Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Vol. 01, 1995, pp. 17–26
Submitted: June 11, 1995. Published: September 5, 1995.

STOCHASTIC CALCULUS AND INITIAL VALUE ANALYSIS ON THE ONE DIMENSIONAL DIFFUSIONS

Dražen Pantić

Abstract. Stochastic calculus and initial value analysis are used for complete description of distribution type of diffusion processes with Lipschitz coefficients. Sufficient conditions so that the solutions of stochastic differential equations posses absolutely continuous one dimensional distribution are given. For stochastic differential equations with uniformly elliptic coefficients probability density is investigated in detail. Also, the distribution of inverse process is given.

(1991) A.M.S. Subject Classification Codes. 60H07, 60H10.

Key Words and Phrases. Malliavin calculus, solutions of stochastic differential equations, point distributions.

INTRODUCTION

Let $\{X_{t,x}, t \geq 0\}$, $x \in \mathbb{R}$, be solution of SDE

$$X_{t,x} - x = \int_0^t a(X_{s,x}) dB_s + \int_0^t b(X_{s,x}) ds \tag{1}$$

where coefficients a and b have bounded and continuous first order derivatives $(a, b \in \mathbb{C}_b^1)$. Under those conditions for $p > 1$ there exists L_p derivative $DX_{t,x}$ of $X_{t,x}$ w.r.t. initial condition x, so that, as in [2, §8],

$$DX_{t,x} - 1 = \int_0^t a'(X_{s,x}) DX_{s,x} dB_s + \int_0^t b'(X_{s,x}) DX_{s,x}\, ds. \tag{2}$$

Using Ito formula, $DX_{t,x}$ could be computed explicitly as

$$DX_{t,x} = \exp\Big\{ \int_0^t a'(X_{s,x}) dB_s + \int_0^t (b'(X_{s,x}) - \frac{1}{2} a'(X_{s,x}))^2 ds \Big\}. \tag{3}$$

The boundeness of derivatives implies that, for $p > 1$, $DX_{t,x}$ as well as its inverse $(DX_{t,x})^{-1}$ are in L_p, [6], while $(DX_{t,x})^{-1}$ satisfies following SDE, [10, Lemma (3.7)]:

$$(DX_{t,x})^{-1} - 1 = -\int_0^t a'(X_{s,x})(DX_{s,x})^{-1} dB_s + \int_0^t \big(a'(X_{s,x})^2 - b'(X_{s,x})\big)(DX_{s,x})^{-1}\, ds.$$

Department of Mathematics, University of Belgrade, Studentski trg 16, 11000 Beograd, Yugoslavia
E-mail Address: epantidr@ubbg.etf.bg.ac.yu

Also, for a Skorohod integrable process $\boldsymbol{u} = \{u_s, 0 \le s \le T\}$ a directional derivative $D_u X_{t,x}$ of functional $X_{t,x}$ could be expressed in terms of derivatives $\{DX_{t,x}, t \ge 0\}$ in the following way, [12],

$$D_u X_{t,x} = DX_{t,x} \int_0^t (DX_{s,x})^{-1} a(X_{s,x})\, u_s \, ds. \tag{4}$$

First we shall treat the general case. Defining process $\underline{\boldsymbol{u}} = \{\underline{u}_s, 0 \le s \le t\}$ as

$$\underline{u}_s = a(X_{s,x}) DX_{s,x}$$

we have

$$D_{\underline{u}} X_{t,x} = DX_{t,x} \int_0^t a(X_{s,x})^2 \, ds. \tag{5}$$

Process $\underline{\boldsymbol{u}}$ is Ito integrable, so for a continuously differentiable function with compact support f it follows, by integration by parts formula, [5], and relation (5), that

$$\mathbb{E}\{f'(X_{t,x}) DX_{t,x} \int_0^t a(X_{s,x})^2 ds\} = \mathbb{E}\{D_{\underline{\boldsymbol{u}}}(f(X_{t,x}))\} = \mathbb{E}\{f(X_{t,x}) \int_0^t a(X_{s,x}) DX_{s,x} dB_s\}. \tag{6}$$

Knowing that $DX_{t,x}$ is a.s. positive, we have to find conditions so that $\int_0^t a(X_{s,x})^2 ds$ is a.s. positive in order to apply next Lemma.

Lemma 1. *Let X be a random variable such that there exists integrable and almost sure positive random variable Y and a constant $C > 0$ so that for each $f \in \mathbb{C}_b^1$ we have*

$$\mid \mathbb{E}\{f'(X) \cdot Y\} \mid \le C \sup\{\mid f(x) \mid : x \in \mathbb{R}\}.$$

Then probability law generated by the random variable X is absolutely continuous with respect to Lebesgue measure.

