a control law that linearizes inverted dynamics is developed. The equation for inverted circular dynamics is derived from differential method and an integral method. By making alternate approximative assumptions of the inverted circular manifold with hyperbolic functions, an inverting control function is derived. Computer simulation results show that the alternate manifold linearizes the inverted dynamics. Analog simulation results show that the developed control law in this paper linearizes inverted dynamics. The method can be applied other nonlinear systems that have a gain term for the position of the angle in the sinusoid nonlinearity such as the space station. Because of torque requirements of the actual inverted systems such as the space station and inverted pendulum, a nonlinear adaptive gain is added. The adaptive gain is derived from a hyperbolic function which also results in a linearizing inverting function.

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Key Words and Phrases. Banach space, perturbed Newton-like methods, nondifferentiable operator.

1 Introduction

The results of this work linearize nonlinear differential equations that are inverted with sinusoid nonlinearity by inputing hyperbolic forcing functions. The applications of the results in this paper can be applied to robotics, aerospace engineering, and machinery that can be described by circular manifold. It will be shown that there are

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hyperbolic functions that approximate the inverse system solution for the inverted circular dynamics. Stability theory is applied to show that this solution is valid. Computer and analog simulation verify these results. The control law is derived from the sliding chain model. An application of the derived control law using hyperbolic functions is attitude control of space vehicles employing control moment gyro (CMG) is an interesting problem. The equations of motion of the space station are described by nonlinear differential equations. Often, attitude control system design using linear control theory [3, 6, 4] is obtained. However linear control systems are designed based on the assumption that the perturbation in attitude angles are small. The controller of [2] is based on the inversion of a nonlinear input-output map. The adaptive nonlinear control law that is derived by making alternate assumption for the potential of the manifold. The adaptive control law inversely proportional to cosh$\alpha\theta$. The resulting control law is ideal for the space station employing control moment gyros because momentum is bounded. In this paper, for simplicity, CMG's are considered as ideal torquers; however, the CMG gimbal dynamics should be included in the further development of control systems [5].

Approximate solution for inverted dynamics is derived in sections 2 and 3. The results are applied to the nonlinear inverted dynamics. Stability proof is given in section 4. Simulation for the claims are done in software and by mechanical analog. Described in this paper the dynamics of circular inverted dynamics and the space station dynamics.

2 Fixed Point Inverted Circular Dynamics

The pendulum dynamics are developed here by integral equation method. The derivation for a pendulum consisting of a mass m suspended on a for a rigid rod of length l, and angle θ. The forces acting are $F_n = ml\omega^2 = ml\dot{\theta}$ and $F_t = ml\ddot{\theta}$. The forces are defined as $F_n = s - mgcos\theta$ and $F_t = -mgsin\theta$. The equations of motion are[7]:

$$s - mgcos\theta = ml\dot{\theta}^2$$

$$-mgsin\theta = ml\ddot{\theta}$$

The first equation gives $s = mgcos\theta + ml\dot{\theta}^2$, which determines s if θ can be found as a function of time. The second equation gives the equation of motion:

$$\ddot{\theta} + \frac{g}{l}sin\theta = 0$$

With forcing function input:

$$\ddot{\theta} + \frac{g}{l}sin\theta = -u$$

An alternate assumption on the tangential force F_t.

$$F_t = ml\ddot{\theta} = -mgsin\theta \approx -mgK_{scaling}(e^{sin\theta} - e^{-sin\theta}) \tag{1}$$

Where for proper choice of scaling factor the approximation is nearly exact. Because $\sin\theta$ is a solution for the circular manifold about a stable equilibrium point, then the feedback control law is able to regulate:

$$u = Ksinh\theta \tag{2}$$

Where u is a feedback that can subtract out the nonlinear $\sin\theta$ term. This assumption was applied in computer simulation to the input of the inverted pendulum dynamics a bounded oscillatory response occured.

