THE THEORY OF HEAPS AND THE CARTIER–FOATA MONOID

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Abstract. We present Viennot's theory of heaps of pieces, show that heaps are equivalent to elements in the partially commutative monoid of Cartier and Foata, and illustrate the main results of the theory by reproducing its application to the enumeration of parallelogram polyominoes due to Bousquet–Mélou and Viennot.

1. Introduction

The purpose of this note is to present an alternative, geometric point of view of the *"monoïde partiellement commutatif"* of Cartier and Foata [7], now known as the *Cartier–Foata monoid*. This alternative point of view is due to Viennot [22], who introduced a combinatorial theory which he coined the theory of "*heaps of pieces*." While theoretically completely equivalent to the theory of Cartier and Foata, its main feature is the visualisation of elements of the monoid in terms of so-called "heaps," which makes it very versatile in combinatorial applications.

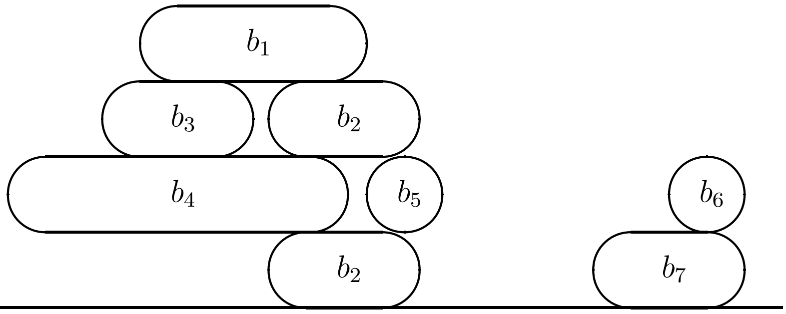
We explain the basic set-up in the next section, and, in Section 3, why this is equivalent to the monoid of Cartier and Foata. The two main theorems (generalising results from [7]) are stated and proved in Section 4, while a beautiful application to parallelogram polyominoes, due to Bousquet–Mélou and Viennot [6], is recalled in Section 5. Other applications include applications to animals, polyominoes, Motzkin paths and orthogonal polynomials, Rogers–Ramanujan identities, Lyndon words, fully commutative elements in Coxeter groups, Bessel functions, and Lorentzian quantum gravity, see [3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24]. The reader is also referred to the survey [2].

2. Heaps of pieces

Informally, a *heap* is what we would imagine. We take a collection of "pieces," say $b_1, b_2, \dots$, and put them one on top of the other, sometimes also sideways, to form a "heap," see Figure 1.

We imagine that pieces can only move vertically (so that the heap in Figure 1 would indeed form a stable arrangement). Note that we allow several copies of a piece to appear in a heap. (This means that they differ only by a vertical translation.) For example, in Figure 1 there appear two copies of b_2. Under these assumptions, there are pieces which can move past each other, and others which cannot. For example, in Figure 1, we can move the piece b_6 higher up, thus moving it higher than b_1 if we wish.

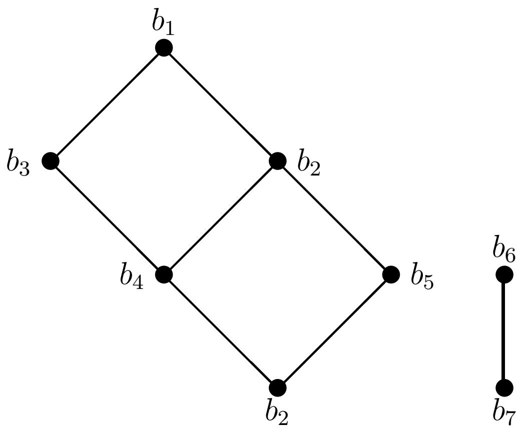
However, we cannot move b_7 higher than b_6, because b_6 blocks the way. On the other hand, we can move b_7 past b_1 (thus taking b_6 with us).



A heap of pieces

Figure 1

To make these considerations mathematically rigorous, consider the "skeleton" of a heap. This is obtained by replacing each piece by a vertex, and by joining two vertices by an edge whenever one vertex blocks the way of the other in the sense described above. The skeleton of the heap in Figure 1 is shown in Figure 2. (There, we have labelled each vertex by the name of the corresponding piece.) Mathematically, a skeleton is a labelled *partially ordered set* or *poset*.