Proof. Let us define positive bounded linear functional on the space of the continuously differentiable functions with compact support with

$$L(f) = \mathbb{E}\{f(X)Y\} = \mathbb{E}\{f(X)\mathbb{E}\{Y|X\}\} = \int f(x)\mathbb{E}\{Y|X = x\} G(dx)$$

where G is probability measure generated by X. Then, we have

$$\mid \int f'(x)\mathbb{E}\{Y|X = x\} G(dx) \mid \le C \sup\{\mid f(x) \mid : x \in \mathbb{R}\}$$

so using [4, §1] it follows that there exists density g so that

$$\int f(x)\mathbb{E}\{Y|X = x\} dG(x) = \int f(x) g(x)\, dx.$$

Clearly, the previous relation could be extended to the space of bounded measurable functions, so for a Borel set B we have

$$\mathbb{E}\{\mathbb{I}\{X \in B\} Y\} = \int \mathbb{I}\{x \in B\} g(x)\, dx.$$

If B is of Lebesgue measure zero then

$$\mathbb{E}\{\mathbb{I}\{X \in B\} Y\} = 0.$$

Knowing that $Y > 0$ almost sure one easily gets that $\mathbb{P}\{X \in B\} = 0$, i.e. probability law generated by the random variable X is absolutely continuous with respect to Lebesgue measure. □

GENERAL CASE

In the next theorem we shall completely describe the first moment τ from which $\int_0^t a(X_{s,x})^2 ds$ is a.s. positive, as well as the behavior of diffusion process (1) up to that moment.

Theorem 1. *Let $\{X_{t,x}, t \geq 0\}, x \in \mathbb{R}$, be solution of SDE (1) with coefficients in $\mathbb{C}_1^b$, and $\{Y_{t,x}, t \geq 0\}$ deterministic process defined by*

$$Y_{t,x} - x = \int_0^t b(Y_{s,x})\, ds.$$

Define

$$\tau_x = \inf\{t \geq 0 :\mid a(X_{t,x}) \mid > 0\} \text{ and } D_x = \inf\{t \geq 0 :\mid a(Y_{t,x}) \mid > 0\}.$$

Then, with probability one, $\tau_x = D_x$ and $X_{t,x} = Y_{t,x}$ for $t \leq D_x$.

Proof. Assume first that $0 < D_x < \infty$, and let $0 < t < D_x$. Then

$$X_{t,x} - Y_{t,x} = \int_0^t \big(a(X_{s,x}) - a(Y_{s,x})\big)\, dB_s + \int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big) ds, \quad \text{and}$$

$$\begin{aligned}
&\mathbb{E}\{(X_{t,x} - Y_{t,x})^2\}^{1/2} \leq \\
&\leq \mathbb{E}\Big\{\Big(\int_0^t \big(a(X_{s,x}) - a(Y_{s,x})\big) dB_s\Big)^2\Big\}^{1/2} + \mathbb{E}\Big\{\Big(\int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big) ds\Big)^2\Big\}^{1/2} \leq \\
&\leq \mathbb{E}\Big\{\int_0^t \big(a(X_{s,x}) - a(Y_{s,x})\big)^2 ds\Big\}^{1/2} + \mathbb{E}\Big\{t \int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big)^2 ds\Big\}^{1/2} \leq \\
&\leq K\Big(\int_0^t \mathbb{E}\{(X_{s,x} - Y_{s,x})^2\} ds\Big)^{1/2} + t^{1/2} K\Big(\int_0^t \mathbb{E}\{(X_{s,x} - Y_{s,x})^2\} ds\Big)^{1/2} \leq \\
&\qquad \leq K(1 + D_x^{1/2})\Big(\int_0^t \mathbb{E}\{(X_{s,x} - Y_{s,x})^2\} ds\Big)^{1/2},
\end{aligned}$$

where K is upper bound for first derivatives of a and b. Then, using Gronwall inequality, we can conclude that with probability one $X \equiv Y$ on $[0, D_x]$, and hence $D_x \leq \tau_x$ a.s.

Assume now that $0 < \tau_x < \infty$, and let $0 < t < \tau_x$. As τ_x is optional time the set

$$A = \{\omega : t < \tau_x(\omega)\}$$

is $\mathcal{F}_t$ measurable, where $\{\mathcal{F}_t, t \geq 0\}$ is standard filtration generated by $\{B_t, t \geq 0\}$. Then we have

$$\mathbb{E}\{\int_0^t a(X_{s,x})^2 ds\ \mathbb{I}_A\} = 0.$$

It is obvious that

$$\Big(\int_0^t a(X_{s,x})\, dB_s\Big)^2 \mathbb{I}\{t < \tau_x\} \leq \Big(\int_0^{t\wedge\tau_x} a(X_{s,x})\, dB_s\Big)^2.$$

