Mixed Moduli of Smoothness in L_p, $1 < p < \infty$: A Survey *

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Abstract

In this paper we survey recent developments over the last 25 years on the mixed fractional moduli of smoothness of periodic functions from L_p, $1 < p < \infty$. In particular, the paper includes monotonicity properties, equivalence and realization results, sharp Jackson, Marchaud, and Ul'yanov inequalities, interrelations between the moduli of smoothness, the Fourier coefficients, and "angular" approximation. The sharpness of the results presented is discussed.

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Contents

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1 Introduction

To open the discussion on mixed moduli of smoothness, we start with function spaces of dominating mixed smoothness. The Sobolev spaces of dominating mixed smoothness were first introduced (on $\mathbb{R}^2$) by Nikol'skii [49, 50]. He defined the space

$$S_p^{r_1,r_2}W(\mathbf{R}^2)=\Bigg\{f\in L_p(\mathbf{R}^2):\|f\|_{S_p^{r_1,r_2}W(\mathbf{R}^2)} = \|f\|_{L_p(\mathbf{R}^2)} +\Big\|\frac{\partial^{r_1}f}{\partial x_1^{r_1}}\Big\|_{L_p(\mathbf{R}^2)} + \Big\|\frac{\partial^{r_2}f}{\partial x_2^{r_2}}\Big\|_{L_p(\mathbf{R}^2)}+\Big\|\frac{\partial^{r_1+r_2}f}{\partial x_1^{r_1}\partial x_2^{r_2}}\Big\|_{L_p(\mathbf{R}^2)}\Bigg\},$$

where $1<p<\infty$, $r_1,r_2=0,1,2$. Here, the mixed derivative $\frac{\partial^{r_1+r_2}f}{\partial x_1^{r_1}\partial x_2^{r_2}}$ plays a dominant role and it gave the name to these scales of function spaces.

Later, the fractional Sobolev spaces with dominating mixed smoothness (see [43] by Lizorkin and Nikol'skii), the Hölder-Zygmund-type spaces (see Nikol'skii [49, 50] and Bakhvalov [4]), and the Besov spaces of dominating mixed smoothness were introduced (see Amanov [1]). We would also like to mention the paper [3] by Babenko which considered Sobolev spaces with dominating mixed smoothness in the context of multivariate approximation. It transpires that spaces with dominating mixed smoothness have several unique properties which can be used in different settings, for example, in multivariate approximation theory of periodic functions (see [69, 1.3] and [81]) or in high-dimensional approximation and computational mathematics (see, e.g., [76]).

To define Hölder-Besov spaces (Nikol'skii-Besov) of dominating mixed smoothness, the notion of the mixed modulus of smoothness is used, i.e.,

$$\omega_{\mathbf{k}}(f,\mathbf{t})_p=\omega_{k_1,\dots,k_d}(f,t_1,\dots,t_d)_p=\sup_{|h_i|\le t_i, i=1,\dots,d}\|\Delta_{\mathbf{h}}^{\mathbf{k}}f\|_p,$$

where the $\mathbf{k}$-mixed difference is given by

$$\Delta_{\mathbf{h}}^{\mathbf{k}}=\Delta_{h_1}^{k_1}\circ\dots\circ\Delta_{h_d}^{k_d},$$

$$\mathbf{k}=(k_1,\dots,k_d),\quad \mathbf{h}=(h_1,\dots,h_d),$$

and $\Delta_{h_i}^{k_i}$ is the difference of order k_i with step h_i with respect to x_i; for example,

$$\Delta_{h_i}^{1} f(x_1,\dots,x_d) = f(x_1,\dots,x_i+h_i,\dots,x_d) - f(x_1,\dots,x_i,\dots,x_d),$$

$$\Delta_{h_i}^{k_i} f(x_1,\dots,x_d) = \sum_{j=0}^{k_i} (-1)^j \binom{k_i}{j} f(x_1,\dots,x_i+(k_i-j)h_i,\dots,x_d), \qquad k_i\in\mathbb{N}.$$