The skeleton of the heap in Figure 1

Figure 2

Definition 2.1. A *partially ordered set (poset)* is a pair $(P, \preceq)$, where P is a set, and where $\preceq$ is a binary relation defined on P which is

(1) reflexive, i.e., $x \preceq x$ for all x in P,
(2) antisymmetric, i.e., if $x \preceq y$ and $y \preceq x$, then $x = y$ for all x, y in P,
(3) transitive, i.e., if $x \preceq y$ and $y \preceq z$ then $x \preceq z$ for all x, y, z in P.

Posets are usually shown graphically in the form of *Hasse diagrams*. The Hasse diagram of a poset is the graph with vertices P, in which x and y are connected by an edge if $x \preceq y$ and there is no z different from x and y with $x \preceq z \preceq y$. Moreover, in the diagram, x is shown at a lower level than y. Clearly, the diagram in Figure 2 is the Hasse diagram of a poset, with vertices labelled by pieces.

Now we can rigorously define what a heap is.

Definition 2.2. Let $\mathcal{B}$ be a set (of pieces) with a symmetric and reflexive binary relation $\mathcal{R}$. A heap is a triple $(P, \preceq, \ell)$, where $(P, \preceq)$ is a poset, and where ℓ is a labelling of the elements of P by elements of $\mathcal{B}$, such that:

(1) If $x, y \in P$ and $\ell(x)\mathcal{R}\ell(y)$, then either $x \preceq y$ or $y \preceq x$.
(2) The relation $\preceq$ is the transitive closure of the relations from (1).

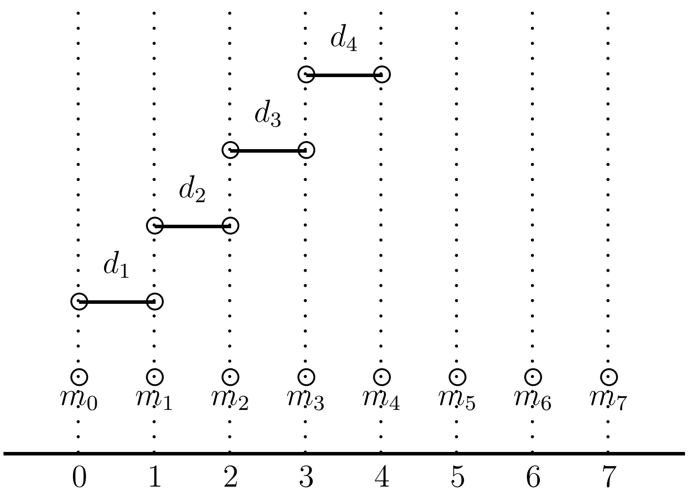
Remark 2.3. The meaning of the relation $\mathcal{R}$ is that it expresses which pieces cannot be moved past each other. That is, a relation $x\mathcal{R}y$ means that x blocks the way of y, and vice versa. Requirement (1) above then says that, hence, in (any realisation of) a heap, either x must be above y, symbolised by $y \preceq x$, or x must be below y, symbolised by $x \preceq y$.

We illustrate this concept with the heap in Figure 1. The pieces are $\mathcal{B} = \{b_1, b_2, \dots, b_7\}$. The relations are (not mentioning the relations of the form $b_i\mathcal{R}b_i$; if a relation $b_i\mathcal{R}b_j$ holds then also $b_j\mathcal{R}b_i$)

$$b_1\mathcal{R}b_2,\ b_1\mathcal{R}b_3,\ b_1\mathcal{R}b_4,\ b_2\mathcal{R}b_4,\ b_3\mathcal{R}b_4,\ b_2\mathcal{R}b_5,\ b_6\mathcal{R}b_7. \tag{2.1}$$

According to Remark 2.3, these relations mean that if $b_i\mathcal{R}b_j$, then in any heap a piece b_i must be either below or above a piece b_j, more precisely, in the corresponding poset a vertex u labelled b_i and a vertex v labelled b_j must either satisfy $u \preceq v$ or $v \preceq u$. In our running example we have $b_2\mathcal{R}b_4$, and indeed there is one piece b_2 which is above b_4, and there is another piece b_2 which is below b_4, see Figure 1.

