CONSTRUCTIVE FUNCTION THEORY ON SETS OF THE COMPLEX PLANE THROUGH POTENTIAL THEORY AND GEOMETRIC FUNCTION THEORY

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Abstract

This is a survey of some recent results concerning polynomial inequalities and polynomial approximation of functions in the complex plane. The results are achieved by the application of methods and techniques of modern geometric function theory and potential theory.

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1 Introduction

Constructive function theory, or more generally, the theory of the representation of functions by series of polynomials and rational functions, may be described as part of the intersection of analysis and applied mathematics. The main feature of the research discussed in this survey concerns new methods based on conformal invariants to solve problems arising in potential theory, geometric function theory and approximation theory.

The harmonic measure, module and extremal length of a family of curves serve as the main tool. A significant part of the work depends on new techniques for the study of the special conformal mapping of the upper half-plane onto the upper half-plane with vertical slits. These techniques have independent value and have already been applied to other areas of mathematics.

This survey is organized as follows. Section 2 is devoted to the properties of the Green function $g_{\overline{\mathbf{C}}\setminus E}$ and equilibrium measure μ_E of a compact set E on the real line $\mathbf{R}$. Recently, Totik [104], Carleson and Totik [39], and the author [13, 14, 16] suggested new methods to approach these objects. We use a new representation of basic notions of potential theory (logarithmic capacity, the Green function, and equilibrium measure) in terms of a conformal mapping of the exterior of the interval $[0,1]$ onto the exterior of the unit disk $\mathbf{D}$ with finite or infinite number of radial slits [12] – [14]. This method provides a number of new links between potential theory and the theory of univalent functions. Later in this section, we describe the connection between uniformly perfect compact sets and John domains. We give a new interpretation (and a generalization) of a recent remarkable result by Totik [104, (2.8) and (2.12)] concerning the smoothness properties of g_Ω and μ_E. We also demonstrate that if for $E\subset[0,1]$ the Green function satisfies the 1/2-Hölder condition locally at the origin, then the density of E at 0, in terms of logarithmic capacity, is the same as that of the whole interval $[0,1]$. We analyze the geometry of Cantor-type sets and propose an extension of the results by Totik [104, Theorem 5.3] on Cantor-type sets possessing the 1/2-Hölder continuous Green function. We also construct two examples of sets of minimum Hausdorff dimension with Green function satisfying the 1/2-Hölder condition either uniformly or locally.

In Section 3, we continue to discuss the properties of the Green function, but now we motivate this investigation by deriving Remez-type polynomial inequalities. We give sharp uniform bounds for exponentials of logarithmic potentials if the logarithmic capacity of the subset, where they are at most 1, is known. We also propose a technique to derive Remez-type inequalities for complex polynomials. The known results in this direction are scarce and they are proved for relatively simple geometrical cases by using methods of real analysis. We propose to use modern methods of complex analysis, such as the application of conformal invariants in constructive function theory and the theory of quasiconformal mappings in the plane, to study metric properties of complex polynomials. Based on this idea, we discuss a number of problems motivated by [50].

In Section 4, we consider several applications of methods and techniques covered in the previous two sections to questions arising in constructive function theory. The main idea of this section is to create a link between potential theory, geometric function theory and approximation theory. We present a new necessary condition and a new sufficient condition for the approximation of the reciprocal of an entire function by reciprocals of polynomials on the non-negative real line with geometric speed of convergence. The Nikol'skii-Timan-Dzjadyk theorem concerning polynomial approximation of functions on the interval $[-1,1]$ is generalized to the case of approximation of functions given on a compact set on the real line. For analytic functions defined on a continuum E in the complex plane, we discuss Dzjadyk-type polynomial approximations in terms of the k-th modulus of continuity ($k \geq 1$) with simultaneous interpolation at given points of E and decaying strictly inside as $e^{-cn^{\alpha}}$, where c and α are positive constants independent of the degree n of the approximating polynomial.

Each section concludes with a list of open problems.

2 Potential theory

2.1 Basic conformal mapping

Let $E \subset \mathbf{C}$ be a compact set of positive logarithmic capacity $\mathrm{cap}(E)$ with connected complement $\Omega := \overline{\mathbf{C}} \setminus E$ with respect to the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, $g_\Omega(z) = g_\Omega(z,\infty)$ be the Green function of Ω with pole at infinity, and μ_E be the equilibrium measure for the set E (see [62] and [89] for further details on logarithmic potential theory). The metric properties of g_Ω and μ_E are of independent interest in potential theory (see, for example, [38, 68, 65, 89, 20, 39, 104, 13, 14]). They also play an important role in problems concerning polynomial approximation of continuous functions on E (see, for example, [99, 47, 55, 93, 19]) and the behavior of polynomials with a known uniform norm along E (see, for example, [107, 77, 78, 32, 50, 37, 101, 102]).

Note that sets in $\mathbf{R}$ present an important special case of general sets in $\mathbf{C}$. This, for instance, is due to the following standard way to simplify problems concerning estimation of the Green function and capacity. For $E \subset \mathbf{C}$ denote by $E_* := \{r : \{|z| = r\} \cap E \neq \emptyset\}$ the circular projection of E onto the non-negative real line $\mathbf{R}^+ := \{x \in \mathbf{R} : x \geq 0\}$. Then

$$\mathrm{cap}(E) \geq \mathrm{cap}(E_*)$$

and

$$g_{\overline{\mathbf{C}}\setminus E}(-x) \leq g_{\overline{\mathbf{C}}\setminus E_*}(-x), \quad x > 0$$

(provided that $\mathrm{cap}(E_*) > 0$). That is, among those sets that have a given circular projection $E_* \subset \mathbf{R}^+$ the smallest capacity occurs for $E = E_*$ and the worst behavior of the Green function occurs for the same $E = E_*$.

In this survey, we discuss a number of problems in potential theory, polynomial inequalities, and constructive function theory for the case where E is a subset of $\mathbf{R}$.

The main idea of our approach is to connect g_Ω, μ_E, and $\mathrm{cap}(E)$ with the special conformal mapping $F = F_E$ described below. This conformal mapping was recently investigated in [12] – [14] (written in another form it was also discussed in [108, 64, 97]).

Let $E \subset [0,1]$ be a regular set such that $0 \in E$, $1 \in E$. Then $[0,1] \setminus E = \sum_{j=1}^{N}(a_j, b_j)$, where N is finite or infinite.

Denote by $\mathbf{H} := \{z : \Im(z) > 0\}$ the upper half-plane and consider the function

$$F(z) = F_E(z) := \exp\left(\int_E \log(z-\zeta)\, d\mu_E(\zeta) - \log \mathrm{cap}(E)\right), \quad z \in \mathbf{H}. \tag{2.1}$$

It is analytic in $\mathbf{H}$.

Since

$$g_\Omega(z) = \log\frac{1}{\operatorname{cap}(E)} - \int \log\frac{1}{|z-t|}d\mu_E(t), \quad z\in\Omega,$$

the function F has the following obvious properties:

$$|F(z)| = e^{g_\Omega(z)} > 1, \quad z\in\mathbf{H},$$

$$\Im(F(z)) = e^{g_\Omega(z)}\sin\left(\int_E \arg(z-\zeta)\,d\mu_E(\zeta)\right) > 0, \quad z\in\mathbf{H}.$$

Moreover, F can be extended from $\mathbf{H}$ continuously to $\overline{\mathbf{H}}$ such that

$$\begin{aligned} |F(z)| &= 1, \quad z\in E,\\ F(x) &= e^{g_\Omega(x)} > 1, \quad x>1,\\ F(x) &= -e^{g_\Omega(x)} < -1, \quad x<0. \end{aligned}$$

Next, for any $1\le j\le N$ and $a_j\le x_1<x_2\le b_j$, we have

$$\arg\left(\frac{F(x_2)}{F(x_1)}\right) = \arg\exp\left(\int_E \log\frac{x_2-\zeta}{x_1-\zeta}\,d\mu_E(\zeta)\right) = 0,$$

that is,

$$\arg F(x_1) = \arg F(x_2), \quad a_j\le x_1<x_2\le b_j.$$

Our next objective is to show that F is univalent in $\mathbf{H}$. We shall use the following simple result. Let $\sqrt{z^2-1}$, $z\in\overline{\mathbf{C}}\setminus[-1,1]$, be the analytic function defined in a neighborhood of infinity as

$$\sqrt{z^2-1} = z\left(1-\frac{1}{2z^2}+\cdots\right).$$

Then, for any $-1\le x\le 1$ and $z\in\mathbf{H}$,

$$u_x(z) := \Re\left(\frac{\sqrt{z^2-1}}{z-x}\right)\ge 0. \tag{2.2}$$

Using the reflection principle, we can extend F to a function analytic in $\overline{\mathbf{C}}\setminus[0,1]$ by the formula

$$F(z) := \overline{F(\overline{z})}, \quad z\in\mathbf{C}\setminus\overline{\mathbf{H}},$$

and consider the function

$$h(w) := \frac{1}{F(J(w))}, \quad w\in\mathbf{D} := \{w:\,|w|<1\},$$

where J is a linear transformation of the Joukowski mapping, namely

$$J(w) := \frac{1}{2}\left(\frac{1}{2}\left(w+\frac{1}{w}\right)+1\right),$$

which maps the unit disk $\mathbf{D}$ onto $\overline{\mathbf{C}}\setminus[0,1]$. Note that the inverse mapping is defined as follows

$$w = J^{-1}(z) = (2z-1) - \sqrt{(2z-1)^2-1}, \quad z\in\overline{\mathbf{C}}\setminus[0,1].$$

Therefore, for $z \in \mathbf{H}$ and $w = J^{-1}(z) \in \mathbf{D}$, we obtain

$$\begin{aligned}
\frac{wh'(w)}{h(w)} &= w(\log h(w))' = -w\left(\int_E \log(J(w)-\zeta)\,d\mu_E(\zeta)\right)' \\
&= -w\,J'(w)\int_E \frac{d\mu_E(\zeta)}{z-\zeta} = -\frac{1}{4}\left(w-\frac{1}{w}\right)\int_E \frac{d\mu_E(\zeta)}{z-\zeta} \\
&= \frac{1}{2}\int_E \frac{\sqrt{(2z-1)^2-1}}{z-\zeta}\,d\mu_E(\zeta) = \int_E \frac{\sqrt{(2z-1)^2-1}}{(2z-1)-(2\zeta-1)}\,d\mu_E(\zeta).
\end{aligned}$$

According to (2.2) for w under consideration, we have

$$\Re\left(\frac{wh'(w)}{h(w)}\right) \geq 0.$$

Because of the symmetry and the maximum principle for harmonic functions we obtain

$$\Re\left(\frac{wh'(w)}{h(w)}\right) > 0, \quad w \in \mathbf{D}.$$

This means that h is a conformal mapping of $\mathbf{D}$ onto a starlike domain (cf. [79, p. 42]).

Hence, F is univalent and maps $\overline{\mathbf{C}} \setminus [0,1]$ onto a (with respect to ∞) starlike domain $\overline{\mathbf{C}} \setminus K_E$ with the following properties: $\overline{\mathbf{C}} \setminus K_E$ is symmetric with respect to the real line and coincides with the exterior of the unit disk with $2N$ slits.

Note that

$$\begin{aligned}
\operatorname{cap}(E) &= \frac{1}{4\operatorname{cap}(K_E)}, \\
g_\Omega(z) &= \log|F(z)|, \quad z \in \Omega, \qquad (2.3) \\
\pi\mu_E([a,b]) &= |F([a,b]\cap E)|,
\end{aligned}$$

where $|A|$ denotes the linear Lebesgue measure (length) of a Borel set $A \subset \mathbf{C}$.

The connection between the geometry of E and the properties of the conformal mapping F can be studied using conformal invariants such as the extremal length and module of a family of curves (see [1, 63, 82]).

Below, we describe some typical results of this investigation.

2.2 Uniformly perfect subsets of the real line and John domains

The uniformly perfect sets in the complex plane $\mathbf{C}$, introduced by Beardon and Pommerenke [28], are defined as follows. A compact set $E \subset \mathbf{C}$ is *uniformly perfect* if there exists a constant c, $0 < c < 1$, such that for all $z \in E$:

$$E \cap \{\zeta : cr \leq |z-\zeta| \leq r\} \neq \emptyset, \quad 0 < r < \operatorname{diam}(E) := \sup_{z,\zeta\in E}|z-\zeta|.$$

Uniformly perfect sets arise in many areas of complex analysis. For example, many results for simply connected domains can be extended to domains with uniformly perfect boundary (see, for example, [80, 81, 109, 33]). Pommerenke [80] has shown that uniformly perfect sets can be described using a density condition expressed in terms of the logarithmic capacity. Namely, E is uniformly perfect if and only if there exists a positive constant c such that for all $z \in E$:

$$\operatorname{cap}(E \cap \{\zeta : |\zeta - z| \leq r\}) \geq c\, r, \quad 0 < r \leq \operatorname{diam}(E). \tag{2.4}$$

It follows immediately from (2.4) that each component of $\overline{\mathbf{C}} \setminus E$ is regular (for the Dirichlet problem).

Note that sets E with connected complement $\overline{\mathbf{C}} \setminus E$ satisfying (2.4) play a significant role in the solution of the inverse problem of the constructive theory of functions of a complex variable. We refer to [99] where they are called *c-dense* sets.

Another remarkable geometric condition used in direct theorems of approximation theory in $\mathbf{C}$ (cf. [55, 7, 27]) defines a John domain [67, 82]. We consider only the case of a simply connected domain $\Omega \subset \overline{\mathbf{C}}$ such that $\infty \in \Omega$. Following [82, p. 96], we call Ω a *John domain* if there exists a positive constant c such that for every rectilinear crosscut $[a, b]$ of Ω,

$$\operatorname{diam}(H) \leq c|a - b|$$

holds for the bounded component H of $\Omega \setminus [a, b]$.

There is a close connection between these two notions if $E \subset \mathbf{R}$.

Theorem 2.1 ([12]) *A set $E \subset \mathbf{R}$ is uniformly perfect if and only if $\overline{\mathbf{C}} \setminus K_E$, defined in Subsection 2.1, is a John domain.*

Since the behavior of a conformal mapping of a John domain onto the unit disk is well-studied (see, for example, [82]), the theorem above can be useful in the investigation of metric properties of the Green function for the complement of a uniformly perfect subset of $\mathbf{R}$.

In particular, Theorem 2.1 can be used to solve the inverse problem of approximation theory of functions that are continuous on a uniformly perfect compact subset of the real line (see, for details, [12]).

2.3 On the Green function for a complement of a compact subset of R

First, we discuss the following recent remarkable result by Totik [104]. Let $E \subset [0, 1]$ be a compact set of positive logarithmic capacity and let Ω be the complement of E in $\overline{\mathbf{C}}$. The smoothness of g_Ω and μ_E at 0 depends on the density of E at 0. This smoothness can be measured by the function

$$\theta_E(t) := |[0, t] \setminus E|, \quad t > 0.$$

Theorem 2.2 (Totik [104, (2.8) and (2.12)]) *There are absolute positive constants C_1, C_2, D_1 and D_2 such that for $0 < r < 1$,*

$$g_\Omega(-r) \leq C_1 \sqrt{r} \exp\left(D_1 \int_r^1 \frac{\theta_E^2(t)}{t^3} dt\right) \log \frac{2}{\operatorname{cap}(E)}\,, \tag{2.5}$$

$$\mu_E([0, r]) \leq C_2 \sqrt{r} \exp\left(D_2 \int_r^1 \frac{\theta_E^2(t)}{t^3} dt\right). \tag{2.6}$$

The results in [104] are formulated and proven for general compact sets of the unit disk. The theorem above is one of the main steps in their verification. Even though the statement of this theorem is rather particular, the theorem has several notable applications, such as Phragmén-Lindelöf-type theorems, Markov- and Bernstein-type, Remez- and Schur-type polynomial inequalities, etc.

Observe that we can simplify the geometrical nature of the compact set E under consideration. Indeed, it is well-known that there exists a sequence of compact sets $E_n \subset [0,1]$, $n \in \mathbf{N} := \{1, 2, \dots\}$, such that

(i) $E \subset E_n$ and each E_n consists of a finite number of closed intervals,

(ii) for $0 < r < 1$, we have

$$g_\Omega(-r) = \lim_{n\to\infty} g_{\Omega_n}(-r), \quad \Omega_n := \overline{\mathbf{C}} \setminus E_n,$$

$$\mu_E([0,r]) = \lim_{n\to\infty} \mu_{E_n}([0,r]).$$

The set $[0,1] \setminus E_n$ is smaller and simpler then $[0,1] \setminus E$. For example,

$$\theta_{E_n}(t) \le \theta_E(t), \quad t > 0.$$

However, g_{Ω_n} and μ_{E_n} can be arbitrarily close to g_Ω and μ_E. Thus, in order to establish Totik-type results it is natural to concentrate only on compact sets consisting of a finite number of real intervals.

Let

$$E = \cup_{j=1}^k [a_j, b_j], \quad 0 \le a_1 < b_1 < a_2 < \cdots < a_k < b_k \le 1,$$

and let

$$E^* := (0,1) \setminus E = \cup_{j=1}^m (\alpha_j, \beta_j), \quad 0 \le \alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_m < \beta_m \le 1.$$

For $0 < r < 1$, we set $E_r^* := E^* \setminus (0, r]$. We are interested in the case when $E_r^* \ne \emptyset$, i.e.,

$$E_r^* = \cup_{j=1}^{m_r} (\alpha_{j,r}, \beta_{j,r}), \quad r \le \alpha_{1,r} < \beta_{1,r} < \alpha_{2,r} < \cdots < \alpha_{m_r,r} < \beta_{m_r,r} \le 1.$$

Theorem 2.3 ([13]) *For* $0 < r < 1$

$$g_\Omega(-r) \ge c_1 \sqrt{r} \exp\left(d_1 \sum_{j=1}^{m_r} \frac{\beta_{j,r} - \alpha_{j,r}}{\beta_{j,r}} \log \frac{\beta_{j,r}}{\alpha_{j,r}} \right), \tag{2.7}$$

where $c_1 = 1/16$, $d_1 = 10^{-13}$.

Theorem 2.3 provides a lower bound for the Green function (cf. [104, (3.5)]). Since in (2.7) only the size of the components of E_r^* influences this bound, one cannot expect to find an upper bound of the same form. We believe that in a Totik-type theorem not only the size of the components $(\alpha_{j,r}, \beta_{j,r})$ but also their mutual position must be important.

We fix $q > 1$. The set of a finite number of closed intervals $\{[\delta_j, \nu_j]\}_{j=1}^n = \{[\delta_j(r,q), \nu_j(r,q)]\}_{j=1}^n$, where $0 \le \delta_1 < \nu_1 \le \delta_2 < \cdots \le \delta_n < \nu_n \le 1$, is called a *q-covering* of E_r^* if

(i) $E_r^* \subset \cup_{j=1}^n [\delta_j, \nu_j]$,

(ii) either $2\delta_j \le \nu_j$, or $q|E_r^* \cap [\delta_j, \nu_j]| \le \nu_j - \delta_j$.