Having in mind that $t \wedge \tau_x$ is optional time we have

$$\mathbb{E}\Big\{\Big(\int_0^t a(X_{s,x})\, dB_s\Big)^2 \mathbb{I}\{t < \tau_x\}\Big\} \leq \mathbb{E}\Big\{\Big(\int_0^{t\wedge\tau_x} a(X_{s,x})\, dB_s\Big)^2\Big\} = \mathbb{E}\Big\{\int_0^{t\wedge\tau_x} a(X_{s,x})^2\, ds\Big\} = 0,$$

and consequently $\int_0^t a(X_{s,x})dB_s = 0$ a.s. on set A. Hence for each $t > 0$ on the set A we have

$$X_{t,x} - Y_{t,x} = \int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big)\, ds, \qquad \text{and}$$
$$\begin{aligned}
&\mathbb{E}\{(X_{t,x} - Y_{t,x})^2\, \mathbb{I}\{t < \tau_x\}\} = \\
&= \mathbb{E}\Big\{ \Big(\int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big) ds\Big)^2\, \mathbb{I}\{t < \tau_x\}\Big\} \leq \\
&\leq t\, \mathbb{E}\Big\{ \int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big)^2 ds\, \mathbb{I}\{t < \tau_x\}\Big\} \leq \\
&\leq t\, \mathbb{E}\Big\{ \int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big)^2\, \mathbb{I}\{s < \tau_x\}\, ds\Big\} \leq \\
&\leq tK^2 \int_0^t \mathbb{E}\{\, (X_{s,x} - Y_{s,x})^2\, \mathbb{I}\{s < \tau_x\}\}\, ds
\end{aligned}$$

Let us mark $\Delta_t = \mathbb{E}\{(X_{t,x} - Y_{t,x})^2 \mathbb{I}\{t < \tau_x\}\}$. Then previous relation could be written as

$$\Delta_t \leq tK^2 \int_0^t \Delta_s\, ds,$$

from where it follows that $\Delta_t = 0$ for each $t > 0$. Hence we can conclude that for each $t > 0$

$$\mathbb{P}\{\omega : t < \tau_x(\omega), X_{t,x} \neq Y_{t,x}\} = 0.$$

Using the continuity of paths we have that

$$\mathbb{P}\{\omega : (\exists t > 0)(t < \tau_x(\omega), X_{t,x} \neq Y_{t,x})\} = 0,$$

so with probability one $X \equiv Y$ on $[0, \tau_x]$, i.e. $\tau_x \leq D_x$ a.s. and the proof is complete. □

The following theorem completely describes the type of one dimensional distributions of the solutions of SDE :

Theorem 2. *Under conditions from* Theorem 1. *the distribution of random variable $X_{t,x}$ is either degenerate or absolutely continuous, whether $t \leq D_x$ or $t > D_x$.*

Proof. The first part when $t \leq D_x$ is described in Theorem 1 so let $t > D_x$. Then $\int_0^t a(X_{s,x})^2 ds$ is positive a.s. and from relation (6) and Lemma 1 we can reach our conclusion. □

UNIFORMLY ELLIPTIC CASE

Now, we shall treat an important special case when that coefficient a is uniformly elliptic in sense that

$$\inf\{\mid a(x) \mid : x \in \mathbb{R}\} > 0. \tag{EL}$$

Theorem 3. *Let the diffusion process $\{X_{t,x}, t \geq 0\}$ defined by SDE* (1) *has continuously differentiable coefficients a and b with bounded derivatives and let*

$$\inf\{|a(x)| : x \in \mathbb{R}\} > 0.$$

Then we have :

1° *for each $t > 0$ and $x \in \mathbb{R}$ the probability law of random variable $X_{t,x}$ is absolutely continuous w.r.t. Lebesgue measure with transition density $p_t(x, \cdot)$;*

2° *for any bounded, measurable function f and $t>0$ the mapping $x \to \mathbb{E}\{f(X_{t,x})\}$ is continuously differentiable and*

$$\frac{\partial}{\partial x}\mathbb{E}\{f(X_{t,x})\} = t^{-1}\,\mathbb{E}\{f(X_{t,x})\int_0^t a(X_{s,x})^{-1} DX_{s,x} dB_s\}; \tag{7}$$

3° *for each $y \in \mathbb{R}$ and $t>0$ transition density $p_t(x,y)$ is differentiable w.r.t. initial condition x and*

$$\frac{\partial}{\partial x} p_t(x,y) = t^{-1}\,\mathbb{E}\{\int_0^t a(X_{s,x})^{-1} DX_{s,x} dB_s | X_{t,x} = y\}\, p_t(x,y). \tag{8}$$