A class of heaps which is of great importance for studying animals, polyominoes, Motzkin paths and orthogonal polynomials (cf. [2, 3, 4, 5, 12, 13, 24]), is the class of heaps of *monomers* and *dimers*, which we now introduce. (A more general class of heaps will be relevant in our sample application in Section 5.)

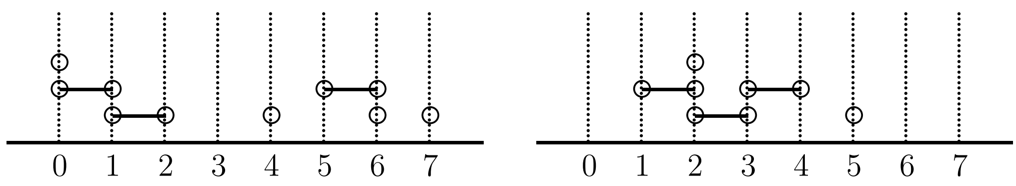


Monomers and dimers

Figure 3

Example 2.4. Let $\mathcal{B} = M \cup D$, where $\mathcal{M} = \{m_0, m_1, \dots\}$ is the set of monomers and $\mathcal{D} = \{d_1, d_2, \dots\}$ is the set of dimers. We think of a monomer m_i as a point, symbolised by a circle, with x-coordinate i, see Figure 3. We think of a dimer d_i as two points, symbolised by circles, with x-coordinates $i-1$ and i which are connected by an edge, see Figure 3.

We impose the relations $m_i \mathcal{R} m_i$, $m_i \mathcal{R} d_i$, $m_i \mathcal{R} d_{i+1}$, $i = 0, 1, \dots$, $d_i \mathcal{R} d_j$, $i-1 \le j \le i$, and extend $\mathcal{R}$ to a symmetric relation. Figure 4 shows two heaps of momomers and dimers.



Two heaps of monomers and dimers

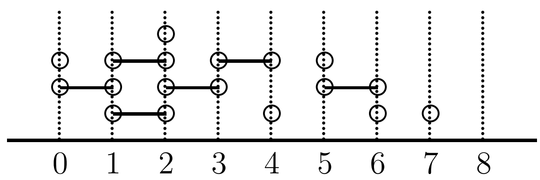
Figure 4

Next we make heaps into a *monoid* by introducing a *composition* of heaps. (A monoid is a set with a binary operation which is associative.) Intuitively, given two heaps H_1 and H_2, the composition of H_1 and H_2, the heap $H_1 \circ H_2$, is the heap which results by putting H_2 on top of H_1. The rigorous definition is the following.

Definition 2.5. Let H_1 and H_2 be heaps, $H_1 = (P_1, \preceq_1, \ell_1)$, $H_2 = (P_2, \preceq_2, \ell_2)$. Then the composition of H_1 and H_2, $H_1 \circ H_2$, is the heap $(H_3, \preceq_3, \ell_3)$ with

(1) $P_3 = P_1 \cup P_2$.
(2) The partial order $\preceq_3$ on P_3 is the transitive closure of
   (a) $v_1 \preceq_3 v_2$ if $v_1 \preceq_1 v_2$,
   (b) $v_1 \preceq_3 v_2$ if $v_1 \preceq_2 v_2$,
   (c) $v_1 \preceq_3 v_2$ if $v_1 \in P_1$, $v_2 \in P_2$ and $\ell_1(v_1) \mathcal{R} \ell_2(v_2)$.

The composition of the two heaps in Figure 4 is shown in Figure 5.



The composition of the heaps in Figure 4

Figure 5

Given pieces $\mathcal{B}$ with relation $\mathcal{R}$, let $\mathcal{H}(\mathcal{B}, \mathcal{R})$ be the set of all heaps consisting of pieces from $\mathcal{B}$, including the empty heap, denoted by $\emptyset$. It is easy to see that Definition 2.5 makes $(\mathcal{H}(\mathcal{B}, \mathcal{R}), \circ)$ into a monoid with unit $\emptyset$.