Theorem 2.4 ([13]) *For $0<r<1$, $q>1$ and any finite q-covering of E_r^* the inequalities*

$$g_\Omega(-r)\le c_2\sqrt{r}\exp\left(d_2\sum_{j=1}^n\frac{\nu_j-\delta_j}{\nu_j}\log\frac{\nu_j}{\delta_j}\right)\log\frac{2}{\mathrm{cap}(E)}\,, \tag{2.8}$$

$$\mu_E([0,r])\le c_3\sqrt{r}\exp\left(d_2\sum_{j=1}^n\frac{\nu_j-\delta_j}{\nu_j}\log\frac{\nu_j}{\delta_j}\right)$$

hold with $c_2=24$, $c_3=5$ and

$$d_2=\max\left(1,\frac{2q^2}{\pi(q-1)^2}\right).$$

Notice that the factor $\log(2/\mathrm{cap}(E))$ on the right of (2.5) and (2.8) appears only to cover pathological cases. It is useful to keep in mind that

$$|E|\le 4\,\mathrm{cap}(E)\le 1.$$

Corollary 2.5 ([13]) *The estimates (2.5) and (2.6) hold with $C_1=384$, $C_2=80$ and $D_1=D_2=120$.*

Corollary 2.6 ([13]) *For the compact set*

$$\tilde E:=\{0\}\cup\bigcup_{n=1}^\infty\bigcup_{j=1}^{n^2}\left[\frac{n^2+j-1}{2^{n+1}n^2},\frac{2n^2+2j-1}{2^{n+2}n^2}\right],$$

we have

$$g_{\overline{\mathbf{C}}\setminus\tilde E}(-r)\le c\sqrt{r},\quad 0<r<1,$$

with some absolute constant $c>0$, which is better than (2.5).

Indeed, let

$$\tilde E_r:=\tilde E\cap[r,1],\quad 0<r<1.$$

For $\tilde E_r^*=(r,1)\setminus\tilde E_r$ with $2^{-k-2}<r\le 2^{-k-1}$, we construct a 2-covering

$$[r,2^{-k}],\left\{\left\{\left[\frac{n^2+j-1}{2^{n+1}n^2},\frac{n^2+j}{2^{n+1}n^2}\right]\right\}_{j=1}^{n^2}\right\}_{n=1}^{k-1},\left[\frac{1}{2},1\right].$$

By the monotonicity of the Green function and Theorem 2.4, for any $0<r<1$ and some absolute constant $c>0$, we obtain

$$g_{\overline{\mathbf{C}}\setminus\tilde E}(-r)\le g_{\overline{\mathbf{C}}\setminus\tilde E_r}(-r)\le c\sqrt{r}.$$

In what follows in this subsection, we assume that 0 is a regular point of E, i.e., $g_\Omega(z)$ extends continuously to 0 and $g_\Omega(0)=0$.

The monotonicity of the Green function yields

$$g_\Omega(z)\ge g_{\overline{\mathbf{C}}\setminus[0,1]}(z),\quad z\in\mathbf{C}\setminus[0,1],$$

that is, if E has the "highest density" at 0, then g_Ω has the "highest smoothness" at the origin. In particular,

$$g_\Omega(-r) \geq g_{\overline{\mathbf{C}}\setminus[0,1]}(-r) > \sqrt{r}, \quad 0 < r < 1. \tag{2.9}$$

In this regard, we would like to explore properties of E whose Green's function has the "highest smoothness" at 0, that is, of E conforming to the following condition

$$g_\Omega(z) \leq c|z|^{1/2}, \quad c = \text{const} > 0,\ z \in \mathbf{C},$$

which is known to be the same as

$$\limsup_{r\to 0} \frac{g_\Omega(-r)}{r^{1/2}} < \infty \tag{2.10}$$

(cf. [89, Corollary III.1.10]). Various sufficient conditions for (2.10) in terms of metric properties of E are stated in [104], where the reader can also find further references.

There are compact sets $E \subset [0,1]$ of linear Lebesgue measure 0 with property (2.10) (see e.g. [104, Corollary 5.2]), hence (2.10) may hold, though the set E is not dense at 0 in terms of linear measure. On the contrary, our first result states that if E satisfies (2.10) then its density in a small neighborhood of 0, measured in terms of logarithmic capacity, is arbitrarily close to the density of $[0,1]$ in that neighborhood.

Theorem 2.7 ([14]) *The condition (2.10) implies*

$$\lim_{r\to 0} \frac{\operatorname{cap}(E\cap[0,r])}{\operatorname{cap}([0,r])} = 1. \tag{2.11}$$

The converse of Theorem 2.7 is slightly weaker.

Theorem 2.8 ([14]) *If E satisfies* (2.11), *then*

$$\lim_{r\to 0} \frac{g_\Omega(-r)}{r^{1/2-\varepsilon}} = 0, \quad 0 < \varepsilon < \frac{1}{2}. \tag{2.12}$$

The connection between properties (2.10), (2.11) and (2.12) is quite delicate. For example, even a slight alteration of (2.10) can lead to the violation of (2.11). As an illustration of this phenomenon, we construct a regular set $E \subset [0,1]$ such that (2.12) holds and

$$\liminf_{r\to 0} \frac{\operatorname{cap}(E\cap[0,r])}{\operatorname{cap}([0,r])} = 0. \tag{2.13}$$

Let

$$b_j := 2^{-2^{j-1}}, \quad a_j := b_{j+1}\log(j+1), \quad j \in \mathbf{N}.$$

Consider

$$E := \{0\} \cup \left(\cup_{j=1}^{\infty}[a_j, b_j]\right).$$

We have

$$\lim_{r\to 0}\left(\log\frac{1}{r}\right)^{-1}\int_r^1 \frac{\theta_E^2(x)}{x^3}\,dx = 0, \tag{2.14}$$

and

$$\lim_{j\to\infty} \frac{b_{j+1}}{a_j} = 0. \tag{2.15}$$

Thus, (2.12) follows from (2.5) and (2.14). Moreover, since

$$\frac{\mathrm{cap}(E\cap[0,a_j])}{a_j}\le\frac{b_{j+1}}{4a_j}\,,$$

(2.15) implies (2.13).

A comprehensive description of E satisfying (2.10) was recently provided by Carleson and Totik [39].

2.4 Cantor-type sets

Let $0<\varepsilon_j<1$ and $K(j)\in\mathbf{N}$, $j\in\mathbf{N}$, be two sequences. Starting from $I=[0,1]$ first, we remove $K(1)$ open intervals $I_1,\dots,I_{K(1)}$ of I such that $I\setminus\cup_{k(1)=1}^{K(1)}I_{k(1)}$ consists of $K(1)+1$ disjoint closed intervals $J_1,\dots,J_{K(1)+1}$ and

$$|I_{k(1)}|=\frac{\varepsilon_1}{K(1)},\quad 1\le k(1)\le K(1),$$

$$|J_{k(1)}|=\frac{1-\varepsilon_1}{K(1)+1},\quad 1\le k(1)\le K(1)+1.$$

Then, for any $1\le k(1)\le K(1)+1$, we remove $K(2)$ open intervals $I_{k(1),1},\dots,I_{k(1),K(2)}$ of $J_{k(1)}$ such that $J_{k(1)}\setminus\cup_{k(2)=1}^{K(2)}I_{k(1),k(2)}$ consists of $K(2)+1$ disjoint closed intervals $J_{k(1),1},\dots,J_{k(1),K(2)+1}$ and

$$|I_{k(1),k(2)}|=\frac{1-\varepsilon_1}{K(1)+1}\frac{\varepsilon_2}{K(2)},\quad 1\le k(2)\le K(2),$$

$$|J_{k(1),k(2)}|=\frac{1-\varepsilon_1}{K(1)+1}\frac{1-\varepsilon_2}{K(2)+1},\quad 1\le k(2)\le K(2)+1,$$

etc.

Denote the Cantor-type set so obtained by $\mathcal{C}=\mathcal{C}(\{\varepsilon_j\},\{K(j)\})$. That is, $\mathcal{C}:=\cap_{n=1}^{\infty}\mathcal{C}_n$, where

$$\mathcal{C}_n=\mathcal{C}_n(\{\varepsilon_j\},\{K(j)\}):=\bigcup_{\mathbf{k}(n)}J_{\mathbf{k}(n)}$$

is the set we obtain after n steps during the construction, and

$$\mathbf{k}(j):=k(1),k(2),\dots,k(j),\quad j\in\mathbf{N}$$

is a multi-index.

Theorem 2.9 ([18]) *The following two conditions are equivalent:*

(i) $g_{\overline{\mathbf{C}}\setminus\mathcal{C}}$ *satisfies (2.10) with* $E=\mathcal{C}$*;*

(ii) $\sum_j\varepsilon_j^2<\infty$.

In the case $K(j)=1$, $j\in\mathbf{N}$, this statement is equivalent to [104, Theorem 5.3], but the latter is stated for the equilibrium measure on $\mathcal{C}$. Interestingly, (ii) does not depend on the sequence $\{K(j)\}$.

2.5 On sparse sets with Green function of the highest smoothness

Let $E \subset \mathbf{R}$ be a compact set with positive logarithmic capacity. For simplicity, we assume that $E \subset [-1,1]$ and $\pm 1 \in E$. Let $\Omega = \overline{\mathbf{C}} \setminus E$. In what follows, we assume that E is a regular set, i.e., g_Ω extends continuously to E where it takes the value 0.

We are going to discuss the metric properties of E such that g_Ω satisfies the 1/2-Hölder condition

$$|g_\Omega(z_2) - g_\Omega(z_1)| \le c|z_2 - z_1|^{1/2}, \quad z_1, z_2 \in \Omega \setminus \{\infty\}, \tag{2.16}$$

where $c > 0$ is some constant.

According to (2.9) the choice of the right-hand side of (2.16) appears to be best suited for this theory. In this regard, we discuss the properties of E whose Green's function has the "highest smoothness".

Recently Totik [103, 104] constructed two examples of a set E whose Green's function satisfies (2.16) and whose linear measure is zero.

We analyze the problem: how sparse can E be, in terms of its Hausdorff dimension $\dim(E)$ [82, p. 224], if it satisfies (2.16).

First, we note that if E satisfies (2.16) then

$$\dim(E) \ge \frac{1}{2}\,. \tag{2.17}$$

Indeed, from (2.16) it follows immediately (for details, see [39], proof of Proposition 1.4) that for any interval $I \subset \mathbf{R}$,

$$\mu_E(I \cap E) \le c_1 |I|^{1/2},$$

where c_1 is a positive constant.

Hence, for any covering of E by intervals $\{I_j\} \subset \mathbf{R}$, we have

$$\sum_j |I_j|^{1/2} \ge c_1^{-1} \sum_j \mu_E(I_j \cap E) \ge c_1^{-1},$$

which proves (2.17).

Theorem 2.10 ([16]) *There exists a regular set $E_0 \subset \mathbf{R}$ with the following properties:*

(i) $g_{\overline{\mathbf{C}}\setminus E_0}$ *satisfies (2.16);*

(ii) $\dim(E_0) = 1/2$.

Next, we describe the construction of E_0 in Theorem 2.10. For $-1 \le a < b \le 1$, we consider two sequences of real numbers

$$\cdots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \cdots, \quad x_k - x_0 = x_0 - x_{-k}$$

and

$$y_0 > y_{\pm 1} > y_{\pm 2} > \cdots, \quad y_k = y_{-k},$$

such that

$$x_0 = \frac{a+b}{2}, \quad y_0 = \frac{b-a}{2} \exp\left\{-\frac{2}{b-a}\right\},$$

$$y_k = (b - x_k)\exp\left\{-\frac{1}{b - x_k}\right\}, \quad k \in \mathbf{N} = \{1, 2, \ldots\},$$
$$\frac{y_k}{x_k - x_{k-1}} = \frac{1}{\pi}\left(\frac{1}{b - x_k} - \log\frac{1}{b - x_k}\right), \quad k \in \mathbf{N}.$$

We have

$$\lim_{k\to\infty} x_{-k} = a,\ \lim_{k\to\infty} x_k = b,\ \lim_{k\to\infty} y_k = 0.$$

Let $z_k = x_k + iy_k$. For $k \in \mathbf{Z} = \{0, \pm1, \pm2, \ldots\}$ consider vertical intervals $J_k = [x_k, z_k]$ and horizontal intervals $I_k = [x_{k-1}, x_k]$. For multi-indices, we use the notation

$$\mathbf{k}(m) = k(1), k(2), \ldots, k(m), \quad \mathbf{k}(m) - 1 = k(1), k(2), \ldots, k(m-1), k(m) - 1,$$

where $m \in \mathbf{N}$ and $k(m) \in \mathbf{Z}$. We inductively define two sequences of intervals

$$\{J_{\mathbf{k}(m)}\}_{\mathbf{k}(m)\in\mathbf{Z}^m} \quad \text{and} \quad \{I_{\mathbf{k}(m)}\}_{\mathbf{k}(m)\in\mathbf{Z}^m}$$

in the following way. Denote by

$$\{J_{\mathbf{k}(1)}\}_{\mathbf{k}(1)\in\mathbf{Z}} \quad \text{and} \quad \{I_{\mathbf{k}(1)}\}_{\mathbf{k}(1)\in\mathbf{Z}}$$

the sequences of vertical and horizontal intervals, which we obtain by the above procedure for $[a, b] = [-1, 1]$.

Next, for $m > 1$ denote by

$$\{J_{\mathbf{k}(m)}\}_{\mathbf{k}(m)\in\mathbf{Z}^m} \quad \text{and} \quad \{I_{\mathbf{k}(m)}\}_{\mathbf{k}(m)\in\mathbf{Z}^m}$$

the sequences of vertical and horizontal intervals, which we obtain by the above procedure for $[a, b] = I_{\mathbf{k}(m-1)}$. The endpoints of $\{J_{\mathbf{k}(m)}\}$ we denote by $x_{\mathbf{k}(m)} \in \mathbf{R}$ and $z_{\mathbf{k}(m)} \in \mathbf{C}$, respectively, so that $I_{\mathbf{k}(m)} = [x_{\mathbf{k}(m)-1}, x_{\mathbf{k}(m)}]$. Since

$$D_0 = \{z = x + iy : |x| < 1, y > 0\} \setminus \left(\bigcup_{m\in\mathbf{N}} \bigcup_{\mathbf{k}(m)\in\mathbf{Z}^m} J_{\mathbf{k}(m)}\right)$$

is a simply connected domain, by the Riemann mapping theorem there exists a conformal mapping ϕ_0 of D_0 onto the upper half plane $\mathbf{H}$.

We interpret the boundary of D_0 in terms of Carathéodory's theory of prime ends (see [79]). Let $P(D_0)$ denote the set of all prime ends of D_0. For a prime end $Z \in P(D_0)$ denote its impression by $|Z|$. By our construction, all prime ends of D_0 are of the first kind, i.e., $|Z|$ is a singleton for any $Z \in P(D_0)$. For the homeomorphism between $D_0 \cup P(D_0)$ and $\overline{\mathbf{H}}$ we preserve the same notation ϕ_0. We denote by $\psi_0 = \phi_0^{-1}$ the inverse homeomorphism. We identify the prime end $\psi_0(w)$, $w \in \mathbf{R}$, with its impression when no confusion can arise. If $z \in \partial D_0$ is the impression of only one prime end it will also cause no confusion if we use the same letter z to designate the prime end and its impression. For example, we write $\infty, -1, z_{\mathbf{k}(m)}, 1$ for prime ends with impressions at those points.

To define ϕ_0 uniquely, we normalize it by the boundary conditions

$$\phi_0(\infty) = \infty,\ \phi_0(-1) = -1,\ \phi_0(1) = 1.$$

Each point of $J_{\mathbf{k}(m)} \setminus \{z_{\mathbf{k}(m)}\}$ is the impression of two prime ends and $z_{\mathbf{k}(m)}$ is the impression of exactly one prime end. Moreover,

$$\phi_0(\{Z \in P(D_0) : |Z| \in J_{\mathbf{k}(m)} \setminus \{x_{\mathbf{k}(m)}\}\})$$

is an open subinterval of $(-1,1)$ which we denote by $J'_{\mathbf{k}(m)} = (\xi^-_{\mathbf{k}(m)}, \xi^+_{\mathbf{k}(m)})$. Let $\xi_{\mathbf{k}(m)} = \phi_0(z_{\mathbf{k}(m)})$.

In [16], we show that the compact set

$$E_0 = [-1,1] \setminus \left(\bigcup_{m \in \mathbf{N}} \bigcup_{\mathbf{k}(m) \in \mathbf{Z}^m} J'_{\mathbf{k}(m)} \right)$$

satisfies the conditions of Theorem 2.10. The crucial fact is that for $w \in \overline{\mathbf{H}} \cap \Omega_0$:

$$g_{\Omega_0}(w) = \frac{\pi}{2} \Im(\psi_0(w)), \tag{2.18}$$

where $\Omega_0 = \overline{\mathbf{C}} \setminus E_0$.

In order to prove (2.18), consider the function

$$h(w) = \begin{cases} \frac{\pi}{2}\Im(\psi_0(w)) & \text{if } w \in \overline{\mathbf{H}} \cap \Omega_0, \\ \frac{\pi}{2}\Im(\psi_0(\overline{w})) & \text{if } w \in \overline{\mathbf{C}} \setminus \overline{\mathbf{H}}. \end{cases}$$

It is continuous in $\Omega_0 \setminus \{\infty\}$ and, according to the distortion properties of ψ_0, the difference

$$h(w) - \log|w|$$

is bounded in a neighborhood of ∞.

The function h is harmonic in $\mathbf{C} \setminus \mathbf{R}$. In order to prove that h coincides with g_{Ω_0} it is sufficient to show that h is harmonic in some neighborhood of each

$$\xi \in (\mathbf{R} \setminus E_0) \setminus \left(\bigcup_{m \in \mathbf{N}} \bigcup_{\mathbf{k}(m) \in \mathbf{Z}^m} \xi_{\mathbf{k}(m)} \right).$$

Let $\varepsilon = \varepsilon(\xi) > 0$ be such that

$$[\xi - \varepsilon, \xi + \varepsilon] \subset (\mathbf{R} \setminus E_0) \setminus \left(\bigcup_{m \in \mathbf{N}} \bigcup_{\mathbf{k}(m) \in \mathbf{Z}^m} \xi_{\mathbf{k}(m)} \right).$$

Since all derivatives of ψ_0 can be extended continuously to $[\xi - \varepsilon, \xi + \varepsilon]$, it is enough to show that for $k = 1, 2; j = 0, 1, 2; j \le k$ and $w = u + iv$:

$$\lim_{\substack{w \to \xi \\ \Im w > 0}} \frac{\partial^k h(w)}{\partial u^j \partial v^{k-j}} = \lim_{\substack{w \to \xi \\ \Im w < 0}} \frac{\partial^k h(w)}{\partial u^j \partial v^{k-j}},$$

which can be easily done.

It is also natural to consider the problem of how sparse can sets E be such that the following local version of (2.16) is valid:

$$g_\Omega(z) = g_\Omega(z) - g_\Omega(-1) \le c\,|z+1|^{1/2}, \quad z \in \Omega \setminus \{\infty\}, \tag{2.19}$$

where $c > 0$ is a constant. The structural properties of compact sets satisfying (2.19) are discussed in [39, 14] (cf. Subsection 2.3), where the density of E near -1 is measured in terms of logarithmic capacity.

Theorem 2.11 ([16]) *There exists a regular set $E_1 \subset \mathbf{R}$ with the following properties:*

(i) $g_{\overline{\mathbf{C}}\setminus E_1}$ *satisfies (2.19);*

(ii) $\dim(E_1) = 0$.