Proof. 1° Taking $\tilde{u}_s = a(X_{s,x})^{-1} DX_{s,x}$, one has, from (4)

$$D_{\tilde{u}} X_{t,x} = DX_{t,x}\, t. \tag{9}$$

Process $\{\tilde{u}\}$ is Ito integrable, so for a continuously differentiable function with compact support f it follows by integration by parts formula and relation (9), that

$$\mathbb{E}\{f'(X_{t,x}) DX_{t,x}\, t\} = \mathbb{E}\{D_{\tilde{u}}(f(X_{t,x}))\} = \mathbb{E}\{f(X_{t,x})\int_0^t a(X_{s,x})^{-1} DX_{s,x} dB_s\}. \tag{10}$$

Now, $DX_{t,x}$ is a.s. positive random variable and its inverse $(DX_{t,x})^{-1}$ is in L_p for each $p>1$. Also the assumptions we have made on coefficient a assure, via a Burgholder theorem, that $\int_0^t a(X_{s,x})^{-1} DX_{s,x} dB_s$ is in L_2, hence by Lemma 1 assertion 1° holds.

2° First, take $f \in \mathbb{C}_b^1$. Then by [2] and relation (10) one has

$$\frac{\partial}{\partial x}\mathbb{E}\{f(X_{t,x})\} = \mathbb{E}\{f'(X_{t,x}) DX_{t,x}\} = t^{-1}\mathbb{E}\{f(X_{t,x})\int_0^t a(X_{s,x})^{-1} DX_{s,x} dB_s\}.$$

Now, for $x, y \in \mathbb{R}$ we have

$$\mathbb{E}\{f(X_{t,y})\} - \mathbb{E}\{f(X_{t,x})\} = \int_x^y \mathbb{E}\{f(X_{t,z})\int_0^t a(X_{s,z})^{-1} DX_{s,z}\, dB_s\}\,\frac{dz}{t}. \tag{11}$$

Both sides of relation (11) are continuous linear functionals on the set $\mathcal{M}_b$ of measurable and bounded functions, so we can conclude that (11) is valid for each $f \in \mathcal{M}_b$. Also, for $f \in \mathcal{M}_b$ and $t>0$ mapping $x \to \mathbb{E}\{f(X_{t,x})\}$ is absolutely continuous.

If we define semigroup $\{P_t, t \geq 0\}$ of operators on the space $\mathcal{M}_b$ of bounded measurable functions as $P_t f(x) = \mathbb{E}\{f(X_{t,x})\}$, then Markov property of diffusion $\{X_{t,x}\}$ yields, for $\delta > 0$

$$P_{t+\delta} f(x) = \mathbb{E}\{P_\delta f(X_{t,x})\}.$$

Substituting t with $t+\delta$ and f with $P_\delta f$, relation (11) reads as

$$P_{t+\delta} f(y) - P_{t+\delta} f(x) = \int_x^y \mathbb{E}\{P_\delta f(X_{t,z})\int_0^t a(X_{s,z})^{-1} DX_{s,z}\, dB_s\}\,\frac{dz}{t}.$$

Then, from continuity of the function $P_\delta f(x)$ it follows that for each $x \in \mathbb{R}$ there exists

$$\lim_{y \to x} \frac{P_{t+\delta} f(y) - P_{t+\delta} f(x)}{y - x}, \quad \text{and}$$

$$\frac{\partial}{\partial x} P_{t+\delta} f(x) = t^{-1}\,\mathbb{E}\{P_\delta(X_{t,x}) \int_0^t a(X_{s,x})^{-1} DX_{s,x} dB_s\}$$

hence $\frac{\partial}{\partial x} P_{t+\delta} f(x)$ is continuous function in x.

Now relation (11) yields that, for each x

$$\frac{\partial}{\partial x} P_t f(x) = t^{-1}\,\mathbb{E}\{f(X_{t,x}) \int_0^t a(X_{s,x})^{-1} DX_{s,x}\, dB_s\}.$$

3° Let us mark

$$W_{t,x} = t^{-1} \int_0^t a(X_{s,x})^{-1} DX_{s,x} dB_s. \tag{12}$$

Then relation (10) reads as

$$\mathbb{E}\{f'(X_{t,x}) DX_{t,x}\} = \mathbb{E}\{f(X_{t,x}) W_{t,x}\},$$

or, having in mind 1°,

$$\int_{-\infty}^{\infty} f'(u)\,\mathbb{E}\{DX_{t,x} | X_{t,x} = u\} p_t(x,u)\, du = \int_{-\infty}^{\infty} f(u)\,\mathbb{E}\{W_{t,x} | X_{t,x} = u\} p_t(x,u)\, du.$$