3 Derivation of Dynamics from Volterra Equation

From analog simulation of inverted pendulum cart dynamics with $\sinh\theta$ terms, it became apparent that some form of gain scheduling would be needed [8]. This is because the cart had a radius of gyration that was $\pm 45\,\deg$, and the driving torque of the motor was limited. The controller was a digital computer with sampling rate was high enough that discretization error was approximately equal to zero. To solve the control problem for a large angle, some form of adaptive model reference gain would need to be proportional to inverse of the position. From section 2, the potential of the inverted system is:

$$V = 1 - mgcos\theta \tag{3}$$

An approximation of this curve for $\pm 90\,\deg$ is $cosh\theta$. The adaptive time varying gain is then:

$$Model\ Reference\ Gain = \frac{1}{cosh\theta} \tag{4}$$

The resulting control law:

$$u = Ktanh(\alpha\theta) \tag{5}$$

Where K and α are constants.

In this section, an integral equation will be derived from sliding chain [1], and prove the above claim. This will be applied to inverted dynamics. By an alternate approximation for the shape of the surface, the control law is developed. The equation of the curve will be referred to as:

$$z = \phi(s) \tag{6}$$

Where $\phi(s)$ is assumed to be a smooth function. Let σ be the length of the arc between the points O and A. Let λ be the distance MA. To describe the curve only for the portion $s = \sigma - \lambda$. This yields

$$z = \phi(\sigma - \lambda) \tag{7}$$

Assume that $f(\lambda)$ is the linear density function of the pendulum at the point M, then $f(\lambda)d\lambda$ is the mass of the pendulum at point M, of length dλ. The work corresponding to the movement from z to z+dz is

$$gf(\lambda)d\lambda = g\phi^{'}(\sigma - \lambda)f(\lambda)d\lambda \tag{8}$$

Where the total work done over the interval 0 to σ

$$W_{tot} = gd\sigma \int_0^{\sigma} \phi^{'}(\sigma - \lambda)f(\lambda)d\lambda dz \tag{9}$$

The kinetic energy energy over the time t to t+dt (over the period necessary to move z to z+dz).

$$T = \frac{1}{2}Mv^2 \tag{10}$$

Where $M = \int_0^a f(\lambda)d\lambda$ the total mass of the pendulum, and since $v = ds/dt = d\sigma/dt$.

$$dT = M\frac{d\sigma}{dt}\frac{d^2\sigma}{dt^2}dt = M\frac{d^2\sigma}{dt^2}d\sigma \tag{11}$$

The resulting equation of motion is

$$M\frac{d^2\sigma}{dt^2} = g\int_0^{\sigma} \phi^{'}(\sigma - \lambda)f(\lambda)d\lambda \tag{12}$$

The law of motion $d^2\sigma = F(\sigma)dt^2$, then

$$\int_0^{\sigma} \phi^{'}(\sigma - \lambda)f(\lambda)d\lambda dz = g^{-1}MF(\sigma) \tag{13}$$

which constitutes an integral equation of Volterra type of the first kind.

To apply these equations to our control problem, assume that λ is small with respect to σ. The width of MA is small compared to the arc OC.

$$d\theta = d\lambda \tag{14}$$

The equation of the curve is

$$z = Rsin^2\theta + Rcos^2\theta \tag{15}$$

$$\phi^{'} = -2sin\theta + 2cos\theta \tag{16}$$

The linear density function $f(\lambda)$ is a constant K. Assume that R=1. The distance OC are regulating to is from 0 to θ. So $\theta = \sigma$.

$$\frac{d^2\theta}{dt^2} = gM^{-1}K\int_0^{\theta}(-2sin\theta + 2cos\theta)d\theta \tag{17}$$

For small angles the $cos\theta$ term is dominant.

$$\frac{d^2\theta}{dt^2} = gM^{-1}K_{proportional}\int_0^\theta cos\theta d\theta \tag{18}$$

Where $K_{proportional}$ equals $K \times 2$.