We describe the construction of E_1 in Theorem 2.11. We begin with two sequences of real numbers

$$1 = x_0 > x_1 > x_2 > \cdots > -1 \text{ and } 4 = y_0 > y_1 > y_2 > \cdots > 0$$

such that

$$y_k = (x_k+1)^2, \quad k \in \mathbf{N},$$

$$\lim_{k\to\infty} x_k = -1, \quad \lim_{k\to\infty} y_k = 0,$$

$$\frac{y_k}{x_{k-1}-x_k} \ge \frac{2}{\pi}\log\frac{1}{x_{k-1}-x_k}, \quad x_{k-1}-x_k < \frac{1}{2}, \quad k \in \mathbf{N}.$$

Starting with the set of intervals

$$I_k = [x_{k-1}, x_k],\ J_k = [x_k, x_k + iy_k] = [x_k, z_k], \quad k = k(1) \in \mathbf{N},$$

we construct the sets of intervals $\{I_{\mathbf{k}(m)}\}$ and $\{J_{\mathbf{k}(m)}\}$ in the following manner.

Let, for $m \ge 2$, intervals $\{I_{\mathbf{k}(m-1)}\}$ and $\{J_{\mathbf{k}(m-1)}\}$ be constructed, and let

$$(A_{\mathbf{k}(m-1)})^2 = \exp\left\{m^2 + \pi\sum_{j=1}^{m-1}\frac{|J_{\mathbf{k}(j)}|}{|I_{\mathbf{k}(j)}|}\right\}.$$

We define $\delta_{\mathbf{k}(m-1)} > 0$ such that

$$\frac{|J_{\mathbf{k}(m-1)}|}{\delta_{\mathbf{k}(m-1)}} \ge \frac{4m}{\pi}\log\frac{A_{\mathbf{k}(m-1)}}{\delta_{\mathbf{k}(m-1)}}.$$

Next, we select a finite number of points

$$x_{\mathbf{k}(m-1)-1} = x_{\mathbf{k}(m-1),0} > x_{\mathbf{k}(m-1),1} > \cdots > x_{\mathbf{k}(m-1),K(m)} = x_{\mathbf{k}(m-1)}$$

such that for any $1 \le k(m) \le K(m)$,

$$\frac{1}{2}\delta_{\mathbf{k}(m-1)} \le x_{\mathbf{k}(m-1),k(m)-1} - x_{\mathbf{k}(m-1),k(m)} \le \delta_{\mathbf{k}(m-1)}.$$

Let

$$y_{\mathbf{k}(m)}=\frac{1}{2}y_{\mathbf{k}(m-1)},\quad z_{\mathbf{k}(m)}=x_{\mathbf{k}(m)}+iy_{\mathbf{k}(m)},\quad 0\le k(m)\le K(m),$$
$$J_{\mathbf{k}(m)}=[x_{\mathbf{k}(m)},z_{\mathbf{k}(m)}],\quad 0\le k(m)\le K(m),$$
$$I_{\mathbf{k}(m)}=[x_{\mathbf{k}(m)},x_{\mathbf{k}(m)-1}],\qquad 1\le k(m)\le K(m).$$

Denote by ϕ_1 a conformal mapping of the simply connected domain

$$D_1=\{z=x+iy:\,|x|<1,y>0\}\setminus\left(\bigcup_{m\in\mathbf{N}}\bigcup_{\substack{1\le k(j)\le K(j)\\ 1\le j\le m}}J_{\mathbf{k}(m)}\right),$$

where $K(1)=\infty$, onto $\mathbf{H}$.

Let $P(D_1)$ be the set of all prime ends of D_1. The reasoning about the structure of $P(D_0)$ applies to $P(D_1)$.

We extend ϕ_1 to the homeomorphism $\phi_1: D_1\cup P(D_1)\to\overline{\mathbf{H}}$ and denote the inverse mapping by $\psi_1=\phi_1^{-1}$. Sometimes, for simplicity, we identify $\psi_1(w),w\in\mathbf{R}$, with the impression of $\psi_1(w)$.

We normalize ϕ_1 by the boundary conditions

$$\phi_1(\infty)=\infty,\ \phi_1(-1)=-1,\ \phi_1(1)=1.$$

For $1\le k(j)\le K(j),1\le j\le m-1$ and $1\le k(m)\le K(m)-1$ define intervals

$$J'_{\mathbf{k}(m)}=(\xi^-_{\mathbf{k}(m)},\xi^+_{\mathbf{k}(m)})=\phi_1(\{Z\in P(D_1):\,|Z|\in J_{\mathbf{k}(m)}\setminus\{x_{\mathbf{k}(m)}\}\})$$

and points $\xi_{\mathbf{k}(m)}=\phi_1(z_{\mathbf{k}(m)})$.

In [16], we show that the compact set

$$E_1=[-1,1]\setminus\left(\bigcup_{m\in\mathbf{N}}\bigcup_{\substack{1\le k(j)\le K(j),1\le j\le m-1\\ 1\le k(m)\le K(m)-1}}J'_{\mathbf{k}(m)}\right)$$

satisfies the conditions of Theorem 2.11. The basic idea is to apply the formula

$$g_{\Omega_1}(w)=\frac{\pi}{2}\Im(\psi_1(w)),\quad w\in\overline{\mathbf{H}}\cap\Omega_1,$$

where $\Omega_1=\overline{\mathbf{C}}\setminus E_1$, whose proof is the same as the proof of (2.18).

We conclude this section with the following remark. One of the natural ways to construct sparse sets with Hölder continuous Green function is to consider (nowhere dense) Cantor-type sets (see [77, 32, 65, 101, 103], [104, Chapter 5]).

Let $\{\varepsilon_j\}$ be a sequence with $0<\varepsilon_j<1$. Starting from $[-1,1]$, we first remove the middle ε_1 part of this interval. Then, in the second step, we remove the middle ε_2 part of both remaining intervals, etc. Denote the so obtained Cantor set by $\mathcal{C}=\mathcal{C}(\{\varepsilon_j\})$. According to [104, Theorem 5.1] and the reasoning in the same monograph [104, p. 48, after Corollary 5.2] the following three conditions are equivalent:

(i) $g_{\overline{\mathbf{C}}\setminus\mathcal{C}}$ satisfies (2.16);

(ii) $g_{\overline{\mathbf{C}}\setminus\mathcal{C}}$ satisfies (2.19);

(iii) $\sum_j \varepsilon_j^2 < \infty$.

At the same time, by [82, Theorem 10.5] each Cantor type set $\mathcal{C}(\{\varepsilon_j\})$ with the property

$$\lim_{j\to\infty} \varepsilon_j = 0$$

has Hausdorff dimension 1. Therefore, Cantor-type sets cannot be used in the proof of either Theorem 2.10 or Theorem 2.11.

2.6 Open problems

We begin with a new construction of nowhere dense sets. It is well-known that Cantor-type sets present a remarkable example of nowhere dense sets which are "thick" from the point of view of potential theory (cf. [38, 73, 104]). Motivated by results of this section, we suggest the following new construction of such sets. Let $a_k > 0$, $k \in \mathbf{N}$, be such that $\lim_{k\to\infty} a_k = 0$. Starting from the half-strip

$$\Sigma_0 := \{z = x + iy : |x| < 1, y > 0\},$$

we first divide the base $I_0 := [-1, 1]$ of Σ_0 into two intervals $I_{1,1} := [-1, 0]$ and $I_{1,2} := [0, 1]$ and remove the vertical slit $J_{1,1} := [0, ia_1]$ (with one endpoint in the middle of I_0). Then, in the second step, we divide each of the two new horizontal intervals from the previous step into two subintervals of the same length 1/2 and remove the vertical slits $J_{2,1} := [-1/2, -1/2 + ia_2]$ as well as $J_{2,2} := [1/2, 1/2 + ia_2]$ (with one endpoint in the middle of the base intervals $I_{1,1}$ and $I_{1,2}$, respectively), etc.

As a result, we have a simply connected domain

$$\Sigma = \Sigma(\{a_k\}) := \Sigma_0 \setminus \left(\bigcup_{k,m} J_{k,m} \right).$$

By the Riemann mapping theorem there exists a conformal mapping ϕ of Σ onto the upper half plane $\mathbf{H}$.

We interpret the boundary of Σ in terms of Carathéodory's theory of prime ends (see [79]). Let $P(\Sigma)$ denote the set of all prime ends of Σ. By our construction, all prime ends of Σ are of the first kind, i.e., $|Z|$ is a singleton for any $Z \in P(\Sigma)$. For the homeomorphism between $\Sigma \cup P(\Sigma)$ and $\overline{\mathbf{H}}$, which coincides with ϕ in $\mathbf{H}$, we preserve the same notation ϕ.

To define ϕ uniquely, we normalize it by the boundary conditions

$$\phi(\infty) = \infty,\ \phi(-1) = -1,\ \phi(1) = 1.$$

Each interior point of the slit $J_{k,m} = [x_{k,m}, x_{k,m} + ia_k]$ is the impression of two prime ends. Moreover,

$$J'_{k,m} := \phi(\{Z \in P(\Sigma) : |Z| \in J_{k,m} \setminus \{x_{k,m}\}\})$$

is an open subinterval of $(-1, 1)$.

Hence,

$$E = E(\{a_k\}) := [-1, 1] \setminus \left(\bigcup_{k,m} J'_{k,m} \right)$$

is a nowhere dense subset of $[-1,1]$.

It seems to be an interesting problem to investigate the connection between the geometry of E (for example, its Hausdorff dimension and Hausdorff measure), the rate of decrease of a_k as $k \to \infty$, and continuous properties of the Green function $g_{\overline{\mathbf{C}}\setminus E}$.

The crucial fact is that for $w \in \overline{\mathbf{H}} \cap \Omega$:

$$g_\Omega(w) = \frac{\pi}{2}\Im(\phi^{-1}(w)),$$

where $\Omega = \overline{\mathbf{C}} \setminus E$.

For example, the following problems can be considered.

Problem 1. *Are the following two conditions*

(i) *g_Ω satisfies the the $1/2$-Hölder property, i.e.,*

$$g_\Omega(z) \le c\,\mathrm{dist}(z,E)^{1/2}, \quad z \in \Omega,$$

where $c = c(E) > 0$ is a constant and

$$\mathrm{dist}(A,B) := \inf_{\zeta\in A, \zeta\in B} |z-\zeta|, \quad A, B \subset \mathbf{C},$$

(ii) $\sum_j a_j^2 < \infty$,

equivalent?

(cf. [104, Theorem 5.1] concerning Cantor-type sets).

Problem 2. *Use the ideas of this section to streamline the proof of the Carleson-Totik [39, Theorem 1.1] characterization of compact sets $E \subset \mathbf{R}$ such that the Green function $g_{\overline{\mathbf{C}}\setminus E}$ satisfies a Hölder condition, i.e., there are constants $c > 0$ and $0 < \alpha \le 1/2$ such that*

$$g_{\overline{\mathbf{C}}\setminus E}(z) \le c\,\mathrm{dist}(z,E)^\alpha, \quad z \in \mathbf{C} \setminus E.$$

We conjecture that a more general choice of horizontal intervals $I_{k,m}$ and slits $J_{k,m}$ in the procedure described above will allow one to construct nowhere dense sets with various extremal properties.

Consider a typical example. Let $h(r), 0 \le r \le 1/2$, be a monotone increasing function and $h(0) = 0$. Denote by $\Lambda_h(E)$ the Hausdorff measure of a set $E \subset \mathbf{C}$ with respect to h (see [82, p. 224]). A well-known metric criterion for sets of zero capacity states that (see [62, Theorem 3.14]) if

$$\Lambda_h(E) < \infty, \quad h(r) = |\ln r|^{-1},$$

then $\mathrm{cap}(E) = 0$.

Problem 3. *Show that for any monotone increasing function $g(r), 0 \le r \le 1/2$, satisfying*

$$\lim_{r\to 0} \frac{g(r)}{h(r)} = 0,$$

there exists a compact set $E_g \subset \mathbf{R}$ such that

$$\mathrm{cap}(E_g) > 0 \quad \text{and} \quad \Lambda_g(E_g) < \infty.$$

(cf. [38, Chapter IV]).

3 Remez-type inequalities

3.1 Remez-type inequalities in terms of capacity

Let $\mathbf{\Pi}_n$ be the set of all real polynomials of degree at most $n \in \mathbf{N}$. The Remez inequality [86] (see also [49, 37, 56]) asserts that

$$\|p_n\|_I \le T_n\left(\frac{2+s}{2-s}\right) \tag{3.1}$$

for every $p_n \in \mathbf{\Pi}_n$ such that

$$|\{x \in I : |p_n(x)| \le 1\}| \ge 2 - s, \quad 0 < s < 2, \tag{3.2}$$

where $I := [-1, 1]$, T_n is the Chebyshev polynomial of degree n, and $\|\cdot\|_A$ means the uniform norm along $A \subset \mathbf{C}$.

Since

$$T_n(x) \le (x + \sqrt{x^2 - 1})^n, \quad x > 1,$$

we have by (3.1) that a polynomial p_n with (3.2) satisfies

$$\|p_n\|_I \le \left(\frac{\sqrt{2} + \sqrt{s}}{\sqrt{2} - \sqrt{s}}\right)^n. \tag{3.3}$$

The last inequality (more precisely its n-th root) is asymptotically sharp.

Remez-type inequalities give bounds for classes of functions on a line segment, on a curve or on a region of the complex plane, given that the modulus of the functions is bounded by 1 on some subset of prescribed measure. Remez-type inequalities play a central role in proving other important inequalities for generalized nonnegative polynomials, exponentials of logarithmic potentials and Müntz polynomials. There are a number of recent significant advances in this area. A survey of results concerning various generalizations and numerous applications of this classical inequality can be found in [49], [37] and [56]. In particular, a pointwise, asymptotic version of (3.1) is also obtained [48, Theorem 4]. Namely

$$|p_n(x)| \le \exp\left(c\, n \min\left\{\frac{s}{\sqrt{1-x^2}}, \sqrt{s}\right\}\right) \tag{3.4}$$

holds for $x \in I$ and every $p_n \in \mathbf{\Pi}_n$ satisfying (3.2), where $c > 0$ is some universal constant.

In this section, we discuss an analogue of (3.2) – (3.3) in which we use logarithmic capacity instead of linear length. Our main results deal not only with polynomials, but also with exponentials of potentials (see [49, 50]).

Given a nonnegative Borel measure ν with compact support in $\mathbf{C}$ and finite total mass $\nu(\mathbf{C}) > 0$ as well as a constant $c \in \mathbf{R}$, we say that

$$Q_{\nu,c}(z) := \exp(c - U^\nu(z)), \quad z \in \mathbf{C},$$

where

$$U^\nu(z) := \int \log \frac{1}{|\zeta - z|}\, d\nu(\zeta), \quad z \in \mathbf{C},$$

is the logarithmic potential of ν, is an *exponential of a potential* of degree $\nu(\mathbf{C})$.

Let

$$E_{\nu,c} := \{z \in \mathbf{C} : \, Q_{\nu,c}(z) \le 1\}.$$

Theorem 2.1 and Corollary 2.11 in [50] assert that for $0 < s < 2$ the condition

$$|E_{\nu,c} \cap I| \ge 2 - s \tag{3.5}$$

implies

$$\|Q_{\nu,c}\|_I \le \left(\frac{\sqrt{2}+\sqrt{s}}{\sqrt{2}-\sqrt{s}}\right)^{\nu(\mathbf{C})}. \tag{3.6}$$

Theorem 3.1 ([9]) *Let $0 < \delta < 1/2$. Then the condition*

$$\operatorname{cap}(E_{\nu,c} \cap I) \ge \frac{1}{2} - \delta \tag{3.7}$$

yields that

$$\|Q_{\nu,c}\|_I \le \left(\frac{1+\sqrt{2\delta}}{1-\sqrt{2\delta}}\right)^{\nu(\mathbf{C})}. \tag{3.8}$$

Since $|E_{\nu,c} \cap I| \le 4 \operatorname{cap}(E_{\nu,c} \cap I)$ [79, p. 337], the assertion (3.5) – (3.6) follows from (3.7) – (3.8).

Furthermore, for $0 < \delta < 1/2$, set

$$\nu = \nu_\delta := \mu_{[-1,1-4\delta]}, \quad c = c_\delta := \log \frac{2}{1-2\delta}.$$

Then $\nu(\mathbf{C}) = 1$, $E_{\nu,c} = [-1, 1-4\delta]$,

$$Q_{\nu,c}(x) = \frac{1}{1-2\delta}\left(x + 2\delta + ((x+2\delta)^2 - (1-2\delta)^2)^{1/2}\right), \quad x \ge 1 - 4\delta.$$

Therefore, in this case

$$\operatorname{cap}(E_{\nu,c} \cap I) = \frac{1}{2} - \delta,$$

$$\|Q_{\nu,c}\|_I = Q_{\nu,c}(1) = \frac{1+\sqrt{2\delta}}{1-\sqrt{2\delta}},$$

which shows the sharpness of Theorem 3.1.

Note that the modulus of any complex polynomial $p_n(z) = c\prod_{j=1}^n (z - z_j)$, $0 \ne c \in \mathbf{C}$, can be written as an exponential of a potential in the following way. Let

$$\nu_n := \sum_{j=1}^n \delta_{z_j}, \tag{3.9}$$

where δ_z is the Dirac unit measure in the point $z \in \mathbf{C}$. For $z \in \mathbf{C}$, we have

$$Q_{\nu_n, \log|c|}(z) = \exp(\log|c| + \log \prod_{j=1}^n |z - z_j|) = |p_n(z)|. \tag{3.10}$$

Therefore, applying Theorem 3.1, we obtain for $0 < \delta < 1/2$: the condition

$$\operatorname{cap}(\{x \in I : \, |p_n(x)| \le 1\}) \ge \frac{1}{2} - \delta$$

implies

$$\|p_n\|_I \le \left(\frac{1+\sqrt{2\delta}}{1-\sqrt{2\delta}}\right)^n$$

(cf. (3.2) – (3.3)).

The previous remark can be rewritten in a form as in [41, Theorem 1.1]. Namely, let $r > 0$ and $p_n \in \mathbf{\Pi}_n$ be such that $\|p_n\|_{[-r,r]} = 1$. Then for $0 < \varepsilon < 1$,

$$\mathrm{cap}(\{x \in [-r, r] : |p_n(x)| \le \varepsilon^n\}) \le \frac{2r\varepsilon}{(1+\varepsilon)^2}.$$

This inequality is asymptotically sharp for any fixed ε and r.

Next, we present an analogue of the above results for complex polynomials. By $\mathbf{P}_n$ we denote the set of all complex polynomials of degree at most $n \in \mathbf{N}$. Let

$$\Pi(p_n) := \{z \in \mathbf{C} : |p_n(z)| > 1\}, \quad p_n \in \mathbf{P}_n.$$

From the numerous generalizations of the Remez inequality, we cite one result which is a direct consequence of the trigonometric version of the Remez inequality (and is equivalent to this trigonometric version, up to constants).

Assume that $p_n \in \mathbf{P}_n$, $\mathbf{T} := \{z : |z| = 1\}$ and

$$|\mathbf{T} \cap \Pi(p_n)| \le s, \quad 0 < s \le \frac{\pi}{2}. \tag{3.11}$$

Then, $q_n(t) := |p_n(e^{it})|^2$ is a trigonometric polynomial of degree at most n and, by the Remez-type inequality on the size of trigonometric polynomials (cf. [48, Theorem 2], [37, p. 230]), we have

$$\|p_n\|_{\mathbf{T}} \le e^{2sn}, \quad 0 < s \le \frac{\pi}{2}. \tag{3.12}$$

Our next objective is to discuss an analogue of (3.11) – (3.12) in which we use logarithmic capacity instead of linear length. As before, our main result deals not only with polynomials, but also with exponentials of potentials.