3. Equivalence with the Cartier–Foata monoid

The monoid which we have just defined in the previous section can be seen to be equivalent to the Cartier–Foata monoid [7]. In order to explain this equivalence, we first observe that heaps could also be encoded by *words* with letters from $\mathcal{B}$, i.e., by sequences of pieces. To obtain a word from a heap $H=(P,\preceq,\ell)$, one considers a *linear extension* of the poset P (i.e., a linear ordering $\le$ of the elements of P in which $x\le y$ whenever $x\preceq y$), and then reads the labels $\ell(x)$ of the elements of P, while x runs through all elements of P in the linear order, bottom to top. Clearly, since there may be several linear extensions of a poset, several different words may be read off from the same heap. For the heap in Figure 1 (see Figure 2 for the corresponding poset), possible such readings are

$$b_2b_7b_4b_5b_6b_3b_2b_1 \text{ and } b_2b_5b_4b_2b_3b_1b_7b_6. \tag{3.1}$$

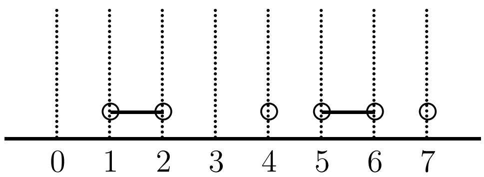
Of course, we want to identify words that are read off from the same heap. Therefore we introduce an equivalence relation on words: We say that the words u and w are equivalent, in symbols $u\sim w$, if w arises from u by a squence of interchanges of two adjacent letters x and y for which $x\not\mathcal{R}y$. For example, given the relation $\mathcal{R}$ as in (2.1), the words in (3.1) arise from each other by the following sequence of interchanges:

$$\begin{aligned}b_2b_7b_4b_5b_6b_3b_2b_1 &\sim b_2b_7b_5b_4b_6b_3b_2b_1 \sim b_2b_5b_7b_4b_6b_3b_2b_1\\ &\sim b_2b_5b_4b_7b_6b_3b_2b_1 \sim b_2b_5b_4b_7b_6b_2b_3b_1 \sim b_2b_5b_4b_7b_2b_6b_3b_1\\ &\sim b_2b_5b_4b_2b_7b_6b_3b_1 \sim b_2b_5b_4b_2b_7b_3b_6b_1 \sim b_2b_5b_4b_2b_3b_7b_6b_1\\ &\sim b_2b_5b_4b_2b_3b_7b_1b_6 \sim b_2b_5b_4b_2b_3b_1b_7b_6.\end{aligned}$$

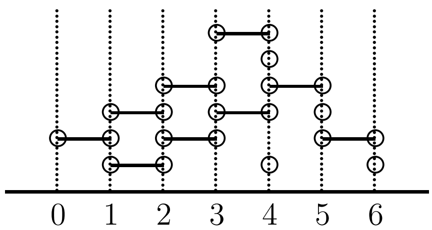
Thus, heaps in $\mathcal{H}(\mathcal{B},\mathcal{R})$ correspond to equivalence classes of words modulo $\sim$. Under this correspondence, the composition of heaps corresponds exactly to the composition of equivalence classes of words induced by concatenation of words. The equivalence of the heap monoid and the Cartier–Foata monoid is now obvious.

4. The main theorems

For the statement of the main theorems in the theory of heaps, we need two more terms. A *trivial heap* is a heap consisting of pieces all of which are pairwise unrelated, i.e., $x\not\mathcal{R}y$ for all pieces x,y in H. Figure 6.a shows a trivial heap consisting of monomers and dimers. A *pyramid* is a heap with exactly one maximal element (in the corresponding poset). Figure 6.a shows a pyramid consisting of monomers and dimers. Finally, if H is a heap, then we write $|H|$ for the number of pieces in H.



a. A trivial heap



b. A pyramid

Figure 6

The following theorem is Proposition 5.3 from [22].