$$\frac{d^2\theta}{dt^2} = gM^{-1}K_{proportional} \times sin\theta \tag{19}$$

An approximation of the cosine term with hyperbolic function for the interval $\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

$$cos\theta = sech^2\theta \tag{20}$$

Substituting:

$$\frac{d^2\theta}{dt^2} = gM^{-1}K_{proportional}\int_0^\theta sech^2\theta d\theta \tag{21}$$

The resulting differential equation is

$$\frac{d^2\theta}{dt^2} = gM^{-1}K_{proportional} \times tanh\theta \tag{22}$$

For dynamics that differ by a gain term in the sinusoid angle. The equation of the curve is

$$z = Rsin^2(\alpha\theta) + Rcos^2(\alpha\theta) \tag{23}$$

The linear density function $f(\lambda)$ is a constant K. Assume that R=1. The distance OC are regulating to is from 0 to θ. So $\theta = \sigma$.

$$\frac{d^2\theta}{dt^2} = gM^{-1}K\int_0^\theta (-2\alpha sin(\alpha\theta) + 2\alpha cos(\alpha\theta))d\theta \tag{24}$$

For small angles:

$$\frac{d^2\theta}{dt^2} = gM^{-1}K_{proportional}\int_0^\theta \alpha cos(\alpha\theta)d\theta \tag{25}$$

$$\frac{d^2\theta}{dt^2} = gM^{-1}\alpha K_{proportional} \times sin(\alpha\theta) \tag{26}$$

The approximation of the cosine and sine term with a sech function in the preceding integral equation:

$$\frac{d^2\theta}{dt^2} = gM^{-1}K_{proportional}\int_0^\theta \alpha sech^2(\alpha\theta)d\theta \tag{27}$$

The resulting differential equation is

$$\frac{d^2\theta}{dt^2} = gM^{-1}\alpha K_{proportional} \times tanh(\alpha\theta) \tag{28}$$

Definition 1 *A Manifold is topological space which is locally Euclidean; there are four types; topological, piecewise linear, differentiable, and complex, depending on whether the local coordinate systems are obtained from continuous, piecewise linear, differentiable, or complex analytic functions of those in Euclidean space; intuitively a surface[9].*

Applying this definition, $\sinh\alpha\theta$ is an approximation of the inverse dynamics of second order inverted manifold systems described in this paper.

4 Stability Proof for the Pitch Axis Dynamics

In this section stability proof for the control law developed in sections 3 and 4 is given [10].

Theorem 1 *The equilibrium point* $\mathbf{0}$ *at time* t_0 *of a nonlinear differential equation is stable if there exists a continuously differentiable l.p.d.f.* V *such that*

$$\dot{V}(t, \mathbf{x}) \leq 0 \ \ \forall t \geq t_0, \forall \mathbf{x} \in \mathbf{B_r} \tag{29}$$

for some ball $\mathbf{B_r}$

Conjecture 1 *The control law*

$$u = K_{gain} sinh(\alpha\theta)$$

for proper choice of K_{gain} *and* α *results in stable dynamics for inverted systems of the type:*

$$\ddot{\theta} - K_{gain} sin(\alpha\theta) = -u$$

Proof 1 *For the control given the potential surface is*

$$V(t, \theta) = \alpha K_{gain} cosh(\alpha\theta) \tag{30}$$

$$\dot{V}(t, \theta) = \alpha K_{gain} sinh(\alpha\theta) \tag{31}$$

For the inverted equilibrium point, the Liapunov function bounds the surface and

$$\dot{V}(t, \mathbf{x}) \leq 0 \ \ \forall t \geq t_0, \forall \mathbf{x} \in \mathbf{B_r}$$