Taking, for $\epsilon > 0$ and $y \in \mathbb{R}$,

$$f_\epsilon(x) = \int_{-\infty}^{x} e^{-(t-y)^2/2\epsilon} \frac{dt}{\sqrt{2\pi\epsilon}},$$

we get, using partial integration

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-(u-y)^2/2\epsilon}\,\mathbb{E}\{DX_{t,x} | X_{t,x} = u\} p_t(x,u) \frac{du}{\sqrt{2\pi\epsilon}} &= \\
= \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{u} e^{-(t-y)^2/2\epsilon} \frac{dt}{\sqrt{2\pi\epsilon}}\Big)\mathbb{E}\{W_{t,x} | X_{t,x} = u\} p_t(x,u)\, du &= \\
= -\int_{-\infty}^{\infty} e^{-(t-y)^2/2\epsilon} \Big(\int_{-\infty}^{t} \mathbb{E}\{W_{t,x} | X_{t,x} = u\} p_t(x,u)\, du\Big) \frac{dt}{\sqrt{2\pi\epsilon}}&.
\end{aligned} \tag{13}$$

¿From relation (11) follows, via [3, Theorem 1.28.], that $\mathbb{E}\{DX_{t,x} | X_{t,x} = y\} p_t(x,y)$ is continuous and bounded function so letting $\epsilon \downarrow 0$ one can conclude that

$$\mathbb{E}\{DX_{t,x} | X_{t,x} = y\} p_t(x,y) = -\int_{-\infty}^{y} \mathbb{E}\{W_{t,x} | X_{t,x} = u\} p_t(x,u) du = -\mathbb{E}\{W_{t,x} \mathbb{I}\{X_{t,x} \le y\}\}. \tag{14}$$

It is obvious that there exist

$$\lim_{y \to \pm\infty} \mathbb{E}\{DX_{t,x} | X_{t,x} = y\} p_t(x,y)$$

and as $\mathbb{E}\{DX_{t,x} | X_{t,x} = \cdot\} p_t(x,\cdot)$ is nonnegative, integrable function it follows that

$$\lim_{y \to \pm\infty} \mathbb{E}\{DX_{t,x} | X_{t,x} = y\} p_t(x,y) = 0.$$

Relation (7) implies that

$$\mathbb{E}\{W_{t,x} \mathbb{I}\{X_{t,x} \le y\}\} = \frac{\partial}{\partial x}\mathbb{E}\{\mathbb{I}\{X_{t,x} \le y\}\} = \frac{\partial}{\partial x}\mathbb{P}\{X_{t,x} \le y\},$$

so relation (14) read as

$$\mathbb{E}\{DX_{t,x} | X_{t,x} = y\} p_t(x,y) = -\frac{\partial}{\partial x}\mathbb{P}\{X_{t,x} \le y\}. \tag{15}$$

Now, as $\mathbb{P}\{X_{t,x} \le y\}$ is continuously differentiable in x, we have for $x_1 < x_2$

$$\begin{aligned}\mathbb{P}\{X_{t,x_2} \le y\} - \mathbb{P}\{X_{t,x_1} \le y\} &= -\int_{x_1}^{x_2} \mathbb{E}\{DX_{t,z} | X_{t,z} = y\} p_t(z,y)\, dz \\ &= \int_{x_1}^{x_2} \int_{-\infty}^{y} \mathbb{E}\{W_{t,z} | X_{t,z} = u\} p_t(z,u)\, du dz,\end{aligned}$$

and by Fubini theorem for almost all u function $\mathbb{E}\{W_{t,\cdot} | X_{t,\cdot} = u\} p_t(\cdot, u)$ is integrable and

$$\int_{x_1}^{x_2} \int_{-\infty}^{y} \mathbb{E}\{W_{t,z} | X_{t,z} = u\} p_t(z,u)\, du dz = \int_{-\infty}^{y} \int_{x_1}^{x_2} \mathbb{E}\{W_{t,z} | X_{t,z} = u\} p_t(z,u)\, dz du.$$

Finally, as $\mathbb{P}\{X_{t,x} \le y\} = \int_{-\infty}^{y} p_t(x,u)\, du$ we have that, for each $y \in \mathbb{R}$,

$$\int_{-\infty}^{y} p_t(x_2,u)\, du - \int_{-\infty}^{y} p_t(x_1,u)\, du = \int_{-\infty}^{y} \int_{x_1}^{x_2} \mathbb{E}\{W_{t,z} | X_{t,z} = u\} p_t(z,u)\, dz du$$

so for almost all y we have

$$p_t(x_2,y) - p_t(x_1,y) = \int_{x_1}^{x_2} \mathbb{E}\{W_{t,z} | X_{t,z} = u\} p_t(z,u) dz$$

hence $p_t(x,y)$ is absolutely continuous function in x and

$$\frac{\partial}{\partial x} p_t(x,y) = \mathbb{E}\{W_{t,x} | X_{t,x} = y\}\, p_t(x,y). \quad \square$$

As an appendix to the previous theorem we shall study in detail partial derivative $\frac{\partial}{\partial x} p_t(x,y)$. Precisely, we shall show that the process $t^{-1}\mathbb{E}\{\int_0^t a(X_{s,x})^{-1} DX_{s,x} dB_s | X_{t,x}\}$ is a backwards martingale.