Theorem 4.1. *Let $\mathcal{M}$ be a subset of the pieces $\mathcal{B}$. Then, in the monoid $\mathcal{H}(\mathcal{B},\mathcal{R})$, the (formal) sum of all heaps with maximal pieces (by which we mean the labels of the maximal elements in the corresponding posets) contained in $\mathcal{M}$ is given by*

$$\sum_{\substack{H\in\mathcal{H}(\mathcal{B},\mathcal{R})\\ \text{maximal pieces }\subseteq\mathcal{M}}} H=\Bigg(\sum_{T\in\mathcal{T}(\mathcal{B},\mathcal{R})}(-1)^{|T|}T\Bigg)^{-1}\Bigg(\sum_{T\in\mathcal{T}(\mathcal{B}\backslash\mathcal{M},\mathcal{R})}(-1)^{|T|}T\Bigg), \tag{4.1}$$

where $\mathcal{T}(\mathcal{B},\mathcal{R})$ denotes the set of all trivial heaps with pieces from $\mathcal{B}$, and similarly for $\mathcal{T}(\mathcal{B}\backslash\mathcal{M},\mathcal{R})$. In particular, the sum of all heaps if given by

$$\sum_{H\in\mathcal{H}(\mathcal{B},\mathcal{R})} H=\Bigg(\sum_{T\in\mathcal{T}(\mathcal{B},\mathcal{R})}(-1)^{|T|}T\Bigg)^{-1}. \tag{4.2}$$

Remark 4.2. The inverse of the series which appears on the right-hand sides of (4.1) and (4.2) exists because it has the form $(1-X)^{-1}=\sum_{j\ge0}X^j$.

Remark 4.3. Equation (4.1) generalises Eq. (1) from [7, Introduction, Part A], the latter being equivalent to (4.2).

Proof. Formula (4.2) is the special case of (4.1) in which $\mathcal{M}=\mathcal{B}$. Therefore it suffices to establish (4.1). By multiplication at the left by $\sum_{T\in\mathcal{T}(\mathcal{B},\mathcal{R})}(-1)^{|T|}T$, the latter is equivalent to

$$\sum_{\substack{H\in\mathcal{H}(\mathcal{B},\mathcal{R}),\ T\in\mathcal{T}(\mathcal{B},\mathcal{R})\\ \text{maximal pieces of }H\ \subseteq\mathcal{M}}}(-1)^{|T|}T\circ H=\Bigg(\sum_{T\in\mathcal{T}(\mathcal{B}\backslash\mathcal{M},\mathcal{R})}(-1)^{|T|}T\Bigg). \tag{4.3}$$

We show that most of the terms on the left-hand side of (4.3) cancel each other pairwise. In order to do this, we first fix an arbitrary linear order on the set of pieces $\mathcal{B}$. Now, let (H,T) be a pair of a heap H in $\mathcal{H}(\mathcal{B},\mathcal{R})$ with maximal pieces contained in $\mathcal{M}$ and a trivial heap T. Consider the minimal pieces in $T\circ H$ (again, this means the labels of the minimal elements in the poset corresponding to $T\circ H$) which are below some maximal piece in $T\circ H$ that belongs to $\mathcal{M}$. Let b be the first such minimal piece in the linear order of pieces. Then we form a new pair (H',T') by:

- If $b\in T$, then $H'=b\circ H$ and $T'=T\backslash b$.
- If $b\notin T$, then $H'=H\backslash b$ and $T'=T\circ b$.

In particular, nothing changes in the composed heap, i.e., we have $T'\circ H'=T\circ H$. Hence, we have $(-1)^{|T'|}T'\circ H'=-(-1)^{|T|}T\circ H$. When the same mapping is applied to (H',T') then we obtain back (H,T). Therefore, all the summands on the left-hand side of (4.3) which are indexed by pairs to which the above map is applicable cancel. The remaining summands are those indexed by pairs (H,T) for which $T\circ H$ does not contain any maximal piece in $\mathcal{M}$. This forces H (which "sits" on top of T in $T\circ H$) to be the empty heap, and T to consist of pieces in $\mathcal{B}\backslash\mathcal{M}$ only (all the pieces in a trivial heap are maximal). Thus, (4.3) is established. □

The second main theorem, Proposition 5.10 from [22], concerns the set of pyramids in $\mathcal{H}(\mathcal{B},\mathcal{R})$, which, for convenience, we denote by $\mathcal{P}(\mathcal{B},\mathcal{R})$. In contrast to Theorem 4.1, in the result we must give up non-commutativity.