Conjecture 2 *The nonlinear feedback* $u = K_{gain} sinh(\alpha\theta)$ *is a feedback linearization of inverted dynamics of the form:*

$$\ddot{\theta} - \frac{g}{l} sin(\alpha\theta) = -u \tag{32}$$

Where g*,* l*, and* α *are constants.*

Proof 2 *From preceding sections it is assumed that* $sinh(\alpha\theta)$ *is an approximation for the surface manifold. The control is for a region* $\frac{\pi}{2}$ *to* $\frac{-\pi}{2}$ *with proper choice of gain value* K_{gain}*, and* α*. For the condition of stability shown in preceding conjecture, a solution can be bounded sinusoid. Therefore for this case:*

$$K_{gain} sinh(\alpha sin(\alpha\theta)) \approx \frac{g}{l} sin(\alpha\theta)$$

Substituting into the nonlinear differential equation for proper choice of gain K_{gain} *and* α*.*

$$\ddot{\theta} + \epsilon\theta = 0 \tag{33}$$

Where ϵ *is assumed to be approximately a constant or a small value with respect to* $\frac{g}{l} sin(\alpha\theta)$*.*

Conjecture 3 *The nonlinear feedback* $u = K_{gain} tanh(\alpha\theta)$ *is a feedback linearization of inverted dynamics of the form:*

$$\ddot{\theta} - \frac{g}{l} sin(\alpha\theta) = -u \tag{34}$$

Where g*,* l*, and* α *are constants.*

Proof 3 *This is a result of for the interval* $-\frac{\pi}{2\alpha}$ *to* $\frac{\pi}{2\alpha}$*:*

$$K_{gain} tanh(\alpha\theta) \approx \frac{g}{l} sin(\alpha\theta) \tag{35}$$

Substituting into the nonlinear differential equation for proper choice of gain K_{gain} *and* α*.*

$$\ddot{\theta} + \epsilon\theta = 0 \tag{36}$$

Where ϵ *is assumed to be approximately a constant or a small value with respect to* $\frac{g}{l} sin(\alpha\theta)$*.*

The resulting output is oscillatory the control laws developed using hyperbolic functions described. Oscillatory dynamics are easily described by 2 poles on the complex axis using describing function method.

4.1 Stability and Control of Resulting Dynamics

Theoretical conditions for stability are discussed and applied to the control problem for oscillatory dynamics. The conditions for stability is a result of La Salle's Theorem for periodic systems. Which is defined as [10]:

Definition 2 *A limit cycle of dynamics such as*

$$\dot{x_1} = f_1[x_1(t), x_2(t)]$$

$$\dot{x_2} = f_2[x_1(t), x_2(t)]$$

is a periodic solution of these same described dynamics.

La Salle's Theorem for periodic systems.

Theorem 2 *Suppose that system $\dot{x}(t) = f[t, x(t)]$ for $t \geq 0$ is periodic, i.e., suppose*

$$f(t, x) = f(t + T, x), \ \ \forall\, t, \ \forall\, x \in \ R^n,$$

for some $T > 0$. Suppose $V : R_+ \times R^n \rightarrow R$ is a continuously differentiable p.d.f., with

$$V(t, x) = V(t + T, x), \ \ \forall\, x \in R^n, \forall\, t \geq 0$$

Define

$$S = \{x \in R^n : \dot{V}(t, x) = 0, \ \forall\, t \geq 0\} \tag{37}$$

Suppose $\dot{V}(t, x) \leq 0$, $\forall\, t \geq 0$, $\forall\, x \in R^n$, and that S contains no nontrivial trajectories of $\dot{x}(t) = f[t, x(t)]$. Under these conditions, the equilibrium point **0** *is globally asymptotically stable.*

In order to regulate the system to **0** the control

$$u = K_1\theta + K_2\dot{\theta} + K_3 sinh(\alpha\theta) \tag{38}$$

or

$$u = K_1\theta + K_2\dot{\theta} + K_3 tanh(\alpha\theta) \tag{39}$$

is used such that K_1, K_2, and K_3 are chosen such that $\dot{V}(t, x) \leq 0$.