As $\{X_{t,x}, t \ge 0\}$ is a Markov process, for a $f \in \mathcal{M}_b$ and $s < t$ we have

$$\mathbb{E}_x\{f(X_t)\} = \mathbb{E}_x\{\mathbb{E}\{f(X_t) | \mathcal{F}_s\}\} = \mathbb{E}_x\{\mathbb{E}_{X_s}\{f(X_{t-s})\}\},$$

where $\{\mathcal{F}_t, t \ge 0\}$ is standard filtration generated by $\{B_t, t \ge 0\}$, and $\mathbb{E}_x$ is expectation operator generated by initial value $X_0 = x$. Now, 1° of Theorem 3 yields

$$\begin{aligned}&\frac{\partial}{\partial x}\mathbb{E}_x\{f(X_t)\} = \mathbb{E}_x\{f(X_t) W_t\} = \mathbb{E}_x\left\{f(X_t)\mathbb{E}\{W_t | X_t\}\right\} \text{ as well as} \\ &\frac{\partial}{\partial x}\mathbb{E}_x\{f(X_t)\} = \frac{\partial}{\partial x}\mathbb{E}_x\left\{\mathbb{E}_{X_s}\{f(X_{t-s})\}\right\} = \mathbb{E}_x\left\{\mathbb{E}_{X_s}\{f(X_{t-s})\} W_s\right\} = \\ &\qquad = \mathbb{E}_x\left\{\mathbb{E}_{X_s}\{f(X_{t-s})\}\mathbb{E}\{W_s | X_s\}\right\} = \mathbb{E}_x\left\{f(X_t)\mathbb{E}\{W_s | X_s\}\right\},\end{aligned}$$

hence, for $s < t$

$$\mathbb{E}\{W_{t,x} | X_{t,x}\} = \mathbb{E}\{\,\mathbb{E}\{W_{s,x} | X_{s,x}\} \,| X_{t,x}\}.$$

If we mark

$$\tilde{W}_{t,x} = \mathbb{E}\{W_{t,x} | X_{t,x}\}$$

then previous relation could be rewritten as

$$\tilde{W}_{t,x} = \mathbb{E}\{\tilde{W}_{s,x} | X_{s,x}\}, \; s < t.$$

Let us define backwards flow of σ-algebras as

$$\overleftarrow{\mathcal{F}}_{t,x} = \sigma\{X_{u,x}, u \ge t\}, \; t > 0, \; x \in \mathbb{R},$$

and let for $\phi(x_1, \dots, x_n)$ be n-dimensional bounded Borel function. For $s < t \leq t_1 < \dots < t_n$ we have

$$\begin{aligned}
\mathbb{E}_x\{\tilde{W}_{s,x}\phi(X_{t_1,x},\dots,X_{t_n,x})\} &= \mathbb{E}_x\left\{ \mathbb{E}\{\tilde{W}_{s,x}\,\phi(X_{t_1,x},\dots,X_{t_n,x})|\mathcal{F}_t\} \right\} = \\
&= \mathbb{E}_x\left\{ \tilde{W}_{s,x}\,\mathbb{E}\{\phi(X_{t_1,x},\dots,X_{t_n,x})|\mathcal{F}_t\} \right\} = \\
&= \mathbb{E}_x\left\{ \tilde{W}_{s,x}\,\mathbb{E}_{X_{t,x}}\{\phi(X_{t_1-t,x},\dots,X_{t_n-t,x})\} \right\} = \\
&= \mathbb{E}_x\left\{ \mathbb{E}\{\tilde{W}_{s,x}|X_{t,x}\}\,\mathbb{E}_{X_{t,x}}\{\phi(X_{t_1-t,x},\dots,X_{t_n-t,x})\} \right\} = \\
&= \mathbb{E}_x\left\{ \mathbb{E}\{\tilde{W}_{s,x}|X_{t,x}\}\,\phi(X_{t_1,x},\dots,X_{t_n,x}) \right\},
\end{aligned}$$

hence

$$\mathbb{E}\{\tilde{W}_{s,x}|\overleftarrow{\mathcal{F}}_{t,x}\} = \mathbb{E}\{\tilde{W}_{s,x}|X_{t,x}\} = \tilde{W}_{t,x}, \quad \text{i.e.}$$

process

$$\tilde{W}_{t,x} = t^{-1}\mathbb{E}\{\int_0^t a(X_{s,x})^{-1}DX_{s,x}dB_s|X_{t,x}\}$$

is a backward martingale, relative to flow of σ-algebras

$$\overleftarrow{\mathcal{F}}_{t,x} = \sigma\{X_{u,x}, u \geq t\},\ t > 0,\ x \in \mathbb{R}.$$

