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THE EXTENDED COMPLEXITY OF THREE-MANIFOLDS

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By *the complexity* of a 3-manifold M we mean the number of true vertices of a minimal almost simple spine of M. Denote by M_F the 3-manifold obtained by cutting M along a proper surface F. We investigate the relation between the complexity of M and the complexity of M_F. In general, the complexity can become greater. For example, let N be a regular neighborhood of a knot $K \subset S^3$ different from the unknot and the trefoil knot. After cutting S^3 along $F = \partial N$ we get the disjoint union of the complement $C(K)$ of K and the solid torus. The complexity of S^3 is equal to 0, but the complexity of $C(k)$ is greater than 0. Hence, $c(S^3) < c(S^3_F)$. The natural explanation of the growth of the complexity is that F is compressible. If F is incompressible, then $c(M) \le c(M_F)$, see [1]. Moreover, if $F = D^2, S^2$, or F is a surface parallel to ∂M, then $c(M) = c(M_F)$.

For the sake of inductive proofs it seems to be desirable to extend the notion of complexity such that the extended complexity would *decrease* under cutting along any essential incompressible surface. One of possible versions of such complexity was introduced in [1]: *the extended complexity* of a 3-manifold M is the triple $\overline{c}(M) = (c(M), c_1(M), c_2(M))$, where $c(M)$ is the usual complexity of M, $c_1(M)$ is the minimum number of triple circles of almost simple spines of M with $c(M)$ vertices, and $c_2(M)$ is the minimum number of 2-components of almost simple spines of M having $c(M)$ vertices and $c_1(M)$ triple circles.

Theorem 1.[1] Let F be a connected proper incompressible surface in a 3-manifold M such that $\partial F \neq \emptyset$. Then $\overline{c}(M_F) \le \overline{c}(M)$. If, in addition, F is a boundary incompressible surface not homeomorphic to a dick, then $\overline{c}(M_F) < \overline{c}(M)$.

The following example shows that the condition $\partial F \neq \emptyset$ is essential.

Example. Let G be the boundary of a handlebody of genus two and L be a middle circle of G (Fig.1). Consider the 2-dimensional polyhedron $P = \partial(G \times I) \cup$

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$(L \times I)$. Note that P admits a natural embedding into R^3 and can be obtained by attaching four punctured 2-dimensional tori $F_i, 1 \le i \le 4$, to an annulus A, see Fig 2. Evidently, P has no true vertices and consists of two triple circles and five two-components.

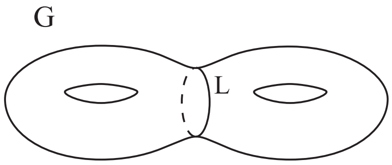

Fig. 1

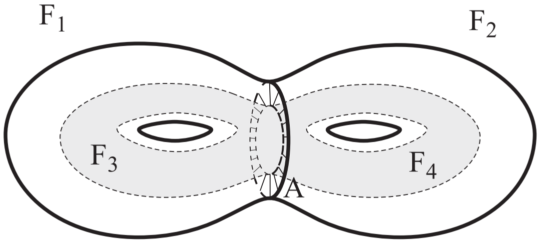

Fig. 2

Consider a regular neighborhood M of P in R^3. Then M is a 3-manifold with the spine P and $\overline{c}(M) = \overline{c}(P) = (0, 2, 5)$. Let us cut M along the closed surface $F = F_1 \cup A \cup F_4$. Then M_F consists of two connected components M_1, M_2, each obtained by identifying the boundary circles of three punctured tori. One can easily show that the extended complexity of each of M_1, M_2 is equal to $(0, 1, 3)$. Therefore, the extended complexity of M_F is $(0, 2, 6)$ and $\overline{c}(M_F) > \overline{c}(M)$.

References

[1] Matveev, S., *Algoritmic Topology and Classification of 3-Manifolds*, Springer-Verlag, Berlin, Heidelberg, 2003.

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