Theorem 4.4. *For the following (formal) sum indexed by pyramids in $\mathcal{H}(\mathcal{B},\mathcal{R})$, we have*

$$\sum_{P\in\mathcal{P}(\mathcal{B},\mathcal{R})}\frac{1}{|P|}P =_{comm} -\log\left(\sum_{T\in\mathcal{T}(\mathcal{B},\mathcal{R})}(-1)^{|T|}T\right), \tag{4.4}$$

where $=_{comm}$ means that the identity holds in the commutative extension of $\mathcal{H}(\mathcal{B},\mathcal{R})$, that is, in the commutative monoid which arises from $\mathcal{H}(\mathcal{B},\mathcal{R})$ by letting all pieces in $\mathcal{B}$ commute.

Proof. We make heaps into *labelled* objects. More precisely, a *labelled heap* is a heap from $\mathcal{H}(\mathcal{B},\mathcal{R})$ with N pieces which are arbitrarily labelled from 1 up to N (with each label between 1 and N appearing exactly once). Given a labelled heap H_1, we decompose it uniquely into labelled pyramids as follows. To begin with, we "push" the piece labelled 1 "downwards." Let this piece be b_1. (In the language of words, we would push b_1 to the left, using partial commutativity of letters.) Since some pieces cannot move past others, thereby we will take several pieces with us. In fact, these will form a pyramid P_1 with maximal piece b_1. Let H_2 be what remains from H_1 after removing P_1. We now repeat the same procedure with the piece b_2 which has the minimal label within all pieces of H_2. Etc. In the end, we will have obtained a set of labelled pyramids with the special property that in each pyramid the label of its maximal element is the smallest of all labels of pieces of the pyramid.

Let $\tilde{\mathcal{H}}(\mathcal{B},\mathcal{R})$ denote the set of all labelled heaps, and let $\tilde{\mathcal{P}}(\mathcal{B},\mathcal{R})$ denote the set of all labelled pyramids in $\tilde{\mathcal{H}}(\mathcal{B},\mathcal{R})$ with the above special property. Then, by the exponential principle for labelled combinatorial objects (cf. [1, Sec. 5.1, Eqs. (20), (31).c], [11, Sec. II.2.1], or [19, Cor. 5.1.6]), we have immediately

$$\sum_{H\in\tilde{\mathcal{H}}(\mathcal{B},\mathcal{R})}\frac{1}{|H|!}H =_{\text{comm}} \exp\left(\sum_{P\in\tilde{\mathcal{P}}(\mathcal{B},\mathcal{R})}\frac{1}{|P|!}P\right).$$

However, for any (unlabelled) heap in $\mathcal{H}(\mathcal{B},\mathcal{R})$ with N pieces there are exactly $N!$ ways to label the pieces to obtain a labelled heap in $\tilde{\mathcal{H}}(\mathcal{B},\mathcal{R})$, while for any (unlabelled) pyramid from $\mathcal{H}(\mathcal{B},\mathcal{R})$ with N pieces there a exactly $(N-1)!$ ways to label the pieces to obtain a labelled pyramid in $\tilde{\mathcal{P}}(\mathcal{B},\mathcal{R})$. (The reader should recall that the maximal element of the pyramid must get the smallest label.) Hence,

$$\sum_{H\in\mathcal{H}(\mathcal{B},\mathcal{R})}H =_{\text{comm}} \exp\left(\sum_{P\in\mathcal{P}(\mathcal{B},\mathcal{R})}\frac{1}{|P|}P\right),$$

which, in view of (4.2), is equivalent to (4.4). □

In applications, heaps will have weights, which are defined by introducing a weight $w(b)$ (being an element in some commutative ring with unit element) for each piece b in $\mathcal{B}$, and by extending the weight w to all heaps H by letting $w(H)$ denote the product of all weights of the pieces in H. Theorems 4.1 and 4.4 immediately imply the following corollary on the corresponding generating function of heaps.

Corollary 4.5. *Let $\mathcal{M}$ be a subset of the pieces $\mathcal{B}$. Then, the generating function for all heaps with maximal pieces contained in $\mathcal{M}$ is given by*

$$\sum_{\substack{H\in\mathcal{H}(\mathcal{B},\mathcal{R})\\ \text{maximal pieces }\subseteq\mathcal{M}}} w(H) = \frac{\sum\limits_{T\in\mathcal{T}(\mathcal{B}\backslash\mathcal{M},\mathcal{R})} (-1)^{|T|}w(T)}{\sum\limits_{T\in\mathcal{T}(\mathcal{B},\mathcal{R})} (-1)^{|T|}w(T)}, \tag{4.5}$$

where again $\mathcal{T}(\mathcal{B},\mathcal{R})$ denotes the set of all trivial heaps with pieces from $\mathcal{B}$. In particular, the generating function for all heaps is given by

$$\sum_{H\in\mathcal{H}(\mathcal{B},\mathcal{R})} w(H) = \frac{1}{\sum\limits_{T\in\mathcal{T}(\mathcal{B},\mathcal{R})} (-1)^{|T|}w(T)}. \tag{4.6}$$