Having in mind the definition of $\tilde{W}_{t,x}$ and 3° of Theorem 3, we deduce

$$\int_{-\infty}^{\infty} \left|\frac{\partial}{\partial x}p_t(x,y)\right|dy = \mathbb{E}\{\left|\tilde{W}_{t,x}\right|\} < \infty.$$

Moreover, for $s < t$

$$\begin{aligned}
\int_{-\infty}^{\infty} \left|\frac{\partial}{\partial x}p_t(x,y)\right|dy = \mathbb{E}\{\left|\tilde{W}_{t,x}\right|\} &= \mathbb{E}\{\ \left|\mathbb{E}\{\tilde{W}_{s,x}|\overleftarrow{\mathcal{F}}_{t,x}\}\right|\ \} \\
&\leq \mathbb{E}\{\ \mathbb{E}\{\left|\tilde{W}_{s,x}\right||\overleftarrow{\mathcal{F}}_{t,x}\}\ \} = \mathbb{E}\{\ \left|\tilde{W}_{s,x}\right|\ \} = \\
&= \int_{-\infty}^{\infty} \left|\frac{\partial}{\partial x}p_s(x,y)\right|dy
\end{aligned}$$

We have proved the following Corollary:

Corollary 1. *For each $t > 0$ and $x \in \mathbb{R}$ process*

$$\tilde{W}_{t,x} = t^{-1}\mathbb{E}\{\int_0^t a(X_{s,x})^{-1}DX_{s,x}dB_s|X_{t,x}\}$$

is a backward martingale, relative to flow of σ-algebras

$$\overleftarrow{\mathcal{F}}_{t,x} = \sigma\{X_{u,x}, u \geq t\},\ t > 0,\ x \in \mathbb{R}.$$

Consequently, function

$$\int_{-\infty}^{\infty} \left|\frac{\partial}{\partial x}p_t(x,y)\right|dy$$

is finite and decreasing with t. $\square$

It is well known [10, Theorem 2.20] that, for each $t > 0$, the mapping $x \to X_{t,x}$ is a.s. homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, hence we can define inverse random transformation

$$X_t^{-1}(\cdot) : \mathbb{R} \to \mathbb{R}$$

such that

$$X_t^{-1}(X_{t,x}) = x, \quad x \in \mathbb{R}.$$

In the next theorem we shall investigate the distribution of random variable $X_t^{-1}(y)$, i.e. the random function $X_t^{-1}(\cdot)$ applied to the point y.

Theorem 4. *Let everything be as in* Theorem 3. *Then:*

1° *for each $t > 0$ and $y \in \mathbb{R}$ function $\mathbb{E}\{DX_{t,\cdot}|X_{t,\cdot} = y\}p_t(\cdot, y)$ is integrable and*

$$\int_{-\infty}^{+\infty} \mathbb{E}\{DX_{t,x}|X_{t,x} = y\}p_t(x, y)\,dx = 1; \tag{16}$$

2° *for each $t > 0$ and $x, y \in \mathbb{R}$*

$$\mathbb{P}\{X_t^{-1}(y) \le x\} = \int_{-\infty}^{x} \mathbb{E}\{DX_{t,z}|X_{t,z} = y\}p_t(z, y)\,dz; \tag{17}$$

3° *probability density*

$$\rho_t(y, x) = \mathbb{E}\{DX_{t,x}|X_{t,x} = y\}\,p_t(x, y) \tag{18}$$

of the random variable $X_t^{-1}(y)$ satisfy Chapman-Kolmogorov equations:

$$\rho_{t+s}(y, x) = \int_{-\infty}^{+\infty} \rho_s(y, z)\,\rho_t(z, x)\,dz \tag{19}$$

for $t, s > 0$ and $x, y \in \mathbb{R}$.

Proof. 1° As for each $t > 0$, $X_{t,\cdot}$ is a.s. homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$ and $DX_{t,x}$, partial derivative in $L_2(\Omega)$ sense of random variable $X_{t,x}$ w.r.t. x, is a.s. positive, we have

$$\lim_{x \to \pm\infty} X_{t,x} = \pm\infty,$$

and from relation (15) we have, for $x_1 < x_2$

$$\mathbb{P}\{X_{t,x_2} \le y\} - \mathbb{P}\{X_{t,x_1} \le y\} = -\int_{x_1}^{x_2} \mathbb{E}\{DX_{t,z}|X_{t,z} = y\}p_t(z, y)\,dz. \tag{20}$$

Now, letting $x_1 \to -\infty$ and $x_2 \to +\infty$ one can conclude that

$$\int_{-\infty}^{+\infty} \mathbb{E}\{DX_{t,z}|X_{t,z} = y\}p_t(z, y)\,dz = 1,$$

what was to be proved in 1°.