The equation is linearized by use of describing function technique. From simulation when these functions are used as control inputs to the dynamics for the pitch axis, bounded oscillation results. By use of describing function method, the oscillation is an approximation of $Asin\beta\theta$ (where A the amplitude and β are constants). No other dominant oscillations exist from this control input. The oscillation is easily described by the 2 dominant poles on the $j\omega$ axis in the complex S plane. So the describing function for this system in the inverted state space:

$$\ddot{\theta} + \beta\theta = 0 \tag{40}$$

This is then a linear system with β term describing the period of the oscillation. In the inverted systems state space:

$$x_1 = \theta \tag{41}$$

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -\beta x_1$$

If terms $K_1\theta + K_2\dot{\theta}$ are negatively fed back into this system:

$$x_1 = \theta \tag{42}$$

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -\beta x_1 - K_1 x_1 - K_2 x_2$$

This is of the form [10]:

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -f(x_2) - g(x_1)$$

Where $f(x_2)$ represents the friction and $g(x_1)$ the restoring properties of the linearized system. Where the Liapunov function is:

$$V(x_1, x_2) = \frac{{x_2}^2}{2} + \int_0^{x_0} g(\sigma) d\sigma \tag{43}$$

For this case $g(\sigma) = (\beta - K_1)x_1$. And

$$\dot{V}(x_1, x_2) = x_2 \dot{x_2} + g(x_1)\dot{x_1} \tag{44}$$

$$= x_2[-f(x_2) - g(x_1)] + g(x_1)x_2 = -x_2 f(x_2)$$

$$\dot{V}(x_1, x_2) = -K_2 {x_2}^2 \tag{45}$$

Where $\dot{V}(x_1, x_2) \leq 0 \; \forall$ stable feedback pole pairs when control feedbacks from $\sinh\alpha\theta$ and $\tanh\alpha\theta$ produce bounded oscillations.

5 Simulation Results

The equations for control were derived mathematically. Computer simulation and analog models were used to test the control laws. The simulation described:

1. Nonlinear pitch axis dynamics of space station with control input $u_2 = sinh(2\theta_2)$.

2. Nonlinear pitch axis dynamics of space station with control input $u_2 = tanh(2\theta_2)$.

3. Nonlinear pitch axis dynamics of space station with control input $u_2 = tanh(2\theta_2) + .2\dot{\theta}$.

4. A nonlinear inverted cart with control input $u = -60tanh(2\theta)$. The $\theta(0) = 4°$. Note that although inverted circular dynamics had a α term of 1, other values can be used so long as the potential function bounds.

5. Computer simulation of the inverted cart with control input $u = 44tanh\theta_{angle} + 37\dot{\theta}_{angle} + 225\theta_{cart} + 67.3\dot{\theta}_{cart}$

The Space Station Parameters are: (Inertia slug-ft^2)

$$I_{11} = 50.28E6,$$
$$I_{22} = 10.80E6,$$
$$I_{33} = 58.57E6,$$

Where $I_i \triangleq I_{ii}$; i=1,2,3 The computer simulation for the space station nonlinear pitch axis dynamics in fortran used the pitch axis dynamics:

$$I_2\ddot{\theta}_2 + 3n^2(I_1 - I_3)sin\theta_2 cos\theta_2 = -u_2 \tag{46}$$

which simplifies to:

$$I_2\ddot{\theta}_2 + \frac{3}{2}n^2(I_1 - I_3)sin(2\theta_2) = -u_2$$

6 Conclusion

By use of hyperbolic functions to describe differential surfaces, it is possible to linearize inverted second order systems with sinusoid nonlinearity in position and small omega. The derivation is for inverted circular dynamics, but the solution is made general by making the hyperbolic function approximate other types of sinusoid nonlinearity. The results were applied to various nonlinear systems, and the dynamics were linearized.

The control laws are the result of alternate assumptions for the potential surface of the dynamics of inverted circular fixed point dynamics. Computer simulation verified these assumptions for the space station dynamics. Simulation on mechanical analog that is similar to the inverted dynamics of the space station, the inverted pendulum cart are presented. An adaptive gain which was proportional to the inverse of the potential surface is suggested, and is a result of analog testing the control law. The technique for feeding back nonlinear terms into a hyperbolic function that approximates the nonlinearity has applications in all areas of circular manifold and related dynamics.