Theorem 3.2 ([10]) *Let $0 < \delta < 1$. Then the condition*

$$\mathrm{cap}(E_{\nu,c} \cap \mathbf{T}) \ge \delta$$

implies that

$$\|Q_{\nu,c}\|_{\mathbf{T}} \le \left(\frac{1+\sqrt{1-\delta^2}}{\delta}\right)^{\nu(\mathbf{C})}.$$

In order to examine the sharpness of Theorem 3.2, we consider the following example.

Let $0 < \alpha < \pi/2$, and let

$$L = L_\alpha := \{e^{i\theta} : 2\alpha \le \theta \le 2\pi - 2\alpha\}. \tag{3.13}$$

Since the function

$$z = \Psi(w) = -w\frac{w-a}{1-aw},$$

where $a = 1/\cos\alpha$, maps $\Delta = \overline{\mathbf{C}} \setminus \overline{\mathbf{D}}$ onto $\Omega := \overline{\mathbf{C}} \setminus L$ (cf. [57]) and since the Green function of Ω with pole at ∞ can be defined via the inverse function $\Phi := \Psi^{-1}$ by the formula

$$g_\Omega(z) = \log|\Phi(z)|, \quad z \in \Omega,$$

we have

$$\operatorname{cap}(L) = \lim_{w\to\infty} \frac{\Psi(w)}{w} = \frac{1}{a} = \cos\alpha, \tag{3.14}$$

as well as

$$\begin{aligned} \max_{z\in\mathbf{T}\setminus L} g_\Omega(z) &= g_\Omega(1) = \log|\Phi(1)| \\ &= \log(a + \sqrt{a^2-1}) = \log\frac{1+\sqrt{1-\operatorname{cap}(L)^2}}{\operatorname{cap}(L)}. \end{aligned} \tag{3.15}$$

Let $c = c_\alpha := -\log\operatorname{cap}(L)$ and let $\nu = \nu_\alpha := \mu_L$ be the equilibrium measure for L; that is, $\nu(\mathbf{C}) = 1$. Since, for $z \in \mathbf{C}$,

$$U^\nu(z) = -g_{\overline{\mathbf{C}}\setminus L}(z) - \log\operatorname{cap}(L),$$

and therefore

$$Q_{\nu,c}(z) = \exp(g_{\overline{\mathbf{C}}\setminus L}(z)),$$

we have $E_{\nu,c} = L$ as well as

$$\|Q_{\nu,c}\|_{\mathbf{T}} = \frac{1+\sqrt{1-\operatorname{cap}(L)^2}}{\operatorname{cap}(L)}.$$

This shows the exactness of Theorem 3.2.

Applying Theorem 3.2 to the exponential of a potential defined by (3.9) – (3.10), we obtain the following: for $p_n \in \mathbf{P}_n$ the condition

$$\operatorname{cap}(\mathbf{T} \setminus \Pi(p_n)) \ge \delta, \quad 0 < \delta < 1, \tag{3.16}$$

yields

$$\|p_n\|_{\mathbf{T}} \le \left(\frac{1+\sqrt{1-\delta^2}}{\delta}\right)^n. \tag{3.17}$$

Since, for any $E \subset \mathbf{T}$, we have $\operatorname{cap}(E) \ge \sin\frac{|E|}{4}$ (see [82]), (3.16) – (3.17) imply the following refinement of (3.11) – (3.12): For $p_n \in \mathbf{P}_n$ the condition

$$|\mathbf{T} \cap \Pi(p_n)| \le s, \quad 0 < s < 2\pi, \tag{3.18}$$

implies

$$\|p_n\|_{\mathbf{T}} \le \left(\frac{1+\sin\frac{s}{4}}{\cos\frac{s}{4}}\right)^n. \tag{3.19}$$

This result is also sharp in the following sense. Let $0 < s < 2\pi$, $\alpha = s/4$, and let $L = L_\alpha$ be defined as in (3.13). By (3.14) and (3.15),

$$g_\Omega(1) = \log\frac{1+\sin\frac{s}{4}}{\cos\frac{s}{4}}.$$

We denote by $f_n(z)$ the n-th Fekete polynomial for a compact set L (see [89]). Hence, condition (3.18) holds for the polynomial $p_n(z) := f_n(z)/\|f_n\|_L$. At the same time, since

$$\lim_{n\to\infty}\left(\frac{|f_n(z)|}{\|f_n\|_L}\right)^{1/n} = \exp(g_\Omega(z)), \quad z \in \Omega \setminus \{\infty\}$$

(see [89, p. 151]), we have

$$\lim_{n\to\infty} |p_n(1)|^{1/n} = \frac{1+\sin\frac{s}{4}}{\cos\frac{s}{4}}$$

(cf. (3.19)).

3.2 Remez-type inequalities in the complex plane

Let $m_2(A)$ be the two-dimensional Lebesgue measure (area) of a set $A \subset \mathbf{C}$. The analogue of (3.1), where the unit interval $[-1,1]$ is replaced by the closure $\overline{G}$ of some bounded Jordan domain $G \subset \mathbf{C}$ and (3.2) by

$$m_2\left(\left\{z \in \overline{G} : |p_n(z)| \le 1\right\}\right) \ge m_2(\overline{G}) - s, \quad 0 < s < m_2(\overline{G}), \tag{3.20}$$

is studied by Erdélyi, Li, and Saff [50]. Let $\mathbf{P}_n(\overline{G}, s)$ denote the subset of polynomials in $\mathbf{P}_n$ satisfying (3.20), and let

$$R_n(z,s) := \sup_{p_n \in \mathbf{P}_n(\overline{G},s)} |p_n(z)|, \quad z \in L := \partial G.$$

If L is a C^2-curve it is established in [50] that there is a constant $c_j = c_j(G) > 0$ where $j = 1, 2$, such that

$$R_n(z,s) \le \exp(c_1 n\sqrt{s}), \quad z \in L, \quad 0 < s \le c_2 < m_2(\overline{G}). \tag{3.21}$$

Actually, this result is established in a more general context of exponentials of logarithmic potentials, where it is used to prove Nikol'skii-type inequalities (cf. [49],[50]). The same problem was investigated recently [61, Theorem 2.3] for domains with smooth boundary (under weaker restrictions on the smoothness rate than in [50]).

We generalize the above results in two directions: we obtain pointwise bounds for $R_n(z,s)$, depending on $z \in L$, and we replace the strong C^2 restriction for the boundaries of G by weaker ones. Our results can easily be generalized to exponentials of logarithmic potentials as well. The method to obtain our (sharp up to constants) estimates differs from the approaches used elsewhere [50],[61]. We make use of properties of Green's functions (cf. [89]), principles of symmetrization (cf. [26]), and the technique of moduli of families of curves (cf. [1], [63]), combined with a useful estimate from [23].

To aid in further discussion, we introduce additional notation. For $z \in \mathbf{C}$ and $r > 0$, let

$$D(z,r) := \{\zeta : |z-\zeta| < r\}, \quad D(r) := D(0,r),$$

$$C(z,r) := \{\zeta : |z-\zeta| = r\}, \quad C(r) := C(0,r).$$

Let $G \subset \mathbf{C}$ be a bounded Jordan domain, $\overline{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ and

$$L := \partial G, \quad \Omega := \overline{\mathbf{C}} \setminus \overline{G}, \quad \gamma_z(r) := \Omega \cap C(z,r).$$

We use the convention that $c_1, c_2, \ldots$ denote positive constants and $\varepsilon_1, \varepsilon_2, \ldots$ sufficiently small positive constants. If not stated otherwise, we assume that both types of constants depend only on G.

Theorem 3.3 ([25]) *Let G be a bounded domain and $z \in L := \partial G$. Suppose that there are constants $\varepsilon_1, \varepsilon_2$ such that*

$$\varepsilon_1 r \le |\gamma_z(r)| \le (2\pi - \varepsilon_1) r, \quad 0 < r < \varepsilon_2. \tag{3.22}$$

Then there exist constants $c_1, c_2, c_3 > c_2$ and $\varepsilon_3 < \left(\frac{\varepsilon_2}{2c_3}\right)^2$ depending only on ε_1 and ε_2 such that

$$R_n(z,s) \le \exp\left(c_1 n \exp\left(-\pi \int_{c_3\sqrt{s}}^{\varepsilon_2/2} \frac{dr}{h_{z,c_2\sqrt{s}}(r)} \right) \right), \quad 0 < s < \varepsilon_3, \tag{3.23}$$

where

$$h_{z,\delta}(r) := \sup_{|t-r|\le\delta} |\gamma_z(t)|, \quad 0 < \delta < r.$$

Moreover, for any arbitrary bounded domain G

$$R_n(z,s) \le \left(\frac{c_4}{m_2(\overline{G}) - s} \right)^n, \quad 0 < s < m_2(\overline{G}), \tag{3.24}$$

where $c_4 > 2\, m_2(\overline{G})$ depends only on the diameter of G.

The inequality (3.23) is the main statement of the theorem. The estimate (3.24) is included for the completeness of the result. The condition (3.22) excludes any domain with a cusp at z.

In the proof of Theorem 3.3, we exploit the following deep connection between estimates which express the possible growth of a polynomial with a known norm on a given compact set $E \subset \mathbf{C}$ and the behavior of the Green function for $\Omega = \overline{\mathbf{C}} \setminus E$.

For $z \in \Omega$ and $u > 0$, the following two conditions are equivalent:

(i) $g_\Omega(z) \le u$;

(ii) for any $p \in \mathbf{P}_n$ and $n \in \mathbf{N}$,

$$|p(z)| \le e^{un} \|p\|_E.$$

Indeed, (i) $\Rightarrow$ (ii) follows from the Bernstein–Walsh lemma [107, p. 77]. (ii) $\Rightarrow$ (i) is a consequence of a result by Myrberg and Leja (see [79, p. 333]).

We study the properties of the Green function by methods of geometric function theory (using symmetrization, moduli of curve families, distortion theorems, harmonic measure, etc.) which allow us, according to the implication (i) $\Rightarrow$ (ii), to get (3.23).

Note that the sharpness of the results for the Green function means, by virtue of the equivalence (i) and (ii), the sharpness (up to constants) of the corresponding Remez-type inequalities.

Since for an arbitrary Jordan domain G, $z \in L$ and $0 < \delta < r$,

$$h_{z,\delta}(r) \le 2\pi(r + \delta),$$

according to (3.23), for each domain satisfying (3.22), we obtain

$$R_n(z,s) \le \exp(c_5 n s^{1/4}), \qquad 0 < s < \varepsilon_3. \tag{3.25}$$

An example below shows that for domains with cusps at z the inequality (3.25) in general does not hold (i.e., the restriction (3.22) in Theorem 3.3 is essential).

Indeed, let $k > 2$ be fixed and let

$$G = G_k := \left\{ z = x + iy : 0 < x < 1, 0 < y < \frac{1}{2} x^{k-1} \right\}. \tag{3.26}$$

For $1/2 < \delta < 1$, we have

$$m_2(\overline{G} \setminus D(1,\delta)) \le \frac{1}{2} \int\limits_0^{2(1-\delta)} x^{k-1} dx = \frac{2^{k-1}}{k}(1-\delta)^k.$$

For $0 < s < 1/(2k) = m_2(\overline{G_k})$, let

$$\delta = \delta(s) := 1 - \left(\frac{ks}{2^{k-1}} \right)^{1/k}.$$

The polynomial

$$p_n(z) := \left(\frac{z-1}{\delta} \right)^n$$

belongs to $\mathbf{P}_n(\overline{G}, s)$. However,

$$|p_n(0)| = \frac{1}{\delta^n} \ge \exp(\varepsilon_4 n s^{1/k}).$$

Hence, for $k > 4$ and G defined by (3.26), the inequality (3.25) is violated.

If more information is known about the geometry of the domain G, the expression in (3.23) can be made more explicit. The following example illustrates this point. A Jordan curve is called *Dini-smooth* (cf. [82, p. 48]) if it is smooth and if the angle $\beta(s)$ of the tangent, considered in terms of the arclength s, satisfies

$$|\beta(s_2) - \beta(s_1)| < h(s_2 - s_1), \quad s_1 < s_2,$$

where $h(x)$ is an increasing function for which

$$\int\limits_0^1 \frac{h(x)}{x} dx < \infty. \tag{3.27}$$

We call a Jordan arc Dini-smooth if it is a subarc of some Dini-smooth curve.

Theorem 3.4 ([25]) *Let $L = \partial G$ consist of finitely many Dini-smooth arcs l_j, which form exterior angles $\alpha_j \pi$, $0 < \alpha_j < 2$, at their junction points $z_j, j = 1, \dots, m$. Let $z \in L$ be arbitrary and let z_k be the nearest point to z among the z_j, i.e.,*

$$|z_k - z| = \min_{1 \le j \le m} |z - z_j|.$$

Then, for $0 < s < m_2(\overline{G})$ the inequality

$$R_n(z,s) \le \exp\left(\frac{c_6\, n\sqrt{s}}{(\sqrt{s} + |z_k - z|)^{1-1/\alpha_k}} \log\left(\frac{c_4}{m_2(\overline{G}) - s} \right) \right) \tag{3.28}$$

holds.

Note that Theorem 3.4 implies (and extends) (3.21) since we can choose $\alpha_1 = \alpha_2 = 1$ and arbitrary distinct points $z_1, z_2 \in L$.

Now, let us proceed with the discussion of the sharpness of (3.28).

Let Φ denote the Riemann function that maps Ω conformally and univalently onto $\Delta := \overline{\mathbf{C}} \setminus \overline{\mathbf{D}}$, where $\mathbf{D} := \{z : |z| < 1\}$ is the unit disk, and is normalized by the conditions

$$\Phi(\infty) = \infty, \quad \Phi'(\infty) > 0.$$

We extend Φ to the homeomorphism $\Phi : \overline{\Omega} \to \overline{\Delta}$. For $z \in \mathbf{C}$ and $\delta > 0$, let

$$L_\delta := \{z \in \Omega : |\Phi(z)| = 1 + \delta\},$$

$$\rho_\delta(z) := \operatorname{dist}(z, L_\delta).$$

Let the function $\delta(z,t)$ be defined by the relation

$$\rho_{\delta(z,t)}(z) = t, \quad z \in L, \quad t > 0.$$

Observe that under the assumptions of Theorem 3.4

$$\delta \asymp \frac{\rho_\delta(z)}{(\rho_\delta(z) + |z - z_k|)^{1-1/\alpha_k}}, \quad 0 < \delta < 1,$$

where $a \asymp b$ denotes the double inequality $\varepsilon_5\, b \le a \le c_7\, b$.

Indeed, let $w := \Phi(z), w_k := \Phi(z_k), w_\delta := (1+\delta)w, z_\delta := \Phi^{-1}(w_\delta)$. According to the distortion properties of conformal mappings of domains with piecewise Dini-smooth boundary (cf. [82, p. 52] or [20, p. 33]), we have

$$\rho_\delta(z) \asymp |z - z_\delta| \asymp \delta(\delta + |w - w_k|)^{\alpha_k - 1} \asymp \delta(\rho_\delta(z) + |z - z_k|)^{1-1/\alpha_k}.$$

Hence (3.28) is equivalent to

$$R_n(z,s) \le \exp\left(c_8\, n\, \delta(z, \sqrt{s}) \log\left(\frac{c_4}{m_2(\overline{G}) - s}\right)\right), \; 0 < s < m_2(\overline{G}). \tag{3.29}$$

In [25] the sharpness of (3.29) is established for an arbitrary quasidisk G, i.e., a Jordan domain bounded by a quasiconformal curve $L := \partial G$.

Recall that, by Ahlfors' theorem (see, for example, [63, p. 100]), a Jordan curve L is *quasiconformal* if and only if there exists a constant c_9 such that for $z_1, z_2 \in L$

$$\min\{\operatorname{diam}(L'), \operatorname{diam}(L'')\} \le c_9 |z_1 - z_2|, \tag{3.30}$$

where L' and L'' denote the two components of $L \setminus \{z_1, z_2\}$. Thus, we exclude from our consideration the regions with cusps on the boundary.

Using Ahlfors' criterion, one can easily verify that convex curves, curves of bounded variation without cusps, and rectifiable Jordan curves which have the same order of arc length and chord length are quasiconformal. At the same time, each part of a quasiconformal curve can be nonrectifiable.

The domain from Theorem 3.4 is a quasidisk.

There is a simple way to show that there exists ε_6 such that for

$$m_2(\overline{G}) - \varepsilon_6 < s < m_2(\overline{G})$$

the inequality (3.29) is sharp (up to constants). Indeed, let $L = \partial G$ be quasiconformal, $z \in L$. Denote by $z^* \in L$ an arbitrary point satisfying

$$|z - z^*| \geq \frac{1}{2}\,\mathrm{diam}(\overline{G}).$$

Choose $0 < r < |z - z^*|$ such that

$$m_2(D(z^*, r) \cap G) = m_2(\overline{G}) - s$$

(this is always possible if ε_6 is sufficiently small). Since G is a quasidisk, for any $z \in L$ (3.22) holds. Therefore,

$$r \leq c_{10}(m_2(\overline{G}) - s)^{1/2}$$

holds as well.

The polynomial

$$p_n(\zeta) := \frac{(\zeta - z^*)^n}{r^n}$$

belongs to $\mathbf{P}_n(\overline{G}, s)$. At the same time

$$R_n(z,s) \geq |p_n(z)| \geq \left(\frac{\varepsilon_7}{m_2(\overline{G}) - s}\right)^{n/2} = \exp\left(\frac{n}{2}\log\frac{\varepsilon_7}{m_2(\overline{G}) - s}\right),$$

which shows the sharpness of (3.29) for values of s close to $m_2(\overline{G})$.

If $0 < s \leq m_2(\overline{G}) - \varepsilon_6$, then (3.29) implies

$$\limsup_{n\to\infty} \frac{\log R_n(z,s)}{n\delta(z, \sqrt{s})} \leq c_{11} = c_{11}(\varepsilon_6), \quad z \in L.$$

Theorem 3.5 ([25]) *Let G be a quasidisk. There exists a constant $\varepsilon_8 = \varepsilon_8(\varepsilon_6)$ such that for $z \in L$ and $0 < s \leq m_2(\overline{G}) - \varepsilon_6$ the inequality*

$$\liminf_{n\to\infty} \frac{\log R_n(z,s)}{n\delta(z, \sqrt{s})} \geq \varepsilon_8 \tag{3.31}$$

holds.

The inequality (3.31) demonstrates that (3.29) is asymptotically sharp (with respect to n) for all $z \in L$ and $0 < s \leq m_2(\overline{G}) - \varepsilon_6$.

3.3 Pointwise Remez-type inequalities in the unit disk

Observe that (3.21) states a uniform bound. Our objective is to derive the pointwise extension of this bound. Note that the pointwise extension (3.4) of the classical Remez inequality (3.1) was established relatively recently by Erdélyi [48, Theorem 4]. The proof of this theorem is based on a Remez-type inequality for trigonometric polynomials (cf. [37, p. 230], [56]). Our approach is quite different. We use ideas from potential theory in the plane [84], [89], principles of circular symmetrization [26], and estimation of conformal invariants such as moduli of families of curves [1], [63].