Furthermore,

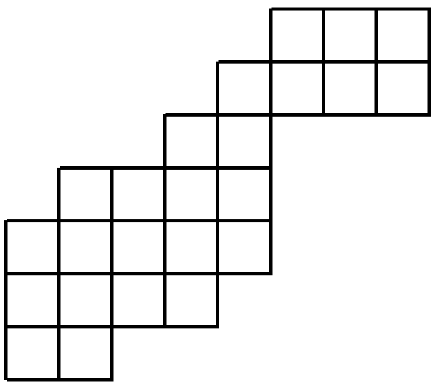
$$\sum_{P\in\mathcal{P}(\mathcal{B},\mathcal{R})} \frac{1}{|P|} w(P) = -\log\left(\sum_{T\in\mathcal{T}(\mathcal{B},\mathcal{R})} (-1)^{|T|}w(T)\right), \tag{4.7}$$

where again $\mathcal{P}(\mathcal{B},\mathcal{R})$ denotes the set of all pyramids in $\mathcal{H}(\mathcal{B},\mathcal{R})$.

5. A sample application

As an illustration, we show how to use the results from Section 4 in order to obtain a formula for a multivariate generating function for *parallelogram polyominoes.* This beautiful application of heaps is due to Bousquet–Mélou and Viennot [6].

A parallelogram polyomino is a non-empty set of square cells in the plane without holes which are enclosed by two paths consisting of unit horizontal and vertical steps in the positive direction, both of which starting in the same point and ending in the same point. An example is shown in Figure 7.



A parallogram polyomino

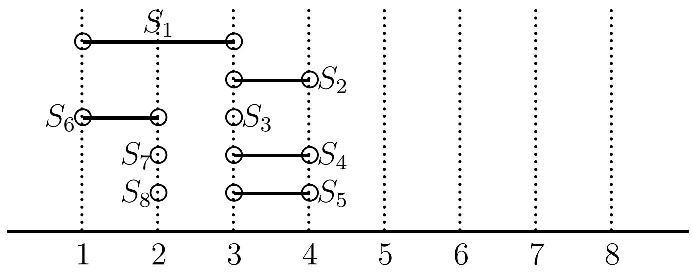
Figure 7

The *area* $a(P)$ of a parallelogram polyomino P is the number of cells of P. The *width* $b(P)$ of a parallelogram polyomino P is the number of columns of cells of P. The *height* $h(P)$ of a parallelogram polyomino P is the number of rows of cells of P. For our parallogram polyomino in Figure 7, P_0 say, we have $a(P_0) = 24$, $b(P_0) = 8$, and $h(P_0) = 7$.

We would like to compute the generating function

$$\sum_P x^{b(P)} y^{h(P)} q^{a(P)},$$

summed over all parallogram polyominoes P. In order to do so, we show that the latter are in bijection with heaps, the pieces of which are segments of the form $[a, c]$, $1 \le a \le c$, with the "obvious" commutation relations: two segments $[a_1, c_1]$ and $[a_2, c_2]$ commute if and only if one segment "is to the left of the other," that is, if $c_1 < a_2$ or if $c_2 < a_1$. See Figure 8 for an example of a heap formed out of pieces of that form. (More precisely, the segments in this figure are $S_1 = [1, 3]$, $S_2 = [3, 4]$, $S_3 = [3, 3]$, $S_4 = [3, 4]$, $S_5 = [3, 4]$, $S_6 = [1, 2]$, $S_7 = [2, 2]$, $S_8 = [2, 2]$.)