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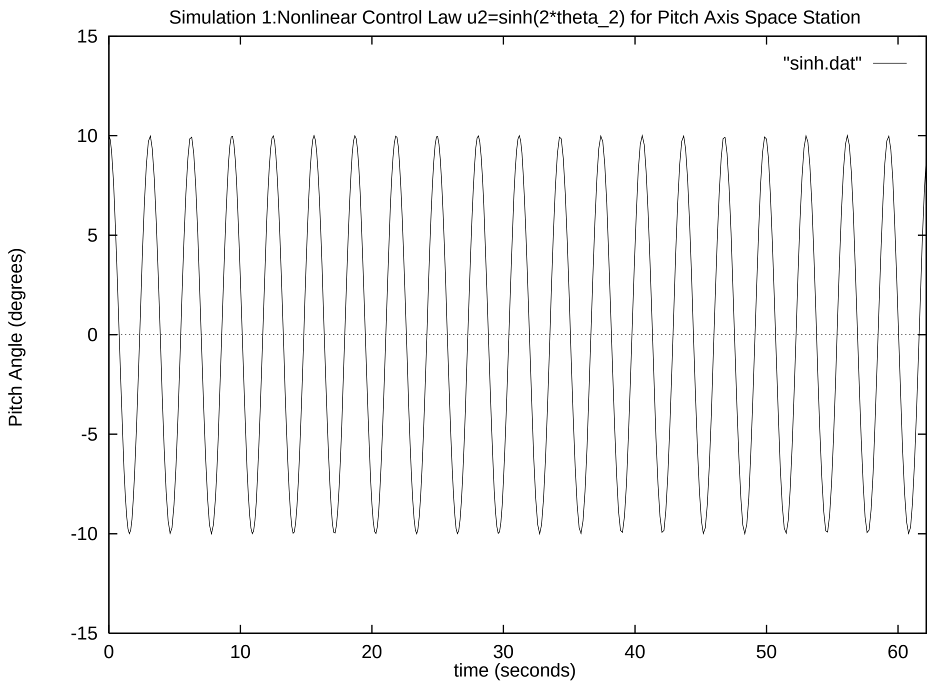