For simplicity, we formulate the appropriate result only for the unit disk.

Theorem 3.6 ([17]) *The condition*

$$m_2(\overline{\mathbf{D}} \cap E_{\nu,c}) \geq \pi - s, \quad 0 < s < \pi,$$

yields

$$Q_{\nu,c}(z) \leq \exp\left(c_1 \nu(\mathbf{C})\sqrt{s}\exp\left(-c_2\frac{(1-|z|)^2}{s}\right)\right), \quad z \in \overline{\mathbf{D}},$$

where c_1 and c_2 are positive absolute constants.

Theorem 3.6 is sharp in the following sense. Given $0 < s < \pi/2$ and $0 \leq x < 1$, let $\delta := 1 - x$. Set, for $0 < r < 1$,

$$\tilde{E}_{r,\delta} := \overline{\mathbf{D}} \setminus (\{\zeta : |\zeta - x| < r\} \cup \{\xi + i\eta : x < \xi \leq 1, |\eta| < r\}). \tag{3.32}$$

Since

$$m_2(\tilde{E}_{r,\delta}) > \pi - \frac{\pi r^2}{2} - 2\delta r,$$

taking r such that $(\pi r^2)/2 + 2\delta r = s$, i.e.,

$$r = r(\delta, s) := \frac{s}{\delta + \sqrt{\delta^2 + (\pi/2)s}},$$

we have for $\tilde{E} := \tilde{E}_{r,\delta}$,

$$m_2(\tilde{E}) > \pi - s. \tag{3.33}$$

In [17] it is shown that

$$g_{\overline{\mathbf{C}}\setminus\tilde{E}}(x, \infty) \geq c_3\sqrt{s}\exp\left(-4\frac{(1-x)^2}{s}\right), \quad c_3 = \frac{e^{-10\pi}}{2\sqrt{2}}. \tag{3.34}$$

Let

$$\nu = m\,\mu_{\tilde{E}}, \quad c = -m\,\log\operatorname{cap}(\tilde{E}),$$

where $m > 0$ is an arbitrary number. Since

$$U^{\mu_{\tilde{E}}}(z) = \begin{cases} -g_{\overline{\mathbf{C}}\setminus\tilde{E}}(z,\infty) - \log\operatorname{cap}(\tilde{E}), & z \in \mathbf{C} \setminus \tilde{E}, \\ -\log\operatorname{cap}(\tilde{E}), & z \in \tilde{E}, \end{cases}$$

we have

$$\nu(\mathbf{C}) = m, \quad E_{\nu,c} = \tilde{E}.$$

Moreover, (3.34) implies

$$Q_{\nu,c}(x) \geq \exp\left(c_3\, m\,\sqrt{s}\exp\left(-4\frac{(1-x)^2}{s}\right)\right). \tag{3.35}$$

Relations (3.33) and (3.35) show the sharpness of Theorem 3.6 for $z \in \mathbf{D}$ (its sharpness for $z \in \mathbf{T}$ is known [50]).

Applying Theorem 3.6 to the exponential of a potential defined by (3.9) – (3.10), we obtain the following: *for any complex polynomial $p_n \in \mathbf{P}_n$, the condition*

$$m_2(\{z \in \overline{\mathbf{D}} : |p_n(z)| \leq 1\}) \geq \pi - s, \quad 0 < s < \pi,$$

implies

$$|p_n(z)| \le \exp\left(c_1\, n\, \sqrt{s}\exp\left(-c_2\frac{(1-|z|)^2}{s}\right)\right), \quad z\in\overline{\mathbf{D}}.$$

The last inequality is sharp for large n. That is, given $0<s<\pi/2$ and $0\le x<1$, let $\tilde{E}$ be defined as in (3.32). It was shown in [17] that for

$$n > \frac{8\log 2}{c_3\sqrt{s}}\exp\left(4\frac{(1-x)^2}{s}\right)$$

there exists a polynomial $P_n\in\mathbf{P}_n$ such that

$$|P_n(z)|\le 1, \quad z\in\tilde{E},$$

$$|P_n(x)| \ge \exp\left(\frac{c_3}{2}\, n\, \sqrt{s}\exp\left(-4\frac{(1-x)^2}{s}\right)\right).$$

3.4 Remez-type inequalities in terms of linear measure

Next, we discuss analogues of (3.1) and (3.12) with an arbitrary Jordan arc or curve instead of $[-1,1]$, and a quasismooth (in the sense of Lavrentiev) curve instead of $\mathbf{T}$, respectively. Our results deal not only with polynomials but also with exponentials of logarithmic potentials (cf. [49, 50]).

Let $L\in\mathbf{C}$ be a bounded Jordan arc or curve. For a (Borel) set $V\subset L$, we consider its covering $U=\cup_j U_j\supset V$ by a finite number or countably many open (i.e., without endpoints) subarcs U_j of L. Let

$$\sigma_L(V) := \inf\sum_j \operatorname{diam}(U_j),$$

where the infimum is taken over all such open coverings of V.

Note that

$$\sigma_L(V)\le\min\{|V|,\operatorname{diam}(L)\}. \tag{3.36}$$

For an exponential of a potential $Q_{\nu,c}$ set

$$E^*_{\nu,c} := \mathbf{C}\setminus E_{\nu,c} = \{z\in\mathbf{C}:\ Q_{\nu,c}(z)>1\}.$$

Theorem 3.7 ([24]) *Let L be an arbitrary Jordan arc or curve, and let*

$$\frac{\sigma_L(E^*_{\nu,c}\cap L)}{\operatorname{diam}(L)} =: u < \frac{1}{2}.$$

Then

$$\|Q_{\nu,c}\|_L \le \left(\frac{1+\sqrt{2u}}{1-\sqrt{2u}}\right)^{\nu(\mathbf{C})}. \tag{3.37}$$

Theorem 3.7 extends [50, Theorem 2.1] from the case where $L=[-1,1]$ to the case where L is an arbitrary Jordan arc or curve.

Theorem 3.9 and the left-hand side of (3.42) below show that (3.37) is sharp (with respect to the degree 1/2 of u) even for the case of Jordan curves. However, if we take into consideration additional information about the geometry of L, the estimate (3.37) can be improved.

Let L be a *quasismooth* (in the sense of Lavrentiev) curve which is defined by the following condition. For any $z_1, z_2 \in L$,

$$\min\{|L'|, |L''|\} \le c_1 |z_1 - z_2|, \quad c_1 = c_1(L) \ge 1, \tag{3.38}$$

where L' and L'' are the connected components of $L \setminus \{z_1, z_2\}$.

According to (3.36) and (3.38), for any quasismooth curve L and a (Borel) set $V \subset L$, we have

$$\varepsilon_1 |V| \le \sigma_L(V) \le |V|.$$

We proceed with the case where E is a *Lavrentiev domain*, i.e., $L = \partial E$ is quasismooth.

Denote by Φ the conformal mapping of $\Omega = \overline{\mathbf{C}} \setminus E$ onto the exterior $\Delta := \overline{\mathbf{C}} \setminus \overline{\mathbf{D}}$ of the unit disk $\mathbf{D}$ normalized by the conditions

$$\Phi(\infty) = \infty, \quad \Phi'(\infty) := \lim_{z \to \infty} \frac{\Phi(z)}{z} > 0.$$

Let

$$L_\delta := \{z \in \Omega : |\Phi(z)| = 1 + \delta\}, \quad \delta > 0,$$

and let the function $\delta(t, L)$, $t > 0$, be defined by the relation

$$\operatorname{dist}(L, L_{\delta(t,L)}) = t.$$

Theorem 3.8 ([24]) *Let L be a quasismooth curve and suppose that*

$$|E^*_{\nu,c} \cap L| \le s < \frac{1}{2} \operatorname{diam}(L).$$

Then

$$\|Q_{\nu,c}\|_L \le \exp(c_2\, \delta(s, L)\, \nu(\mathbf{C}))$$

holds with $c_2 = c_2(L)$.

In order to discuss the sharpness of the bound of Theorem 3.8, we consider an important particular case of exponentials of potentials. Let $V \subset L$ consist of a finite number of open subarcs of L whose closures are disjoint, $J := L \setminus V$, and let $c = c(V) := -\log \operatorname{cap}(J)$. Since

$$U^{\mu_J}(z) = -g_{\overline{\mathbf{C}} \setminus J}(z) - \log \operatorname{cap}(J), \quad z \in \mathbf{C},$$

we have

$$Q_{\mu_J, c}(z) = \exp(g_{\overline{\mathbf{C}} \setminus J}(z)), \quad E^*_{\mu_J, c} \cap L = V.$$

For $0 < s < \operatorname{diam}(L)$, we set

$$\mathcal{U}(s, L) := \{V \subset L : V = \cup_{j=1}^m V_j, V_j \text{ is an open arc }, \overline{V_j} \cap \overline{V_k} = \emptyset, \sum_{j=1}^m \operatorname{diam}(V_j) \le s\},$$

$$\lambda(s, L) := \sup_{V \in \mathcal{U}(s,L)} \sup_{z \in V} g_{\overline{\mathbf{C}} \setminus J}(z).$$

For any quasismooth curve L, Theorem 3.8 implies that

$$\lambda(s, L) \le c_2\, \delta(s, L), \quad 0 < s < \frac{1}{2} \operatorname{diam}(L). \tag{3.39}$$

Theorem 3.9 *Let L be a quasismooth curve. Then*

$$\lambda(s,L) \geq \varepsilon_2 \delta(s,L), \quad 0 < s < \mathrm{diam}(L), \tag{3.40}$$

holds with $\varepsilon_2 = \varepsilon_2(L)$.

Theorem 3.8 and inequalities (3.39) and (3.40) show that the growth of the exponentials of logarithmic potentials can be related to the function $\delta(s,L)$ which depends only on the geometry of L. Below, we state some remarks concerning its properties.

By the Ahlfors criterion (3.30), any quasismooth curve is quasiconformal. Therefore, Φ can be extended to a quasiconformal homeomorphism $\Phi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$. Taking into account the distortion properties of conformal mappings with a quasiconformal extension (cf. [79, pp. 289, 347]), for any quasismooth curve L, we have

$$\delta(s,L) \leq c_3 s^{\alpha}, \quad 0 < s < \mathrm{diam}(L),$$

where $c_3 = c_3(L)$ and $\alpha = \alpha(L) > 1/2$. Thus, for any particular quasismooth curve L, Theorem 3.8 presents better estimates (with respect to the order of s) than Theorem 3.7.

Next, we introduce the notion of Dini-convex curves. In the remainder of this section, we assume that L is a quasismooth curve. The set $\overline{\mathbf{C}} \setminus L$ consists of two Jordan domains: a bounded one $G := \mathrm{int}(L)$ and an unbounded one $\Omega := \mathrm{ext}(L)$. Let h be a positive nondecreasing function satisfying the Dini-condition (3.27) and let, for $0 < \varepsilon \leq 1$,

$$W(h,\varepsilon) := \{\, \zeta = re^{i\theta} : 0 < r < \varepsilon,\ \pi h(r) < \theta < \pi(1 - h(r)) \,\}.$$

We say that L is *Dini-convex* with respect to G if there exist $0 < \varepsilon = \varepsilon(L) \leq 1$ and a function $h = h_L$ satisfying (3.27) such that $h(\varepsilon) < 1/2$ and for any $z \in L$

$$\{\zeta = z + e^{i\theta}\xi : \xi \in W(h,\varepsilon)\} \subset G$$

holds with some $0 \leq \theta = \theta(z) < 2\pi$.

For example, if there is $0 < r < 1$ depending only on L such that for each $z \in L$ there exists an open disk D_z with radius r such that $D_z \subset G$ and $\overline{D}_z \cap L = \{z\}$, then L is Dini-convex with respect to G (with $h(x) = c_4\, x$ and $\varepsilon = r$).

Theorem 3.10 ([24]) *Let L be a quasismooth curve which is Dini-convex with respect to G. Then*

$$\varepsilon_3 s \leq \delta(s,L) \leq c_5 s, \quad 0 < s < \mathrm{diam}(L). \tag{3.41}$$

Comparing Theorem 3.8 with the right-hand side of (3.41), we obtain an analogue of (3.12) for curves L instead of the unit circle $\mathbf{T}$ (see also [11]). This result is sharp because of Theorem 3.9 and the left-hand side of (3.41).

If L consists of a finite number of Dini-smooth arcs which meet in the angles $\alpha_1\pi, \ldots, \alpha_m\pi$ (with respect to Ω), $0 < \alpha_j < 2$,

$$\alpha := \max(1, \alpha_1, \ldots, \alpha_m),$$

then according to the distortion properties of a conformal mapping of a domain with piecewise Dini-smooth boundary onto a disc (cf. [82, Chapter 3]), we have

$$\varepsilon_4 s^{1/\alpha} \leq \delta(s,L) \leq c_6 s^{1/\alpha}, \quad 0 < s < \mathrm{diam}(L). \tag{3.42}$$

Notice that Theorem 3.8 and the right-hand side of (3.42) imply a new Remez-type inequality for domains with a piecewise Dini-smooth boundary which is sharp because of Theorem 3.9 and the left-hand side of (3.42).

3.5 Open problems

We conjecture that the following sharp pointwise extension of (3.1) and (3.4) is valid.

Let $0 < s < 2$ and $-1 < x < 1$. We define $a = a(s, x)$ and $b = b(s, x)$ such that:

(i) $-1 < a < x < b < 1$ and $b - a = s$;

(ii) the conformal mapping $F = F_E$ defined by (2.1) for $E = [0, \frac{a}{2} + \frac{1}{2}] \cup [\frac{b}{2} + \frac{1}{2}, 1]$ maps $\frac{x}{2} + \frac{1}{2}$ into the point $w_0 = \max_{w \in K_E} |w|$.

Let $\tilde{T}_{n,s}(\xi) = \xi^n + \cdots \in \mathbf{\Pi}_n$ be the *Chebyshev-Akhiezer polynomial* deviating least from zero on E, that is,

$$\|\tilde{T}_{n,s}\|_E = \min_{p \in \mathbf{\Pi}_{n-1}} \| \cdot^n + p(\cdot)\|_E.$$

Let $T_{n,s}(\xi) := \tilde{T}_{n,s}(\xi)/\|\tilde{T}_{n,s}\|_E$.

Problem 4. *Is it true that the inequality*

$$|p_n(x)| \le |T_{n,s}(x)|$$

holds for every $p_n \in \mathbf{\Pi}_n$ satisfying (3.2)?

Our next problem concerns the Remez-type inequality for polynomials on a quasidisk.

Problem 5. *Let G be a quasidisk, i.e., a Jordan domain bounded by a quasiconformal curve L. Is it true that for $z \in L$ and arbitrary sufficiently small positive constant ε the inequalities*

$$\exp(c_1 n \delta(z, \sqrt{s})) \le R_n(z, s) \le \exp(c_2 n \delta(z, \sqrt{s}))$$

hold for any $0 < s \le m_2(\overline{G}) - \varepsilon$ with some constants $c_j = c_j(G, \varepsilon), j = 1, 2$?

4 Polynomial Approximation

4.1 Approximation on an unbounded interval

We consider functions f, continuous and real valued on the non-negative real line $\mathbf{R}^+$ and possessing also the properties

$$f > 0 \text{ on } \mathbf{R}^+, \quad \lim_{x \to \infty} f(x) = \infty \,. \tag{4.1}$$

For every positive integer $n \in \mathbf{N}$, we define

$$\rho_n(f) := \inf_{p_n \in \mathbf{\Pi}_n} \|\frac{1}{f} - \frac{1}{p_n}\|_{\mathbf{R}^+} \,.$$

In the present section, we discuss necessary and sufficient conditions for the geometric convergence of reciprocals of polynomials to the reciprocal of the function f on $\mathbf{R}^+$, i.e., the inequality

$$\limsup_{n \to \infty} \rho_n(f)^{1/n} = \frac{1}{q} < 1 \,. \tag{4.2}$$

The first results in this area were due to Cody, Meinardus and Varga [40] concerning the function $\exp(x)$. Later, Meinardus and Varga [69] extended these results to the class of entire functions of completely regular growth. The paper [71] gave rise to investigations devoted to enlarging the class of functions that admit geometric approximation by reciprocals of polynomials on $\mathbf{R}^+$.

We introduce some notations. Given two numbers $r > 0$ and $s > 1$, denote by $\mathcal{E}_r(s)$ the closed ellipse with foci at the points $x = 0$ and $x = r$ such that the ratio between the semimajor axis and semiminor axis equals $(s^2 + 1)/(s^2 - 1)$.

The following theorem states remarkable necessary conditions for geometric convergence.

Theorem 4.1 (Meinardus [70], Meinardus, Reddy, Taylor, Varga [71]) *Let f satisfy (4.2). Then*

(i) *the function f can be extended from $\mathbf{R}^+$ to an entire function of finite order;*

(ii) *for every number $s > 1$, there exist positive constants $c_1 = c_1(s, q)$, $\theta = \theta(s, q)$ and $r_0 = r_0(s, q)$ such that the inequality*

$$\|f\|_{\mathcal{E}_r(s)} \le c_1 \|f\|^{\theta}_{[0,r]} \tag{4.3}$$

holds for all $r \ge r_0$.

After the appearance of [71], a lot of work was done to find sufficient conditions for (4.2) (cf. [34] – [36], [85], [87], [59]). The most general known result in this direction is the following statement.

Theorem 4.2 (Blatt, Kovacheva [36]) *Assume that f is an entire function with (4.1) and, in addition to condition (4.3), the inequality*

$$\|f\|_{[0,r]} \le \mu(r)^{\lambda}, \tag{4.4}$$

where $\mu(r) := \min_{x \ge r}\{f(x)\}$, holds for some number $\lambda > 1$ and for every $r > r_0$. Then (4.2) is true.

On the other hand, Henry and Roulier [59] have shown that the conditions (i) and (ii) of Theorem 4.1 are not sufficient for geometric convergence. For example, in [59] it was proved that

$$f(x) = 1 + x + e^x \sin^2 x \tag{4.5}$$

cannot be approximated with geometric speed. Their proof was based on the fact that f satisfying (4.2) cannot oscillate too often.

The main goal of this section is to discuss a new necessary and a new sufficient condition for geometric convergence found in [21].

We begin with a necessary condition. Let f be as above, i.e., f is an entire function with (4.1). For $r > 0$, we define the set

$$Z_r := \{\, 0 < x < \infty : f(x) < r \,\}\,.$$

Then Z_r is the union of a finite number of disjoint open intervals. This follows from (4.1) and the uniqueness theorem for analytic functions. Now, we consider the closure $\bar{Z}_r$ of Z_r, which is regular and possesses a Green's function

$$g_r(z) := g_{\overline{\mathbf{C}} \setminus \bar{Z}_r}(z)$$

with respect to the region $\overline{\mathbf{C}} \setminus \bar{Z}_r$ with pole at infinity, where $g_r := 0$ on $\bar{Z}_r$. For $s > 1$, we denote by $\mathcal{E}_r(f, s)$ the set which consists of the interior of the level set of $g_r(z)$ and the level set itself for a fixed parameter s, i.e.,

$$\mathcal{E}_r(f, s) := \{\, z \in \mathbf{C} \,:\, 0 \le g_r(z) \le \log s \,\}\,.$$

Then the new necessary condition for geometric convergence can be formulated as follows.