In their turn, moduli of smoothness of an integer order can be naturally extended to fractional order moduli. The one-dimensional fractional modulus of smoothness was introduced in the 1970's (see [11, 78, 92] and the monograph [67]). Moreover, moduli of smoothness of positive orders play an important role in Fourier analysis, approximation theory, theory of embedding theorems and some other problems (see, e.g., [67, 75, 82, 84, 85, 91, 92]). Note that one of the key results in this area of research — an equivalence between the modulus of smoothness and the K-functional — was proved for the one-dimensional fractional modulus in [11] and for the multivariate (non-mixed) fractional modulus in [101, 102]; see also [16, 37, 72]. Clearly, mixed moduli of smoothness are closely related to mixed directional derivatives. Inequalities between the mixed and directional derivatives are given in, e.g., [12].

In general, the modulus of smoothness is an important concept in modern analysis and there are many sources providing information on the one dimensional and multivariate (non-mixed) moduli from different perspectives. Concerning mixed moduli of smoothness, two old monographs [1] and [86] can be mentioned where several basic properties are listed. Also, there is vast literature on the theory of function spaces with dominating mixed smoothness (see Section 1.2 below). The main goal of this paper is to collect the main properties of the mixed moduli of smoothness of periodic functions from $L_p(\mathbb{T}^d)$, $1<p<\infty$, from the point of view of approximation theory and Fourier analysis.

This paper attempts to give a self-contained development of the theory. Since many of the sources where the reader can find these results are difficult to obtain and many of the results are stated without proofs, we will provide complete proofs of all the main results in this survey. Moreover, there are several results in Sections 4-11 that are new, to the best of our knowledge.

For the sake of clarity, in this survey we deal with periodic functions on $\mathbb{T}^2$. We limit ourselves to this case to help the reader follow the discussion and to put the notation and results in a more compact form. All the results of this survey can be extended to the case of $\mathbb{T}^d$, $d>2$.

Let us also mention that since any L_p-function f on $\mathbb{T}^2$ can be written as

$$f(x,y) = F(x,y) + \phi(x) + \psi(y) + c,$$

where $F\in L_p^0(\mathbb{T}^2)$, i.e., $\int_{\mathbb{T}} F\,\mathrm{d}x = \int_{\mathbb{T}} F\,\mathrm{d}y = 0$ and since $\omega_{\alpha_1,\alpha_2}(f;\delta_1,\delta_2)_p = \omega_{\alpha_1,\alpha_2}(F;\delta_1,\delta_2)_p$, it suffices to deal with functions from $L_p^0(\mathbb{T}^2)$.

1.1 How this survey is organized

After auxiliary results and notation given in Sections 2 and 3, in Section 4 we collect the main properties of the mixed moduli, mainly, various monotonicity properties and direct and inverse type approximation theorems. In Section 5 we prove a constructive characterization of the mixed moduli of smoothness which is a realization result (see, e.g., [24]). This result provides us with a useful tool to obtain the results of the later sections. In particular, this allows us to show the equivalence between the mixed modulus of smoothness and the corresponding K-functional in Section 6.

In Section 7 two-sided estimates of the mixed moduli of smoothness in terms of the Fourier coefficients are given. In Section 8 we deal with sharp inequalities between the mixed moduli of smoothness of functions and their derivatives, i.e., $\omega_{\mathbf{k}}(f, \mathbf{t})_p$ and $\omega_{\mathbf{l}}(f^{\mathbf{r}}, \mathbf{t})_p$. Section 9 gives sharp order two-sided estimates of the mixed moduli of smoothness of L_p functions in terms of their "angular" approximations.