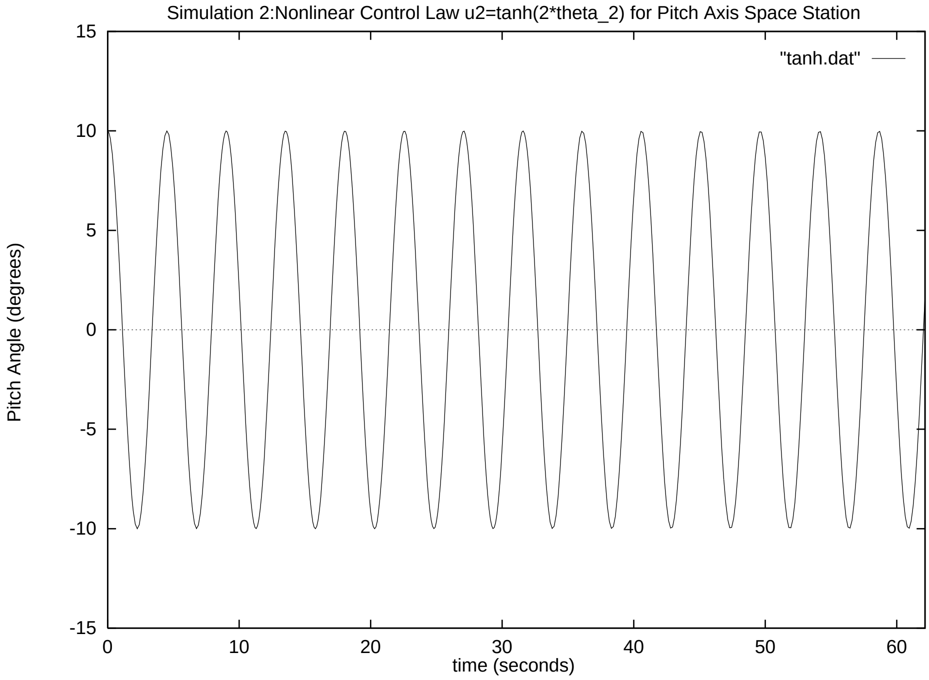
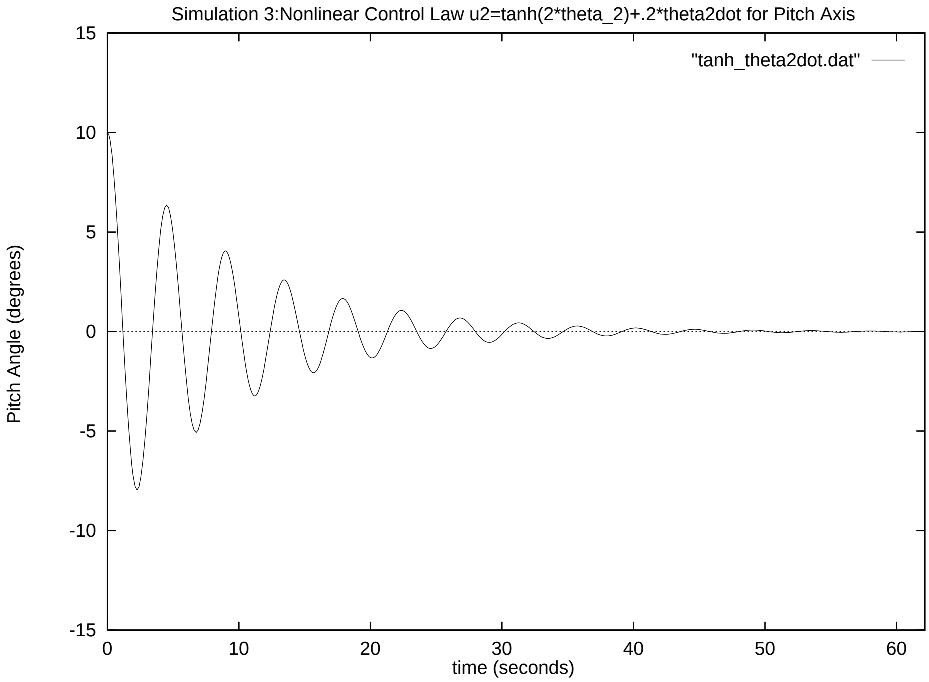
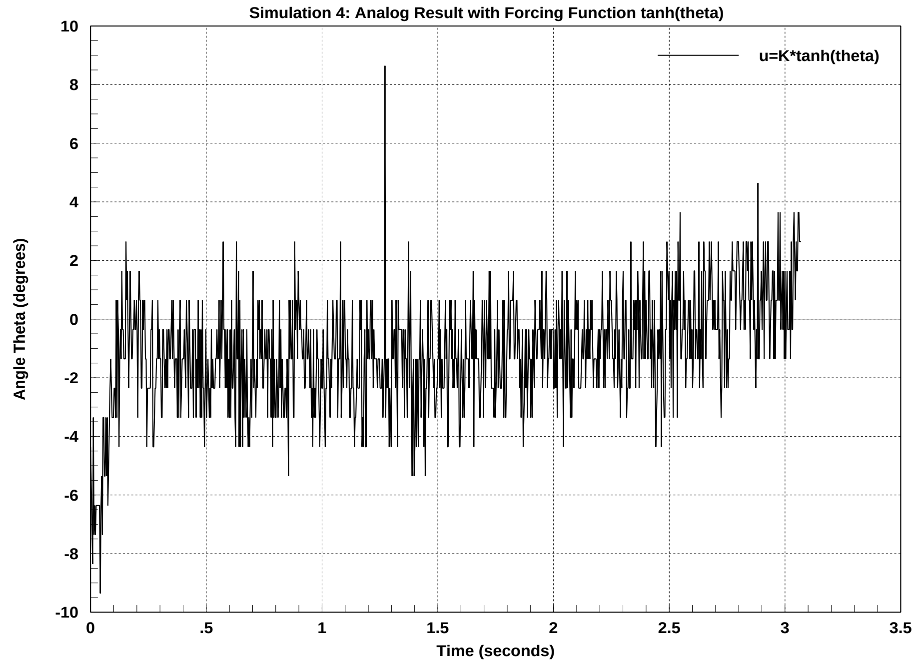