Theorem 4.3 ([21]) *Let f satisfy (4.2). Then for every $1 < s < q$ there exist positive constants $c = c(s,q)$, $\theta = \theta(s,q)$ and $r_0 = r_0(s,q)$ such that*

$$\|f\|_{\mathcal{E}_r(f,s)} \le c\, r^{\theta}, \quad r \ge r_0\,. \tag{4.6}$$

Next, we are going to discuss the geometrical meaning of condition (4.6). For $\infty > H > h > \min_{x\in\mathbf{R}^+} f(x) > 0$, we introduce the strip domain

$$S(h,H) := \{\, (x,y) \,:\, -\infty < x < \infty,\ h < y < H \,\}$$

as well as the intersection of this strip with the graph of f, i.e.,

$$Y(f,h,H) := S(h,H) \cap \{\, (x,y) \,:\, x \ge 0,\ y = f(x) \,\}\,,$$

and define $N(f,h,H)$ to be the number of connected components of $Y(f,h,H)$ joining the line $\{\Im(z) = h\}$ with the line $\{\Im(z) = H\}$. Since f satisfies (4.1), the number $N(f,h,H)$ is finite and, moreover, it is odd.

Theorem 4.4 ([21]) *Let f be an entire function satisfying (4.1). If, in addition, for some $s > 1$ and $\theta > 1$, the function f satisfies (4.6), then, for each $M > \theta$,*

$$\limsup_{h\to\infty} \frac{N(f,h,h^M)}{\log h} < \infty\,. \tag{4.7}$$

Note that the result of Theorem 4.4 is sharp in the following sense: For each $M > 1$ there exists an entire function $f = f_M$ which satisfies (4.6) with some $s > 1$ and $1 < \theta < M$ and

$$\limsup_{h\to\infty} \frac{N(f,h,h^M)}{\log h} > 0\,. \tag{4.8}$$

Indeed, consider the function

$$f_M(x) := e^x + e^{2Mx} \sin^2 \pi x\,.$$

Obviously, it satisfies the conditions of Theorem 4.2. Therefore, f_M guarantees the geometrical convergence of best approximants in the sense of (4.2), and, by Theorem 4.3, f satisfies (4.6) in which case we can take s so close to 1 that $\theta < M$. The relation (4.8) immediately follows if we set $h = e^k$, $k \in \mathbf{N}$, and let $k \to \infty$.

The new sufficient condition for geometrical convergence of best approximants can be stated in the following form.

Theorem 4.5 ([21]) *Let f be an entire function satisfying (4.1) and (4.6) with some $s > 1$ and $\theta > 1$. In addition, assume that there exists a constant $M = M(f) > 1$ such that*

$$\limsup_{h\to\infty} N(f,h,h^M) < \infty\,.$$

Then f satisfies (4.2).

It is easy to see that Theorem 4.2 follows from Theorem 4.5, because under the assumptions of Theorem 4.2, for $M > \lambda$ and h sufficiently large, we have $N(f, h, h^M) = 1$. At the same time, condition (4.3) is weaker than (4.6). The example of the function (4.5) shows that conditions (4.3) and (4.6) are not equivalent. Indeed, f given by (4.5) obviously satisfies (4.3). On the other hand, some straightforward calculations show that relation (4.7) does not hold for this function. Thus, f does not possess (4.6).

The fact that Theorem 4.5 is essentially stronger than Theorem 4.2 is not so obvious.

Theorem 4.6 ([21]) *There exists an entire function f satisfying the assumptions of Theorem 4.5, but not possessing property (4.4).*

The proof of Theorem 4.5 is based on an analogue of the classical result due to Bernstein concerning polynomial approximation of functions analytic in the neighborhood of a subinterval of the real axis, for the case of several intervals.

Let $E = \bigcup_{j=1}^{k} I_j$ be the union of k disjoint intervals $I_j = [\alpha_j, \beta_j]$ of the real axis $\mathbf{R}$ and let $\Omega := \overline{\mathbf{C}} \setminus E$. The set

$$E^s := \{z \in \Omega \, : \, g_\Omega(z) = \log s\}, \quad s > 1,$$

consists of at most k (mutually exterior) curves. Denote by $\mathrm{ext}(E^s)$ the unbounded component of $\overline{\mathbf{C}} \setminus E^s$ and set $\mathrm{int}(E^s) := \overline{\mathbf{C}} \setminus \overline{\mathrm{ext}(E^s)}$. Denote by $C(E)$ the class of all real functions continuous on E.

Theorem 4.7 ([21]) *For each $f \in C(E)$ satisfying the following two conditions:*

$$\text{for some } s > 1, \ f \text{ can be extended analytically into } \overline{\mathrm{int}(E^s)}\,, \tag{4.9}$$

$$f \quad \text{has at least one zero on each } \ I_j\,, \tag{4.10}$$

there exist constants $q > 1$ and $c > 0$ depending only on s and k such that

$$\inf_{p_n \in \mathbf{\Pi}_n} \|f - p_n\|_E =: E_n(f, E) \le c\, \|f\|_{E^s}\, q^{-n}, \quad n \in \mathbf{N}\,. \tag{4.11}$$

Note that (4.11) can be interpreted as a result concerning geometric convergence of the polynomials of best approximation to the function f, independent of the geometry of E.

The proof of Theorem 4.7 is based on a new concept of Faber-type polynomials for E which can be described as follows. Let E be as defined in Theorem 4.7. Denote by $g_\Omega(z, z_0)$, $z, z_0 \in \Omega := \overline{\mathbf{C}} \setminus E$, the Green function for Ω with pole at z_0. It has a multiple-valued harmonic conjugate $\tilde{g}_\Omega(z, z_0)$. Thus, the analytic function

$$\Phi(z, z_0) := \exp(g_\Omega(z, z_0) + i\tilde{g}_\Omega(z, z_0))$$

is also multiple-valued.

Let $\Phi(z) := \Phi(z, \infty)$ and let $n \in \mathbf{N}$ be arbitrary. If $\Phi(z)^n$ is single-valued in Ω, we set

$$W_n(z) := \Phi(z)^n, \quad z \in \Omega\,.$$

If $\Phi(z)^n$ is not single-valued in Ω, then according to [108, pp. 159, 227], there exist $q \le k - 1$ points $x_{1,n}, \ldots, x_{q,n} \in [\alpha_1, \beta_k] \setminus E$ such that the function

$$W_n(z) := \frac{\Phi(z)^n}{\Pi_{i=1}^{q} \Phi(z, x_{i,n})}, \quad z \in \Omega,$$

is single-valued in Ω. In both cases the function

$$F_n(z) := \frac{1}{2\pi i}\int_\Gamma \frac{W_n(\zeta)}{\zeta - z}d\zeta, \quad z \in \mathrm{int}(\Gamma),$$

where $\Gamma \in \mathbf{C}\setminus E$ is any curve surrounding E, is a polynomial of degree n, which coincides with the classical Faber polynomial in the case of connected E.

4.2 The Nikol'skii-Timan-Dzjadyk-type theorem

Let $E \subset \mathbf{R}$ be a compact set, and let $\omega(\delta), \delta > 0$, be a function of modulus of continuity type, i.e., a positive nondecreasing function with $\omega(0+) = 0$ such that for some constant $c \geq 1$,

$$\omega(t\delta) \leq c\, t\, \omega(\delta), \quad \delta > 0, t > 1.$$

Let $C_\omega(E)$ consist of all $f \in C(E)$ such that

$$|f(x_1) - f(x_2)| \leq c_1\, \omega(|x_2 - x_1|), \quad x_1, x_2 \in E,$$

with some $c_1 = c_1(f) > 0$.

For $\omega(\delta) = \delta^\alpha, 0 < \alpha \leq 1$, we set $C_\omega(E) =: C^\alpha(E)$.

One of the central problems in approximation theory is to describe the relation between the smoothness of functions and the rate of decrease of their approximation by polynomials when the degree of these polynomials tends to infinity. The following well-known statement is the starting point of our consideration.

Theorem 4.8 (Nikol'skii [76], Timan [100], Dzjadyk [44]) *Let $f \in C([-1,1])$ and let ω be a function of modulus of continuity type satisfying the inequality*

$$\delta \int_\delta^1 \frac{\omega(t)}{t^2}\, dt \leq c_2\, \omega(\delta), \quad 0 < \delta < 1, \tag{4.12}$$

with some constant $c_2 > 0$. Then the following assertions are equivalent:

(i) $f \in C_\omega([-1,1])$*;*

(ii) *for any $n \in \mathbf{N}$ there exists $p_n \in \mathbf{\Pi}_n$ such that the inequality*

$$|f(x) - p_n(x)| \leq c_3\, \omega\left(\frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n}\right), \quad -1 \leq x \leq 1, \tag{4.13}$$

holds with some constant $c_3 > 0$.

In the late 50s – early 60s Dzjadyk [45], [46] laid the foundation for a new constructive theory of functions on continua in the complex plane (a survey of the results and a bibliography can be found in the monographs [99], [47], [55], [91], [19]). He used the following simple but fundamental idea.

Denote by $I_{1/n}, n \in \mathbf{N}$, the ellipse with foci at ± 1 and sum of semiaxes equal to $1 + 1/n$. Such an ellipse is the image of the circle $\{w : |w| = 1 + 1/n\}$ under the conformal mapping $z = \frac{1}{2}(w + \frac{1}{w})$ of $\Delta := \{w : |w| > 1\}$ onto $\overline{\mathbf{C}} \setminus [-1,1]$, i.e., $I_{1/n}$ is the level line of the conformal mapping

$$\Phi(z) = z + \sqrt{z^2 - 1}$$

of $\overline{\mathbf{C}}\setminus[-1,1]$ onto Δ, where the square root is chosen so that $\Phi(z)=2z+O(\frac{1}{|z|})$ in a neighborhood of ∞.

Then for $-1\le x\le 1$ and $n\in\mathbf{N}$,

$$\frac{1}{n^2}+\frac{\sqrt{1-x^2}}{n}\asymp\rho_{1/n}(x),$$

where

$$\rho_{1/n}(x):=\operatorname{dist}(x,I_{1/n}).$$

The concepts of $C_\omega,\Phi,I_{1/n}$ and $\rho_{1/n}(x)$ are also meaningful for an arbitrary bounded continuum in the complex plane. This is the key to a generalization of the Nikol'skii-Timan-Dzjadyk theorem to classes of functions on continua in $\mathbf{C}$.

If $E\subset\mathbf{C}$ is a compact set, then the interpretation of the Nikol'skii-Timan-Dzjadyk theorem above can be rephrased by consideration of the Green function g_Ω and its level lines. The case when $\Omega=\overline{\mathbf{C}}\setminus E$ is multiply connected is discussed in [74, 75, 93, 95, 72, 8]. Each time the extension of a result from the case of a continuum to the case of a compact set uses quite specific and non-trivial constructions.

In [15], we found how, in the case of finitely connected Ω, the extension of the Nikol'skii-Timan-Dzjadyk theorem can be obtained by using the well-known Bernstein-Walsh lemma on the growth of a polynomial outside the compact set and Walsh's theorem on polynomial approximation of a function analytic in a neighborhood of a compact set with connected complement. Our approach is based on the following theorem.

Theorem 4.9 ([15]) *Let $E=\cup_{j=1}^m E_j$ consist of $m\in\mathbf{N}, m\ge 2$, disjoint continua E_j, $f\in A(E)$, $\|f\|_E\le 1$, and let $z_1,\dots,z_N\in E$ be distinct points. Let, for any $n>n_0\in\mathbf{N}$ and $j=1,\dots,m$, there be a polynomial $p_{n,j}\in\mathbf{P}_n$ such that*

$$|f_j(z)-p_{n,j}(z)|\le\varepsilon_j\left(\frac{1}{n},z\right),\quad z\in\partial E_j,$$

$$p_{n,j}(z_l)=f_j(z_l),\quad z_l\in E_j,$$

where $f_j:=f|_{E_j}$ is the restriction of f to E_j, and the function $\varepsilon_j(\delta,z)$, $0<\delta\le 1$, $z\in\partial E_j$, satisfies, for any $j=1,\dots,m$ and $z\in\partial E_j$, the properties:

(i) $\varepsilon_j(\delta,z)$ is monotonically increasing in δ;

(ii) $|\varepsilon_j(\delta,z)|\le 1,\quad \delta\le\delta_0\le 1$.

Then for any $n\in\mathbf{N}$, $n>c_4(n_0+1/\delta_0)$ there exists a polynomial $p_n\in\mathbf{P}_n$ such that

$$|f(z)-p_n(z)|\le\varepsilon_j\left(\frac{c_5}{n},z\right)+c_6\,e^{-c_7 n},\quad z\in\partial E_j,\quad j=1,\dots,m,$$

$$p_n(z_l)=f(z_l),\quad l=1,\dots,N,$$

where c_k, $k=4,5,6,7$, depend only on E and the choice of points $z_1,\dots,z_N$.

The case of infinitely connected Ω is extremely difficult to handle. This can be seen from a recent paper by Shirokov [95].

In what follows, we are going to discuss the case $E\subset\mathbf{R}$, where the number of components of E can be infinite. It turns out that the appropriate analogue of the Nikol'skii-Timan-Dzjadyk theorem is valid for some E that are not "too scarce" (see Theorem 4.11) and that in general a result of such kind is not true (see Theorem 4.10).

More precisely, let $E \subset \mathbf{R}$ be a regular compact set. For $\delta > 0$ and $z \in \mathbf{C}$ set

$$E_\delta := \{z \in \Omega \,:\, g_\Omega(z) = \delta\},$$

$$\rho_\delta(z) := \operatorname{dist}(z, E_\delta).$$

It turns out that even for $f \in C^\alpha(E)$, polynomials satisfying an analogue of (4.13) cannot be constructed for any E under consideration.

Theorem 4.10 ([8]) *There exist a regular compact set $E_0 \subset \mathbf{R}$ and, for any $0 < \alpha \le 1$, a function $f_\alpha \in C^\alpha(E_0)$ such that the following assertion is false: for any $n \in \mathbf{N}$ there is a polynomial $p_n \in \mathbf{\Pi}_n$ with the property:*

$$|f_\alpha(x) - p_n(x)| \le c\,\rho_{1/n}^\alpha(x), \quad x \in E_0, \tag{4.14}$$

where the constant $c > 0$ is independent of n and x.

The construction of E_0 in Theorem 4.10 uses ideas from Section 2. That is, let

$$U_0 := \{w = \xi + i\eta :\, -\frac{\pi}{2} < \xi < \frac{\pi}{2},\, \eta > 0\} \setminus \bigcup_{\substack{k=-\infty \\ k \ne 0}}^{\infty} J_k',$$

where

$$J_k' := \left[\frac{1}{k|k|}, \frac{1}{k|k|} + \frac{6i}{|k|}\right].$$

Consider the conformal mapping ψ_0 of U_0 onto $\mathbf{H}$, normalized by the boundary conditions

$$\psi_0(\infty) = \infty, \quad \psi_0\left(\pm\frac{\pi}{2}\right) = \pm 1.$$

We extend the inverse mapping $\phi_0 := \psi_0^{-1}$ continuously to $\overline{\mathbf{H}}$ (because of the symmetry of U_0, we have $\phi_0(0) = 0$) and set

$$J_k := \{x \in \mathbf{R} :\, \phi_0(x) \in J_k'\},$$

$$I_k = [x_k', x_k''] := \begin{cases} \psi_0([-\frac{\pi}{2}, -1] \cup [1, \frac{\pi}{2}]), & k = 0, \\ \psi_0([\frac{1}{(k+1)^2}, \frac{1}{k^2}]), & k \in \mathbf{N}, \\ \psi_0([-\frac{1}{k^2}, -\frac{1}{(k-1)^2}]), & -k \in \mathbf{N}. \end{cases}$$

Then

$$E_0 := \left(\bigcup_{k=-\infty}^{\infty} I_k\right) \cup \{0\}$$

satisfies the conditions of Theorem 4.10.

The analysis of the construction above shows that E_0 is "too scarce" in a neighborhood of $0 \in E_0$. Hence, to admit estimates like (4.13) or (4.14), E has to be "thick enough" in a neighborhood of each of its points. In order to formulate the appropriate restrictions, we need some notations.

The set $\mathbf{R} \setminus E$ consists of a finite or infinite number of components, i.e., disjoint open intervals.

We say that $E \in \mathcal{E}(\alpha, c)$, $\alpha > 0, c > 0$, if for any bounded component J of $\mathbf{R} \setminus E$ the inequality

$$\operatorname{dist}(J, (\mathbf{R} \setminus E) \setminus J) \geq c\,|J|^{1/(1+\alpha)} \tag{4.15}$$

holds.

By definition, we relate a single closed interval to $\mathcal{E}(\alpha, c)$.

We can now state the analogue of the Nikol'skii-Timan-Dzjadyk theorem for functions continuous on a compact subset of the real line.

Theorem 4.11 ([8]) *Let the regular set $E \subset \mathbf{R}$ consist of a finite number of disjoint compact sets, each of which belongs to the class $\mathcal{E}(\alpha, c)$ with some $\alpha, c > 0$. Suppose that $f \in C(E)$ and that the function ω of modulus of continuity type satisfies (4.12). Then the following conditions are equivalent:*

(i) *$f \in C_\omega(E)$;*

(ii) *for any $n \in \mathbf{N}$ there exists a polynomial $p_n \in \mathbf{\Pi}_n$ such that*

$$|f(x) - p_n(x)| \leq c_8\, \omega(\rho_{1/n}(x)), \quad x \in E,$$

where the constant $c_8 > 0$ does not depend on x and n.

The simplest example of E satisfying the assumptions of Theorem 4.11 is the union of a finite number of disjoint closed intervals. The compact set

$$E_\alpha := \{0\} \cup \bigcup_{n=n_\alpha}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n} - \frac{1}{n^{2+\alpha}}\right], \quad \alpha > 0,\, n_\alpha > 2^{1/\alpha},$$

which obviously satisfies the conditions of Theorem 4.11, illustrates a nontrivial extension of (4.13) to compact subsets of the real line.

The proof of Theorem 4.11 uses the results and ideas concerning approximation of functions by complex polynomials on continua of the special class H^* introduced and discussed in the next subsection. We outline the main steps of this proof. As usual, we use $c_1, c_2, \cdots$ to denote positive constants that depend on parameters inessential to the argument.

(ii)$\Rightarrow$(i). Since by our assumption (4.15), for any $x \in E$ and $0 < \delta < 1$ the length of the set $E_{x,\delta} := E \cap \{\zeta : |\zeta - x| \leq \delta\}$ satisfies $|E_{x,\delta}| \geq c_1\, \delta$, the compact set E is uniformly perfect. Hence, (ii)$\Rightarrow$(i) follows from Tamrazov's inverse theorem (see [99, p. 138]).

(i)$\Rightarrow$(ii). Let $f \in C_\omega(E)$. Applying the procedure described, for example, in [19, Chapter 1], we extend f continuously to $\mathbf{R}$ such that

$$|f(x_2) - f(x_1)| \leq c_2 \omega(x_2 - x_1), \quad -\infty < x_1 < x_2 < \infty,$$

$$f(x) = 0, \quad \operatorname{dist}(x, E) > 3.$$

Further, we consider the Poisson integral

$$f(z) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\, f(s)\, ds}{(x-s)^2 + y^2}, \quad z = x + iy \in \mathbf{H},$$

which extends f harmonically to the upper half-plane $\mathbf{H}$.