A heap of segments

Figure 8

Given a parallelogram polyomino P consisting of n columns, we obtain a heap of segments in the following way: Let $(c_1, c_2, \ldots, c_n)$ be the sequence of column lengths of P (considering the columns from left to right). In our example in Figure 7, these are $(3, 4, 3, 4, 4, 2, 2, 2)$. Furthermore, define $(a_1, a_2, \ldots, a_n)$ to be the sequence with $a_1 = 1$ and, for $i > 1$, a_i being equal to the length of the segment along which the $(i-1)$-st and the i-th column of P touch each other. In our example in Figure 7, these are $(1, 3, 3, 3, 3, 1, 2, 2)$. Now form the heap by piling the segments $[a_i, c_i]$, $i = n, n-1, \ldots, 1$, on each other, that is, we form the heap

$$[a_n, c_n] \circ [a_{n-1}, c_{n-1}] \circ \cdots \circ [a_1, c_1].$$

It can be checked that the heap in Figure 8 corresponds to the parallelogram polyomino in Figure 7 under this correspondence.

It can be shown that this correspondence is, in fact, a bijection between parallelogram polyominoes P and heaps of segments H with a maximal piece of the form $[1, c]$. Moreover, under this correspondence,

(1) $b(P)$ is the number of pieces of H;
(2) $h(P)$ is one more than the sum of all the lengths of pieces of H;
(3) $a(P)$ is the sum of the right abscissae of the pieces of H (i.e., the sum of the c_i's).

For details, we refer the reader to [6].

Now we apply Corollary 4.5 with $\mathcal{H}(\mathcal{B}, \mathcal{R})$ being our heaps of segments, $\mathcal{M}$ being the set of all pieces of the form $[1, c]$, and with the weight w being defined as

$$w(H) = x^{|H|} y^{\sum(\text{lengths of pieces of } H)} q^{\sum(\text{right abscissae of pieces of } H)}.$$

In order to do so, first of all, we must compute the sum

$$\sum_{T\in\mathcal{T}(\mathcal{B},\mathcal{R})}(-1)^{|T|}w(T).$$

Now, a trivial heap consisting of n pieces has the form

$$[a_1,c_1]\circ[a_2,c_2]\circ\cdots\circ[a_n,c_n],$$

where $1\le a_1\le c_1<a_2\le c_2<\cdots<a_n\le c_n$. Therefore,

$$\sum_{T\in\mathcal{T}(\mathcal{B},\mathcal{R})}(-1)^{|T|}w(T)=\sum_{n=0}^{\infty}(-1)^n x^n\sum_{1\le a_1\le c_1<a_2\le c_2<\cdots<a_n\le c_n}y^{\sum_{i=1}^{n}(c_i-a_i)}q^{\sum_{i=1}^{n}c_i}.$$

Now the sums over $c_n,a_n,c_{n-1},a_{n-1},\dots,c_1,a_1$ can be evaluated, in this order, all of them being geometric sums. The result is

$$\sum_{T\in\mathcal{T}(\mathcal{B},\mathcal{R})}(-1)^{|T|}w(T)=\sum_{n=0}^{\infty}\frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q;q)_n\,(yq;q)_n},$$

where $(\alpha;q)_k$ is the *q-shifted factorial*, given by $(\alpha;q)_0:=1$ and

$$(\alpha;q)_k:=(1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{k-1})$$

if k is a positive integer. Similary, we obtain

$$\sum_T{}'(-1)^{|T|}w(T)=-\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1} q^{\binom{n+2}{2}}}{(q;q)_n\,(yq;q)_{n+1}},$$

the sum being over all trivial heaps T in $\mathcal{T}(\mathcal{B},\mathcal{R})$ containing *at least one* piece from $\mathcal{M}$. Hence, remembering that parallelogram polyominoes are *non-empty* sets of cells, we infer that

$$\begin{aligned}\sum_P x^{b(P)}y^{h(P)}q^{a(P)}&=y\left(\frac{\sum\limits_{T\in\mathcal{T}(\mathcal{B}\backslash\mathcal{M},\mathcal{R})}(-1)^{|T|}w(T)}{\sum\limits_{T\in\mathcal{T}(\mathcal{B},\mathcal{R})}(-1)^{|T|}w(T)}-1\right)\\&=y\frac{\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^{n+1} q^{\binom{n+2}{2}}}{(q;q)_n\,(yq;q)_{n+1}}}{\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q;q)_n\,(yq;q)_n}},\end{aligned}\tag{5.1}$$

the sum over P being over all parallogram polyominoes P.

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