In Section 10 we study sharp relationships between $\omega_{\mathbf{k}}(f, \mathbf{t})_p$ and $\omega_{\mathbf{l}}(f, \mathbf{t})_p$. One part of this relation is usually called the sharp Marchaud inequality (see, e.g., [15, 22, 23]), another is equivalent to the sharp Jackson inequality ([14, 15]). It is well known that these results are closely connected to the results of Section 8 because of Jackson and Bernstein-Stechkin type inequalities. Finally, in Section 11, we discuss sharp Ul'yanov's inequality, i.e., sharp relationships between $\omega_{\mathbf{k}}(f, \mathbf{t})_p$ and $\omega_{\mathbf{l}}(f, \mathbf{t})_q$ for $p < q$ (see [75]).

In Sections 7-10, we deal with two-sided estimates for the mixed moduli of smoothness. In order to show sharpness of these estimates we will introduce special function classes so that for functions from these classes the two-sided estimates become equivalences.

1.2 What is not included in this survey

In this paper, we restrict ourselves to questions which were not covered by previous expository papers and which were actively developed over the last 25 years. For example, we do not discuss questions which are quite naturally linked to the mixed moduli of smoothness such as

- Different types of convergence of multiple Fourier series (see Chapter I in the surveys [29, 104] and the papers [17, 28]);
- Absolute convergence of multiple Fourier series (see Chapter X in the surveys [29, 104] and the papers [48, 47]);
- Summability theory of multiple Fourier series (see the paper [103] and the monograph [105]);
- Interrelations between the total, partial and mixed moduli of smoothness; derivatives (see [10, 21, 39, 58, 88]);
- Fourier coefficients of functions from certain smooth spaces (see, e.g., [2, 7, 29]);
- Conjugate multiple Fourier series (see, e.g., the book [106] and Chapter VIII in the survey [29]);
- Representation and approximation of multivariate functions (see, e.g., [5, 17, 18, 20, 64, 77] and Chapter 11 of the recent book [95]); in particular, for Whitney type results see [27];
- Approximate characteristics of functions, entropy, and widths (see [35, 63, 38, 79, 80, 100]).

Also, we do not deal with the questions of

- The theory of function spaces with dominating mixed smoothness, in particular, with characterization, representation, embeddings theorems, characterization of approximation spaces, m-term approximation, which are fast growing topics nowadays. Let us only mention a few basic older papers [42, 43, 44], the monograph [9] by Besov, Il'in and Nikol'skii, the monograph by Schmeisser and Triebel [71], the recent book by Triebel [93], and the 2006's survey [69] on this topic. The reader might also be interested in the recent work of researchers from the Jena school [34, 40, 68, 70, 94, 98, 99]; see also [32, 36]. The coincidence of the Fourier-analytic definition of the spaces of dominating mixed smoothness and the definition in terms of differences is given in the paper [96].

2 Definitions and notation

Let $L_p = L_p(\mathbb{T}^2)$, $1 < p < \infty$, be the space of measurable functions f of two variables that are 2π-periodic in each variable and such that

$$\|f\|_{L_p(\mathbb{T}^2)} = \left(\int_0^{2\pi}\int_0^{2\pi} |f(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} < \infty.$$

Let also $L_p^0(\mathbb{T}^2)$ be the collection of $f \in L_p(\mathbb{T}^2)$ such that

$$\int_0^{2\pi} f(x,y)\,\mathrm{d}y = 0 \qquad \text{for} \quad \text{a.e. } x$$

and

$$\int_0^{2\pi} f(x,y)\,\mathrm{d}x = 0 \qquad \text{for} \quad \text{a.e. } y.$$

If $F(f,\delta_1,\delta_2) > 0$ and $G(f,\delta_1,\delta_2) > 0$ for all $\delta_1,\delta_2 > 0$, then writing $F(f,\delta_1,\delta_2) \lesssim G(f,\delta_1,\delta_2)$ means that there exists a constant C, independent of f,δ_1,δ_2 such that $F(f,\delta_1,\delta_2) \le CG(f,\delta_1,\delta_2)$. Note that C may depend on unessential parameters (clear from context), and may change form line to line. If $F(f,\delta_1,\delta_2) \lesssim G(f,\delta_1,\delta_2)$ and $G(f,\delta_1,\delta_2) \lesssim F(f,\delta_1,\delta_2)$ simultaneously, then we will write $F(f,\delta_1,\delta_2) \asymp G(f,\delta_1,\delta_2)$.