It can be shown that for any $z_1, z_2 \in \mathbf{H}$ such that $|z_1| < c$, $|z_2| < c$, $|z_1 - z_2| \le \delta < c$, where $c > 0$ is an arbitrary (fixed) constant, we have the inequality

$$|f(z_1) - f(z_2)| \le c_3 \omega(\delta). \tag{4.16}$$

First, we consider the case $E \in \mathcal{E}(\alpha, c)$. Without loss of generality, we assume that E consists of a finite number of components, that is, $E = \cup_{k=1}^m I_k$, $I_k = [x_k', x_k'']$,

$$x_1' < x_1'' < \cdots < x_m' < x_m''.$$

Important is that the estimates below do not depend on m. Using a linear transformation, if necessary, we can also assume that $x_1' = -1$ and $x_m'' = 1$.

For $k = 1, \ldots, m-1$, set

$$J_k := [x_k'', x_{k+1}'],$$

$$\begin{aligned} S_k \quad := \quad & [x_k'' + 2it_k, x_{k+1}' + 2it_k] \\ & \bigcup \{z = x + iy : \, x_k'' - t_k \le x \le x_k'', \, |z - (x_k'' + it_k)| = t_k\} \\ & \bigcup \{z = x + iy : \, x_{k+1}' \le x \le x_{k+1}' + t_k, \, |z - (x_{k+1}' + it_k)| = t_k\}, \end{aligned}$$

where

$$t_k := \frac{1}{3} \min(|J_k|, |I_k|, |I_{k+1}|).$$

According to our assumption (4.15), for sufficiently large positive constants c and α the curve

$$l := \{x + iy : \, -1 \le x \le 1, \, y = c\,(1 - |x|)^{1+\alpha}\}$$

satisfies the condition

$$\operatorname{dist}(S_k, l) \ge 2 \operatorname{diam}(S_k).$$

We denote by G the Jordan domain bounded by

$$\partial G = L := E \cup (\bigcup_{k=1}^{m-1} S_k) \cup l,$$

and let $\Omega^* := \overline{\mathbf{C}} \setminus \overline{G}$.

It can be proved that $\overline{G} \in H^*$ (for the definition of H^*, see Subsection 4.3). Therefore by [4] and (4.16), for any $n \in \mathbf{N}$ there exists a harmonic polynomial

$$t_n(z) = \operatorname{Re} \sum_{j=0}^{n} a_j \, z^j, \quad a_j \in \mathbf{C},$$

(of degree at most n) such that

$$|f(z) - t_n(z)| \le c_4 \omega(\rho_{1/n}^*(z)), \quad z \in L, \tag{4.17}$$

where for $z \in \mathbf{C}$ and $\delta > 0$,

$$\rho_\delta^*(z) := \operatorname{dist}(z, L_\delta),$$

$$L_\delta := \{\zeta : |\Phi(\zeta)| = 1 + \delta\},$$

and Φ is the conformal mapping of Ω^* onto $\Delta := \{w : |w| > 1\}$ normalized by the conditions

$$\Phi(\infty) = \infty, \quad \Phi'(\infty) > 0.$$

A calculation shows that

$$\rho^*_\delta(x) \le c_5 \rho_\delta(x), \quad x \in E,\ 0 < \delta < 1. \tag{4.18}$$

Thus, (4.17) and (4.18) imply (ii).

Let now $E = \cup_{j=1}^s E_j$, where $E_j \in \mathcal{E}(\alpha, c)$ and $E_j \cap E_k = \emptyset$ for $j \ne k$. For each E_j (consisting of a finite number of components), we construct an auxiliary domain G_j as above and join all G_j in $\mathbf{H}$ by smooth arcs so that a new set $K \supset \cup_{j=1}^s \overline{G_j}$ belongs to H^*.

The distances from $x \in E_j$ to the δ-level sets of the Green function for E and for E_j, denoted by $\rho_{\delta,E}(x)$ and $\rho_{\delta,E_j}(x)$, are equivalent, i.e.,

$$\rho_{\delta,E}(x) \asymp \rho_{\delta,E_j}(x), \quad x \in E_j,\ 0 < \delta < 1.$$

The same property holds for the distances $\rho^*_{\delta,E}(x)$ and $\rho^*_{\delta,E_j}(x)$ from $x \in E_j$ to the $(1+\delta)$-level line for the Riemann mapping function Φ constructed for $\overline{\mathbf{C}} \setminus K$ and $\overline{\mathbf{C}} \setminus \overline{G_j}$ respectively. That is,

$$\rho^*_{\delta,K}(x) \asymp \rho^*_{\delta,\overline{G_j}}(x), \quad x \in E_j,\ 0 < \delta < 1.$$

Since by (4.18),

$$\rho^*_{\delta,\overline{G_j}}(x) \le c_6 \rho_{\delta,E_j}(x), \quad x \in E_j,\ 0 < \delta < 1,$$

we have

$$\rho^*_{\delta,K}(x) \le c_7 \rho_{\delta,E}(x), \quad x \in E_j,\ 0 < \delta < 1.$$

Hence, applying (4.16) and [4], we obtain (ii).

4.3 Simultaneous approximation and interpolation of functions on continua in the complex plane

Let $E \subset \mathbf{C}$ be a compact set with connected complement $\Omega := \overline{\mathbf{C}} \setminus E$. Denote by $A(E)$ the class of all functions continuous on E and analytic in E^0, the interior of E (the case $E^0 = \emptyset$ is not excluded). For $f \in A(E)$ and $n \in \mathbf{N}_0 := \{0, 1, 2, \cdots\}$ define

$$E_n(f, E) := \inf_{p_n \in \mathbf{P}_n} \|f - p_n\|_E.$$

By the Mergelyan theorem (see [47]),

$$\lim_{n\to\infty} E_n(f, E) = 0, \quad f \in A(E).$$

The following assertion on "simultaneous approximation and interpolation" quantifies a result of Walsh [107, p. 310]: Let $z_1, \cdots, z_N \in E$ be distinct points, $f \in A(E)$. Then for any $n \in \mathbf{N}$, $n \ge N$, there exists a polynomial $p_n \in \mathbf{P}_n$ such that

$$\|f - p_n\|_E \le c\, E_n(f, E), \tag{4.19}$$

$$p_n(z_j) = f(z_j), \quad j = 1, \dots, N,$$

where $c > 0$ is independent of n and f.

A suitable polynomial has the form

$$p_n(z) = p_n^*(z) + \sum_{j=1}^{N} \frac{q(z)}{q'(z_j)(z - z_j)}(f(z_j) - p_n^*(z_j)),$$

where

$$q(z) := \prod_{j=1}^{N}(z - z_j),$$

and $p_n^* \in \mathbf{P}_n$ satisfies

$$\|f - p_n^*\|_E = E_n(f, E).$$

It is natural to ask whether it is possible to interpolate the function f as before at arbitrary prescribed points and to simultaneously approximate it in a "more subtle way" than in (4.19). The theorem of Gopengauz [58] about simultaneous polynomial approximation of real functions continuous on the interval $[-1, 1]$ and their interpolation at ± 1 is a useful example. For a recent account of improvements and generalizations of this remarkable statement (for real functions), we refer the reader to [83], [98], [60].

We are going to make use of the D-approximation (named after Dzjadyk, who found in the late 50s – early 60s a constructive description of Hölder classes requiring a nonuniform scale of approximation) as a substitute for (4.19). In [5] it is shown that for the D-approximation to hold for a continuum E it is sufficient, and under some mild restrictions also necessary, that E belongs to the class H^* which can be defined as follows (cf. [3], [6]).

From now on, we assume that E is a continuum, i.e., $\Omega := \overline{\mathbf{C}} \setminus E$ is simply connected. Let $\operatorname{diam}(E) > 0$ and $L := \partial E$ be the boundary of E.

We say that $E \in H$ if any points $z, \zeta \in E$ can be joined by an arc $\gamma(z, \zeta) \subset E$ whose length $|\gamma(z, \zeta)|$ satisfies the condition

$$|\gamma(z, \zeta)| \le c\,|z - \zeta|, \quad c = c(E) \ge 1. \tag{4.20}$$

Let us compactify the domain Ω by prime ends in the sense of Carathéodory (see [82]). Let $\tilde{\Omega}$ be this compactification, and let $\tilde{L} := \tilde{\Omega} \setminus \Omega$. Supposing that $E \in H$, then all the prime ends $Z \in \tilde{L}$ are of the first kind, i.e. have singleton impressions $|Z| = z \in L$. The circle $\{\xi : |\xi - z| = r\}$, $0 < r < \frac{1}{2}\operatorname{diam}(E)$, contains one arc, or finitely many arcs, dividing Ω into two subdomains: an unbounded subdomain and a bounded subdomain such that Z can be defined by a chain of cross-cuts of the bounded subdomain. Let $\gamma_Z(r)$ denote that of these arcs for which the unbounded subdomain is as large as possible (for given Z and r). Thus, the arc $\gamma_Z(r)$ separates the prime end Z from ∞ (cf. [29], [19]).

If $0 < r < R < \frac{1}{2}\operatorname{diam}(E)$, then $\gamma_Z(r)$ and $\gamma_Z(R)$ are the sides of some quadrilateral $Q_Z(r, R) \subset \Omega$ whose other two sides are parts of the boundary L. Let $m_Z(r, R)$ be the module of this quadrilateral, i.e., the module of the family of arcs that separate the sides $\gamma_Z(r)$ and $\gamma_Z(R)$ in $Q_Z(r, R)$ (see [1], [63]).

We say that $E \in H^*$ if $E \in H$ and there exist $c = c(E) < \frac{1}{2}\operatorname{diam}(E)$ and $c_1 = c_1(E)$ such that

$$|m_Z(|z - \zeta|, c) - m_{\mathcal{Z}}(|z - \zeta|, c)| \le c_1 \tag{4.21}$$

for any prime ends $Z, \mathcal{Z} \in \tilde{L}$ with their impressions $z = |Z|$, $\zeta = |\mathcal{Z}|$ satisfying $|z-\zeta| < c$.

In particular, H^* includes domains with quasiconformal boundary (see [1], [63]) and the classes B_k^* of domains introduced by Dzjadyk [47]. For a more detailed investigation of the geometric meaning of conditions (4.20) and (4.21), see [6].

We study functions defined by their k-th modulus of continuity ($k \in \mathbf{N}$). There are a number of different definitions of these moduli in the complex plane (see [106], [99], [43], [90]). The definition by Dyn'kin [43] is the most convenient for our purpose.

From now on suppose $E \in H^*$. The quantity

$$\omega_{f,k,z,E}(\delta) := E_{k-1}(f, E \cap \overline{D(z,\delta)}),$$

where $f \in A(E)$, $k \in \mathbf{N}$, $z \in E$, $\delta > 0$ and $D(z,\delta) := \{\zeta : |\zeta - z| \leq \delta\}$ is called the k-th *local modulus of continuity*, and

$$\omega_{f,k,E}(\delta) := \sup_{z \in E} \omega_{f,k,z,E}(\delta)$$

the k-th (global) *modulus of continuity* of f on E.

By definition, the function $w = \Phi(z)$ maps Ω conformally and univalently onto $\Delta := \{w : |w| > 1\}$ and is normalized by the conditions

$$\Phi(\infty) = \infty, \ \Phi'(\infty) > 0.$$

Let

$$L_\delta := \{\zeta : |\Phi(\zeta)| = 1 + \delta\}, \quad \delta > 0,$$

$$\rho_\delta(z) := \mathrm{dist}(z, L_\delta), \quad z \in \mathbf{C},\ \delta > 0.$$

Theorem 4.12 ([22]) *Let $E \in H^*$, $f \in A(E)$, $k \in \mathbf{N}$, and let $z_1, \cdots, z_N \in E$ be distinct points. Then for any $n \in \mathbf{N}$, $n \geq N + k$, there exists a polynomial $p_n \in \mathbf{P}_n$ such that*

$$|f(z) - p_n(z)| \leq c_1\, \omega_{f,k,E}(\rho_{1/n}(z)), \quad z \in L, \tag{4.22}$$

$$p_n(z_j) = f(z_j), \quad j = 1, \cdots, N \tag{4.23}$$

with $c_1 > 0$ independent of n.

Moreover, if $E^0 \neq \emptyset$ and for $0 < \delta < 1$,

$$\int_0^\delta \omega_{f,k,E}(t)\,\frac{dt}{t} \leq c_2\, \omega_{f,k,E}(\delta), \quad c_2 = \text{ constant } > 0, \tag{4.24}$$

then in addition to (4.22) and (4.23),

$$\|f - p_n\|_K \leq c_3 \exp(-c_4 n^\alpha) \tag{4.25}$$

for every compact set $K \subset E^0$, where the constants c_3, c_4 and $\alpha \leq 1$ are independent of n.

The existence of a polynomial p_n satisfying (4.22) is called a *D-approximation* of the function f (*D-property* of E, *Dzjadyk-type theorem*). For $k > 1$, (4.22) generalizes the corresponding direct theorems of Belyi and Tamrazov [30] (E is a quasidisk) and Shevchuk [90] (E belongs to the Dzjadyk class B_k^*). More detailed history can be found in these papers.

It was first noticed by Shirokov [92] that the rate of D-approximation may admit significant improvement strictly inside E. Saff and Totik [88] proved that if L is an analytic curve, then an exponential rate is achievable strictly inside E, while on the boundary the approximation is "near-best". However, even for domains with piecewise smooth boundary without cusps (and therefore belonging to H^*) the error of approximation strictly inside E cannot be better than e^{-cn^α} (cf. (4.25)) with α which may be arbitrarily small (see [66], [96]). In the results from [66], [96] and [94] containing estimates of the form (4.25) it is usually assumed that Ω satisfies the wedge condition. For a continuum $E \in H^*$ this condition can be violated.

We denote by $A^r(E)$, $r \in \mathbf{N}$, the class of functions $f \in A(E)$ which are r-times continuously differentiable on E and set $A^0(E) := A(E)$. Keeping in mind the Gopengauz result [58], we generalize Theorem 4.12 to the case of Hermite interpolation and simultaneous approximation of a function $f \in A^r(E)$ and its derivatives. For simplicity we formulate this assertion only for the case of boundary interpolation points and without the analog of (4.25).

Theorem 4.13 ([22]) *Let $E \in H^*$, $f \in A^r(E)$, $r \in \mathbf{N}$, $k \in \mathbf{N}$, and let $z_1, \dots, z_N \in \partial E$ be distinct points. Then for any $n \in \mathbf{N}$, $n \geq Nr + k$, there exists a polynomial $p_n \in \mathbf{P}_n$ such that for $l = 0, \dots, r$,*

$$|f^{(l)}(z) - p_n^{(l)}(z)| \leq c\,\rho_{1/n}^{r-l}(z)\,\omega_{f^{(r)},k,E}(\rho_{1/n}(z)), \quad z \in L,$$

$$p_n^{(l)}(z_j) = f^{(l)}(z_j), \quad j = 1, \dots, N$$

with c independent of n.

Our next goal is to allow the number of interpolation nodes N in Theorem 4.12 to grow infinitely with the degree of the approximating polynomial n. To this end, we specify the choice of points $z_1, \dots, z_N$. In order to do it optimally from the point of view of interpolation theory, we have to require that the discrete measure

$$\nu_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{z_j},$$

where δ_z denotes the unit mass placed at z, is close to the equilibrium measure for E (for details, see [89]). Fekete points (see [79], [89]) are natural candidates for our purpose. Even in this case the number $N-1$ cannot be equal to the degree n of the approximating polynomial (cf. Faber's theorem [53] claiming that for $E = [-1, 1]$ there is no universal set of nodes at which to every continuous function the Lagrange interpolating polynomials converge in the uniform norm). However, it was first observed by Bernstein [31] that for any continuous function on $[-1, 1]$ and any small $\varepsilon > 0$, there exists a sequence of polynomials interpolating in the Chebyshev nodes and uniformly convergent on $[-1, 1]$, such that $n \leq (1 + \varepsilon)N$. This result was developed in several directions. In particular, Erdös (see [51] and [52]) found a necessary and sufficient condition on the system of nodes, for this type of simultaneous approximation and interpolation to be valid. We generalize the results of Bernstein and Erdös in the following theorem.

Theorem 4.14 ([22]) *Let E be a closed Jordan domain bounded by a quasiconformal curve L. Let f, r, k be as in Theorem 4.12 and let $z_1, \dots, z_N \in E$ be the points of an N-th Fekete point set of E. Then for any $\varepsilon > 0$ there exists a polynomial $p_n \in \mathbf{P}_n$, $n \leq (1 + \varepsilon)N$, satisfying conditions (4.22) and (4.23). Moreover, if (4.24) holds then in addition to (4.22) and (4.23) we have (4.25), and the constants c_1, c_3, c_4 and α are independent of N.*

4.4 Open problems

We begin with a question that would be a complete resolution of the Meinardus-Varga problem on the structure of an entire function with geometric convergence on the positive real axis of reciprocals of polynomials to the reciprocal of the function.

Problem 6. *Let an entire function f satisfy (4.1). Are the following two conditions*

(i) *f satisfies (4.2),*

(ii) *There exist $s > 1$ and positive constants c, θ, and r_0 such that*

$$\|f\|_{\mathcal{E}_r(f,s)} \le c\, r^{\theta}, \quad r \ge r_0,$$

equivalent?

Note that $(i) \Rightarrow (ii)$ is proved in [21] (cf. Theorem 4.3). The inverse implication $(ii) \Rightarrow (i)$ appears to be much more complicated to prove. One of the possibilities is to use an extension of the classical result of Bernstein on polynomial approximation of functions analytic in a neighborhood of a subinterval of the real axis (see [42]) to the case of several intervals.

Problem 7. *Let $E = \bigcup_{j=1}^{k} I_j$ be the union of k disjoint intervals $I_j = [\alpha_j, \beta_j] \subset \mathbf{R}$. Is it true that for each function f satisfying (4.9) and (4.10), there exists a constant $c > 0$ depending only on $s > 1$ such that*

$$E_n(f,E) \le c\, \|f\|_{E^s}\, s^{-n}, \quad n \in \mathbf{N}\,? \tag{4.26}$$

Note that (4.26) does not depend on the geometry of E. This fact makes Problem 7 much more difficult to study compared to the known results on polynomial approximation of functions on a finite number of intervals (cf. [54]).

Our prior research (see Theorem 4.11) indicates that there exists a connection between the Nikol'skii-Timan-Dzjadyk approximation theorem and the concept of uniformly perfect sets. We propose to investigate the details of this connection.

Problem 8. *Let $E \subset \mathbf{R}$ be uniformly perfect. Suppose that $f \in C(E)$ and that the function ω of modulus of continuity type satisfies (4.12). Are the following conditions*

(i) *$f \in C_\omega(E)$,*

(ii) *For any $n \in \mathbf{N}$, there exists a polynomial $p_n \in \mathbf{\Pi}_n$ such that*

$$|f(x) - p_n(x)| \le c\, \omega(\rho_{1/n}(x)), \quad x \in E,$$

where the constant $c > 0$ does not depend on x and n,

equivalent?

This conjecture is motivated by the connection between uniformly perfect sets and John domains described in Subsection 2.2. Since the behavior of a conformal mapping of a John domain onto the disk is well-studied (see, for example, [82]), this can be used for constructing polynomials with the desired properties.

Moreover, we conjecture that uniformly perfect sets present exactly the class of sets to which Theorem 4.8 can be generalized as in Problem 8.