2° ¿From the fact that a.s. $X_{t,x}$ is homeomorphism and that $X_t^{-1}(y)$ is its inverse we have the following equivalence

$$X_t^{-1}(y) \le x \Leftrightarrow X_{t,x} \ge y,$$

so, using relation (20), with $x_1 = -\infty$ and $x_2 = x$

$$\mathbb{P}\{X_t^{-1}(y) \le x\} = \mathbb{P}\{X_{t,x} \ge y\} = 1 - \mathbb{P}\{X_{t,x} < y\} = \int_{-\infty}^{x} \mathbb{E}\{DX_{t,z}\,|X_{t,z} = y\}p_t(z,y)dz.$$

3° Denote with $\mathbb{P}_x\{\cdot\}$ the probability measure generated by the Markov process $\{X_{t,x}, t \ge 0\}$ starting from x. Then for $t, s > 0$ the Markov property yields

$$\mathbb{P}\{X_{t+s,x} \le y\} = \mathbb{P}_x\{X_{t+s} \le y\} = \int_{-\infty}^{+\infty} \mathbb{P}_z\{X_s \le y\}p_t(x,z)\,dz = \mathbb{E}\{\mathbb{P}_{X_{t,x}}\{X_{s,z} \le y\}\}.$$

Using relation (15) we have

$$\begin{aligned}
\rho_{t+s}(y,x) &= -\frac{\partial}{\partial x}\mathbb{P}\{X_{t+s,x} \le y\} = -\frac{\partial}{\partial x}\mathbb{E}\{\mathbb{P}_{X_{t,x}}\{X_{s,z} \le y\}\} = \\
&= -\mathbb{E}\{\frac{\partial}{\partial x}\mathbb{P}_{X_{t,x}}\{X_{s,z} \le y\}\;DX_{t,x}\} = \\
&= -\mathbb{E}\{-\mathbb{E}_{X_{t,x}}\{DX_s|X_s = y\}p_s(X_{t,x},y)\,\mathbb{E}\{D_{t,x}|X_{t,x}\}\} = \\
&= \int_{-\infty}^{+\infty} \mathbb{E}_z\{DX_s|X_s = y\}p_s(z,y)\,\mathbb{E}\{D_{t,x}|X_{t,x} = z\}p_t(x,z)\,dz = \\
&= \int_{-\infty}^{+\infty} \mathbb{E}\{DX_{s,z}\,|X_{s,z} = y\}p_s(z,y)\,\mathbb{E}\{D_{t,x}\,|X_{t,x} = z\}p_t(x,z)\,dz = \\
&= \int_{-\infty}^{+\infty} \rho_s(y,z)\,\rho_t(z,x)\,dz,
\end{aligned}$$

and the theorem is completely proved. □

References

1. N. Bouleau and F. Hirsch, *Formes de Dirichlet generales et densite des variables aleatoires reelles sur l'espace de Wiener.*, J. Func. Anal. **69** (1986), 229–259.
2. I.I. Gihman and A.V. Skorohod, *Stochastic differential equations*, Naukova Dumka, Kiev, 1968. (Russian)
3. S. Kusuoka and D. Stroock, *Applications of the Malliavin Calculus.* I , Proc of the Taniguchi Inter. Symp. on Stoch. Analysis. Katata, North-Holland, 1982, pp. 271–308.
4. P. Malliavin, *Stochastic calculus of variations and hypoelliptic operators*, Proc. Inter. Symp. on Stoch. Diff. Equations, Kyoto 1976, pp. 195–263.
5. D. Nualart, *Noncausal stochastic integrals and calculus*, Proc. of Workshop on Stoch. Anal. and Rel. Topics. Silivri 1986, 1988, pp. 80–129.
6. D. Nualart and M. Zakai, *Generalized stochastic integrals and Malliavin Calculus*, Probability Theory and Rel. Fields **73** (1986), 255–280.
7. D. Pantić, *Stochastic Calculus on Distorted Brownian Motion*, J.Math.Phys. **29** (1988), 207-210.
8. D. Pantić, *Stochastic Calculus and Schroedinger Semigroups Admitting Ground State*, J. Math. Phys. **31** (1990), 1221-1225.
9. D. Pantić, *Toward the Existence of Density for the Distribution of a Brownian Functional*, submitted for publication.
10. D. W. Stroock, *Topics in stochastic differential equations*, Tata Institute of Fundamental Research, 1982.
11. S. Watanabe, *Lectures on stochastic differential equations and Malliavin Calculus*, Tata Institute of Fundamental Research, 1984.
12. M. Zakai, *The Malliavin Calculus*, Acta Appl. Math **3** (1985), 175–207.