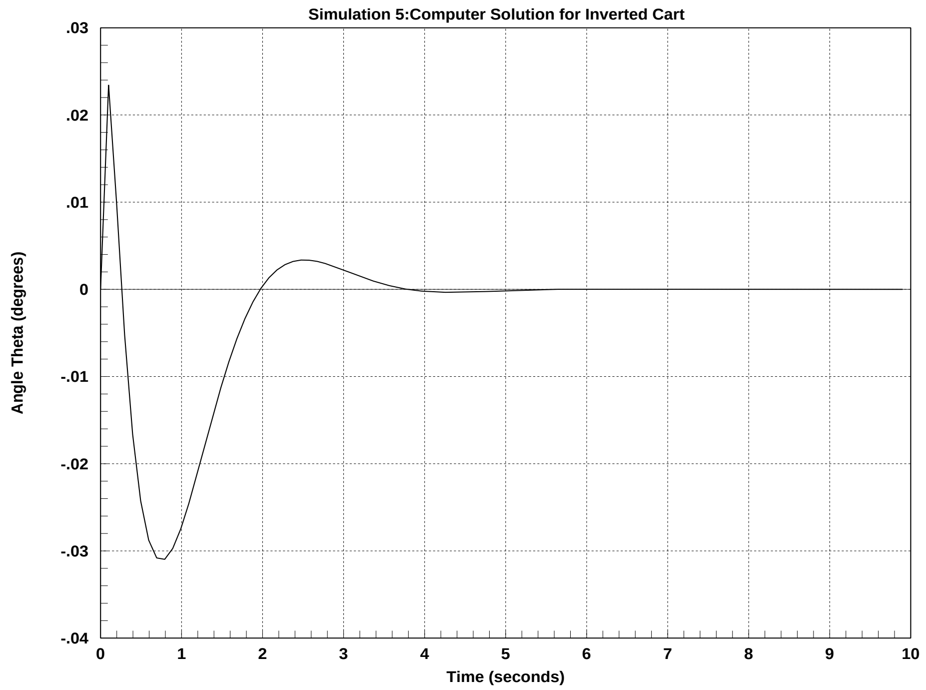
Simulation 1:Nonlinear Control Law u2=sinh(2*theta_2) for Pitch Axis Space Station
"sinh.dat"
Pitch Angle (degrees)
15
10
5
0
-5
-10
-15
0
10
20
30
40
50
60
time (seconds)


Simulation 2:Nonlinear Control Law u2=tanh(2*theta_2) for Pitch Axis Space Station
"tanh.dat"
Pitch Angle (degrees)
time (seconds)
15
10
5
0
-5
-10
-15
0
10
20
30
40
50
60


Simulation 3:Nonlinear Control Law u2=tanh(2*theta_2)+.2*theta2dot for Pitch Axis
"tanh_theta2dot.dat"
Pitch Angle (degrees)
time (seconds)
15
10
5
0
-5
-10
-15
0
10
20
30
40
50
60

2


Simulation 4: Analog Result with Forcing Function tanh(theta)
u=K*tanh(theta)
Angle Theta (degrees)
Time (seconds)
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
.5
1
1.5
2
2.5
3
3.5


Simulation 5:Computer Solution for Inverted Cart
Angle Theta (degrees)
Time (seconds)
.03
.02
.01
0
-.01
-.02
-.03
-.04
0
1
2
3
4
5
6
7
8
9
10