Problem 9. *Suppose that the compact set $E \subset \mathbf{R}$ is not uniformly perfect. Does it follow that, for any $0 < \alpha < 1$, there exists a function $f_\alpha \in C^\alpha(E)$ such that the following assertion is false? For any $n \in \mathbf{N}$, there is a polynomial $p_n \in \mathbf{\Pi}_n$ with the property*

$$|f_\alpha(x) - p_n(x)| \le c\,\rho_{1/n}^\alpha(x), \quad x \in E,$$

where the constant $c > 0$ is independent of n and x.

Next, we discuss the description of classes of functions with a given rate of decrease of their uniform polynomial approximations. Let $E \subset \mathbf{R}$ be a regular compact set and let $f \in C(E)$. The following fundamental problem of approximation theory is another example of the interplay between smoothness properties of a function $f \in C(E)$, the rate of decrease of its best polynomial approximations $E_n(f, E)$, and the geometry of the set E: for fixed $\alpha > 0$ describe all functions $f \in C(E)$ such that

$$E_n(f, E) = O(n^{-\alpha}), \quad n \to \infty. \tag{4.27}$$

For $x \in E$ and $t > 0$ let the function $\delta(x, t)$ be defined by the equality

$$\rho_{\delta(x,t)}(x) = t.$$

Problem 10. *Let $E \subset \mathbf{R}$ be uniformly perfect, $f \in C(E)$, and let $0 < \alpha < 1$. Are the following two conditions*

(i) *The inequality (4.27) holds,*

(ii) *For all $x_1, x_2 \in E$,*

$$|f(x_2) - f(x_1)| \le c\,\delta(x_1, |x_2 - x_1|)^\alpha,$$

where $c > 0$ is independent of x_1 and x_2,

equivalent?

So far, a positive answer to this question is confirmed for $E = [a, b]$ being a closed interval [42, p. 265]. In general, the problem is open. The proof of $(i) \Rightarrow (ii)$ is given in [12]. The inverse implication $(ii) \Rightarrow (i)$ needs new ideas.

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References

[1] Ahlfors, L. V., "Lectures on Quasiconformal Mappings", Princeton, N.J., Van Nostrand, 1966.

[2] Akhiezer, N. I., Über einige Funktionen, welche in zwei gegebenen Intervallen am wenigsten von Null abweichen. I, II, III, *Izv. Akad. Nauk SSSR Otdel. Mat. Estestv. Nauk* (1932/33) 1163–1202, 309–344, 499–536.

[3] Andrievskii, V. V., Geometric properties of Dzjadyk domains, *Ukr. Math. J.* **33**(1982), 543–547.

[4] Andrievskii, V. V., On approximation of functions by harmonic polynomials, *Math. USSR Izvestija* **30**(1982), 1–13.

[5] Andrievskii, V. V., The geometric structure of regions, and direct theorems of the constructive theory of functions, *Math. USSR Sbornik* **54**(1986), 39–56.

[6] Andrievskii, V. V., Metric properties of Riemann's mapping function for the region supplemented to continuum without external zero angles, *Sov. J. Contemp. Math. Anal., Arm. Acad. Sci.* **24**(1989), 57–68.

[7] Andrievskii, V. V., Approximation of harmonic functions on compact sets in **C**, *Ukrain. Mat. Zh.* **45**(1993), 1467–1475.

[8] Andrievskii, V. V., The Nikol'skii-Timan-Dzjadyk theorem for functions on compact sets of the real line, *Constr. Approx.* **17**(2001), 431–454.

[9] Andrievskii, V. V., A Remez-type inequality in terms of capacity, *Complex Variables* **45**(2001), 35–46.

[10] Andrievskii, V. V., A note on a Remez-type inequality for trigonometric polynomials, *J. Approx. Theory* **116**(2002), 416–424.

[11] Andrievskii, V. V., A Remez-type inequality for rational functions considered on a smooth curve, *East J. Approx.* **8**(2002), 197–207.

[12] Andrievskii, V. V., Uniformly perfect subsets of the real line and John domains, *Comput. Methods Funct. Theory* **3**(2003), 385–396.

[13] Andrievskii, V. V., On the Green function for a complement of a finite number of real intervals, *Constr. Approx.* **20**(2004), 565–583.

[14] Andrievskii, V. V., The highest smoothness of the Green function implies the highest density of a set, *Ark. Mat.* **42**(2004), 217–238.

[15] Andrievskii, V. V., Polynomial approximation of analytic functions on a finite number of continua in the complex plane, *J. Approx. Theory* **133**(2005), 238–244.

[16] Andrievskii, V. V., On sparse sets with the Green function of the highest smoothness, *Comput. Methods Funct. Theory* **5**(2005), 347–363.

[17] Andrievskii, V. V., Pointwise Remez-type inequalities in the unit disk, *Constr. Approx.* **22**(2005), 297–308.

[18] Andrievskii, V. V., On optimal smoothness of the Green function for the complement of a Cantor-type set, *Constr. Approx.*, to appear.

[19] Andrievskii, V. V., Belyi, V. I., and Dzjadyk, V. K., "Conformal Invariants in Constructive Theory of Functions of Complex Variable", World Federation Publisher, Atlanta, Georgia, 1995.

[20] Andrievskii, V. V. and Blatt, H.-P., "Discrepancy of Signed Measures and Polynomial Approximation", Springer-Verlag, Berlin/New York, 2002.

[21] Andrievskii, V. V., Blatt, H.-P., and Kovacheva, R. K., Approximation on an unbounded interval, *Analysis* **17**(1997), 197–218.

[22] Andrievskii, V. V., Pritsker, I. E., and Varga, R. S., Simultaneous approximation and interpolation of functions on continua in the complex plane, *J. Math. Pures Appl.* **80**(2001), 373–388.

[23] Andrievskii, V. V. and Ruscheweyh, St., Maximal polynomial subordination to univalent functions in the unit disk, *Constr. Approx.* **10**(1994), 131–144.

[24] Andrievskii, V. V. and Ruscheweyh, St., Remez-type inequalities in terms of linear measure, *Comput. Methods Funct. Theory* **5**(2005), 347–363.

[25] Andrievskii, V. V. and Ruscheweyh, St., Remez-type inequalities in the complex plane, submitted to *Constr. Approx.*

[26] Baernstein, A., Integral means, univalent functions and circular symmetrization, *Acta Math.* **133**(1974), 139–169.

[27] Bagby, T., Bos, L., and Levenberg, N., Quantitative approximation theorems for elliptic operators, *J. Approx. Theory* **85**(996), 69–87.

[28] Beardon, A. and Pommerenke, Ch., The Poincaré metric of plane domains, *J. London Math. Soc.* **18**(1978), 475–483.

[29] Belyi, V. I., Conformal mappings and the approximation of analytic functions in domains with a quasiconformal boundary, *Math. USSR Sbornik* **31**(1977), 289–317.

[30] Belyi, V. I. and Tamrazov, P. M., Polynomial approximations and smoothness moduli of functions in regions with quasiconformal boundary, *Sib. Math. J.* **21**(1981), 434–445.

[31] Bernstein, S. N., Sur une modification de la formule d'interpolation de Lagrange, *Comm. Khark. Math. Soc.* **5**(1932), 49–57.

[32] Białas, L. and Volberg, A., Markov's property of the Cantor ternary set, *Studia Math.* **104**(1993), 259–268.

[33] Bishop, Ch. J., Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture, *Ark. Mat.* **40**(2002), 1–26.

[34] Blatt, H.-P., Ein Störungssatz zur rationalen Tschebyscheff-Approximation über $[0,\infty)$, *J. Approx. Theory* **21**(1977), 246–256.

[35] Blatt, H.-P., Ein hinreichendes Kriterium für geometrische Konvergenz rationaler Tschebyscheff-Approximationen auf $[0,\infty)$, *J. Approx. Theory* **23**(1978), 229–241.

[36] Blatt, H.-P. and Kovacheva, R. K., Geometric rational approximation on $[0,\infty)$, *East J. Approx.* **2**(1996), 31–40.

[37] Borwein, P. and Erdélyi, T., "Polynomials and Polynomial Inequalities", Springer, New York, 1995.

[38] Carleson, L., "Selected Problems on Exceptional Sets", Princeton, 1967.

[39] Carleson, L. and Totik, V., Hölder continuity of Green's functions, *Acta Sci. Math. (Szeged)* **70**(2004), 557–608.

[40] Cody, W. J., Meinardus, G., and Varga, R. S., Chebyshev rational approximations to $\exp(-x)$ in $[0, \infty)$ and applications to heat conduction problems, *J. Approx. Theory* **2**(1969), 50–69.

[41] Cuyt, A., Driver, K., and Lubinsky, D. S., On the size of lemniscates of polynomials in one and several variables, *Proc. Amer. Math. Soc.* **124**(1996), 2123–2135.

[42] DeVore, R. A. and Lorentz, G. G., "Constructive Approximation", Springer-Verlag, New York/Berlin, 1993.

[43] Dyn'kin, E. M., On the uniform approximation of functions in Jordan domains, *Sibirsk. Mat. Zh.* **18**(1977), 775–786 (in Russian).

[44] Dzjadyk, V. K., Constructive characterization of functions satisfying the condition Lipα $(0 < \alpha < 1)$ on a finite segment of the real axis, *Izv. Akad. Nauk SSSR Ser. Mat.* **20**(1956), 623–642 (in Russian).

[45] Dzjadyk, V. K., On a problem of S. M. Nikol'skii in a complex region, *Izv. Akad. Nauk SSSR Ser. Mat.* **23**(1959), 697–736 (in Russian).

[46] Dzjadyk, V. K., On the approximation of continuous functions in a closed domain with corners and on a problem of S. M. Nikol'skii, I, *Izv. Akad. Nauk SSSR Ser. Mat.* **26**(1962), 797–824 (in Russian).

[47] Dzjadyk, V. K., "Introduction to the Theory of Uniform Approximation of Functions by Polynomials", Nauka, Moscow (in Russian), 1977.

[48] Erdélyi, T., Remez-type inequalities on the size of generalized polynomials, *J. London Math. Soc.* **45**(1992), 255–264.

[49] Erdélyi, T., Remez-type inequalities and their applications, *J. Comput. Appl. Math.* **47**(1993), 167–209.

[50] Erdélyi, T., Li X., and Saff, E. B., Remez and Nikolskii type inequalities for logarithmic potentials, *SIAM J. Math. Anal.* **25**(1994), 365–383.

[51] Erdös, P., On some convergence properties of the interpolation polynomials, *Ann. of Math.* **44**(1943), 330–337.

[52] Erdös, P., On the boundedness and unboundedness of polynomials, *J. Anal. Math.* **19**(1967), 135–148.

[53] Faber, G., Über die interpolatorische Darstellung stetiger Funktionen, *Jahresber. Deutsch. Math. Verein* **23**(1914), 192–210.

[54] Fuchs, W. H. J., On Chebychev approximation on several disjoint intervals, *Complex Approximation (Proc. Conf., Quebec, 1978)*, Birkhäuser, Boston, Mass., 1980, 67–74.

[55] Gaier, D., "Lectures on Complex Approximation", Birkhäuser, Basel, 1987.

[56] Ganzburg, M. I., Polynomial inequalities on measurable sets and their applications, *Constr. Approx.* **17**(2001), 275–306.

[57] Goodman, A. W., A note on the zeros of Faber polynomials, *Proc. Amer. Math. Soc.* **49**(1975), 407–410.

[58] Gopengauz, I. E., A theorem of A. F. Timan on the approximation of functions by polynomials on a finite segment, *Mat. Zametki* **1**(1967), 163–172 (in Russian).

[59] Henry, M. S. and Roulier, J. A., Geometric convergence of Chebyshev rational approximants on $[0, \infty)$, *J. Approx. Theory* **21**(1977), 361–374.

[60] Kilgore, T. and Prestin, J., A theorem of Gopengauz type with added interpolatory conditions, *Numer. Funct. Anal. Optim.* **15**(1994), 859–868.

[61] Kroó, A., On Remez-type inequalities for polynomials in $\mathbf{R}^m$ and $\mathbf{C}^m$, *Analysis Math.* **27**(2001), 55–70.

[62] Landkof, N. S., "Foundations of Modern Potential Theory", Grundlehren Math. Wiss., **180**, Springer-Verlag, New York, 1972.

[63] Lehto, O. and Virtanen, K. I., "Quasiconformal Mappings in the Plane", 2nd ed., Springer-Verlag, New York, 1973.

[64] Levin, B. Ja., Connection of a majorant with the conformal mapping, *in "Theory of Functions, Functional Analysis, and Applications"* **52**(1986), 3–21 (in Russian).

[65] Lithner, J., Comparing two versions of Markov's property, *J. Approx. Theory* **77**(994), 202–211.

[66] Maimeskul, V. V., Degree of approximation of analytic functions by nearly best polynomial approximants, *Constr. Approx.* **11**(1995), 1–21.

[67] Martio, O. and Sarvas, J., Injectivity theorems in plane and space, *Ann. Acad. Sci. Fenn. Ser AI Math.* **4**(1978/79), 383–401.

[68] Maz'ja, V. G., On the modulus of continuity of a harmonic function at a boundary point, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **135**(1984), 87–95.

[69] Meinardus, G. and Varga, R. S., Chebyshev rational approximations to certain entire functions in $[0, \infty)$, *J. Approx. Theory* **3**(1970), 300–309.

[70] Meinardus, G., Converse theorems in rational approximation, *International Conference in Constructive Function Theory, Varna, 1970, Proceedings*, 1970, 217–220.

[71] Meinardus, G., Reddy, A. R., Taylor, G. D., and Varga, R. S., Converse theorems and extensions in Chebyshev rational approximation to certain entire functions in $[0,\infty)$, *Trans. Amer. Math. Soc.* **170**(1972), 171–185.

[72] Mezhevich, K. G. and Shirokov, N. A., Polynomial approximations on disjoint segments, *J. Math. Sci. (New York)* **98**(2000), 706–716.

[73] Nevanlinna, R., "Analytic Functions", Grundlehren Math. Wiss., **162**, Springer-Verlag, Berlin, 1970.

[74] Nguen Tu Than', On the approximation of functions by polynomials on a closed set with angles whose complement is finitely connected, *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* **14**(1981), 293–298 (in Russian).

[75] Nguen Tu Than', Approximation of functions by polynomials on sets with a multiply connected complement, *Dokl. Akad. Nauk SSSR* **299**(1988), 555–558 (in Russian).

[76] Nikol'skii, S. M., On the best approximation of functions satisfying Lipschitz conditions by polynomials, *Izv. Akad. Nauk SSSR Ser. Mat.* **10**(1946), 295–322 (in Russian).

[77] Pleśniak, W., A Cantor regular set which does not have Markov's property, *Ann. Pol. Math.* **LI**(1990), 269–274.

[78] Pleśniak, W., Markov's inequality and the existence of an extension operator for C^∞ functions, *J. Approx. Theory* **61**(1990), 106–117.

[79] Pommerenke, Ch., "Univalent functions", Vandenhoeck and Ruprecht, Göttingen, 1975.

[80] Pommerenke, Ch., Uniformly perfect sets and the Poincaré metric, *Arch. Math. (Basel)* **32**(1979), 192–199.

[81] Pommerenke, Ch., On uniformly perfect sets and Fuchsian groups, *Analysis* **4**(1984), 299–321.

[82] Pommerenke, Ch., "Boundary Behaviour of Conformal Maps", Springer-Verlag, Berlin/New York, 1992.

[83] Privalov, A. A., On the simultaneous interpolation and approximation of continuous functions, *Mat. Zametki* **35**(1984), 381–395 (in Russian).

[84] Ransford, T., "Potential Theory in the Complex Plane", Cambridge University Press, Cambridge, 1995.

[85] Reddy, A. R. and Shisha, O., A class of rational approximants on the positive real axis, a survey, *J. Approx. Theory* **12**(1974), 425–434.

[86] Remez, E. J., Sur une propriété des polynômes de Tchebycheff, *Comm. l'Inst. Sci., Kharkow* **13**(1936), 93–95.

[87] Roulier, J. A. and Taylor, G. D., Rational Chebyshev approximation on $[0,\infty)$, *J. Approx. Theory* **11**(1974), 208–215.

[88] Saff, E. B. and Totik, V., Behaviour of polynomials of best uniform approximation, *Trans. Amer. Math. Soc.* **316**(1989), 567–593.

[89] Saff, E. B. and Totik, V., “Logarithmic Potentials with External Fields”, Springer, New York/Berlin, 1997.

[90] Shevchuk, I. A., Constructive characterization of continuous functions on a set $\mathcal{M} \subset \mathbf{C}$ for the k-th modulus of continuity, *Math. Notes* **25**(1979), 117–129.

[91] Shevchuk, I. A., “Approximation by Polynomials and Traces of Functions Continuous on a Segment”, Naukova Dumka, Kiev (in Russian), 1992.

[92] Shirokov, N. A., On uniform approximation of functions on closed sets with nonzero external angles, *Izv. Akad. Nauk Armyan. SSR Ser. Mat.* **9**(1974), 62–80 (in Russian).

[93] Shirokov, N. A., Approximation in the sense of Dzyadyk on compact sets with complement of infinite connectivity, *Russian Acad. Sci. Dokl. Math.* **49**(1994), 431–433.

[94] Shirokov, N. A., Polynomial D-approximations decaying exponentially strictly inside a continuum, *St. Petersburg Math. J.* **6**(1995), 1249–1273.

[95] Shirokov, N. A., Approximation by polynomials on compact sets with complement of infinite connectivity, *St. Petersburg Math. J.* **10**(1999), 197–210.

[96] Shirokov, N. A. and Totik, V., Polynomial approximation on the boundary and strictly inside, *Constr. Approx.* **11**(1995), 145–152.

[97] Sodin, M. L. and Yuditskii, P. M., Functions deviating least from zero on closed subsets of the real axis, *St. Petersburg Math. J.* **4**(1993), 201–249.

[98] Szabados, J. and Vértesi, P., “Interpolation of Functions”, World Scientific Publishing Co., Singapore, 1990.

[99] Tamrazov, P. M., “Smoothnesses and Polynomial Approximations”, Naukova Dumka, Kiev (in Russian), 1975.

[100] Timan, A. F., A strengthening of Jackson’s theorem on the best approximation of continuous functions by polynomials on a finite segment of the real axis, *Dokl. Akad. Nauk SSSR* **78**(1951), 17–20 (in Russian).

[101] Totik, V., Markoff constants for Cantor sets, *Acta Sci. Math. (Szeged)* **60**(1995), 715–734.

[102] Totik, V., Polynomial inverse images and polynomial inequalities, *Acta Math.* **187**(2001), 139–160.

[103] Totik, V., On Markoff’s inequality, *Constr. Approx.* **18**(2002), 427–441.

[104] Totik, V., “Metric Properties of Harmonic Measure”, Memoirs of the American Mathematical Society (to appear), 2006.

[105] Tsuji, M., “Potential Theory in Modern Function Theory”, Chelsea Publ. Co., New York, 1950.

[106] Vorob'ev, N. N. and Polyakov, R. V., Constructive characteristic of continuous functions defined on smooth arcs, *Ukr. Math. J.* **20**(1968), 647–654.

[107] Walsh, J. L., "Interpolation and Approximation by Rational Functions in the Complex Plane", 5th ed., American Mathematical Society, Providence, 1969.

[108] Widom, H., Extremal polynomials associated with a system of curves in the complex plane, *Adv. in Math.* **3**(1969), 127–232.

[109] Juan-Hua Zheng, Uniformly perfect sets and distortion of holomorphic functions, *Nagoya Math. J.* **164**(2001), 17–33.

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