2.1 The best angular approximation

By $s_{m_1,\infty}(f)$, $s_{\infty,m_2}(f)$, and $s_{m_1,m_2}(f)$ we denote the partial sums of the Fourier series of a function $f \in L^p(\mathbb{T}^2)$, i.e.,

$$s_{m_1,\infty}(f) = \frac{1}{\pi}\int_0^{2\pi} f(x+t_1,y)D_{m_1}(t_1)\,\mathrm{d}t_1,$$

$$s_{\infty,m_2}(f) = \frac{1}{\pi}\int_0^{2\pi} f(x,y+t_2)D_{m_2}(t_2)\,\mathrm{d}t_2,$$

$$s_{m_1,m_2}(f) = \frac{1}{\pi^2} \int\limits_0^{2\pi} \int\limits_0^{2\pi} f(x+t_1, y+t_2) D_{m_1}(t_1) D_{m_2}(t_2)\, \mathrm{d}t_1\, \mathrm{d}t_2,$$

where D_m is the Dirichlet kernel, i.e.,

$$D_m(t) = \frac{\sin\,(m+\frac{1}{2})t}{2\sin\frac{t}{2}}, \qquad m = 0, 1, 2, \dots$$

As a means of approximating a function $f \in L^p(\mathbb{T}^2)$, we will use the so called best (two-dimensional) angular approximation $Y_{m_1,m_2}(f)_{L_p(\mathbb{T}^2)}$ which is also sometimes called "approximation by an angle" ([53]). By definition,

$$Y_{m_1,m_2}(f)_{L_p(\mathbb{T}^2)} = \inf_{T_{m_1,\infty}, T_{\infty,m_2}} \|f - T_{m_1,\infty} - T_{\infty,m_2}\|_{L_p(\mathbb{T}^2)},$$

where the function $T_{m_1,\infty} \in L_p(\mathbb{T}^2)$ is a trigonometric polynomial of degree at most m_1 in x, and the function $T_{\infty,m_2} \in L_p(\mathbb{T}^2)$ is a trigonometric polynomial of degree at most m_2 in y.

2.2 The mixed moduli of smoothness

For a function $f \in L_p(\mathbb{T}^2)$, the difference of order $\alpha_1 > 0$ with respect to the variable x and the difference of order $\alpha_2 > 0$ with respect to the variable y are defined as follows:

$$\Delta_{h_1}^{\alpha_1}(f) = \sum_{\nu_1=0}^{\infty} (-1)^{\nu_1} \binom{\alpha_1}{\nu_1} f(x + (\alpha_1 - \nu_1)h_1, y)$$

and, respectively,

$$\Delta_{h_2}^{\alpha_2}(f) = \sum_{\nu_2=0}^{\infty} (-1)^{\nu_2} \binom{\alpha_2}{\nu_2} f(x, y + (\alpha_2 - \nu_2)h_2),$$

where $\binom{\alpha}{\nu} = 1$ for $\nu = 0$, $\binom{\alpha}{\nu} = \alpha$ for $\nu = 1$, $\binom{\alpha}{\nu} = \frac{\alpha(\alpha-1)\dots(\alpha-\nu+1)}{\nu!}$ for $\nu \geq 2$.

Denote by $\omega_{\alpha_1,\alpha_2}(f, \delta_1, \delta_2)_{L_p(\mathbb{T}^2)}$ the mixed modulus of smoothness of a function $f \in L_p(\mathbb{T}^2)$ of orders $\alpha_1 > 0$ and $\alpha_2 > 0$ with respect to the variables x and y, respectively, i.e.,

$$\omega_{\alpha_1,\alpha_2}(f, \delta_1, \delta_2)_{L_p(\mathbb{T}^2)} = \sup_{|h_i| \leq \delta_i, i=1,2} \|\Delta_{h_1}^{\alpha_1}(\Delta_{h_2}^{\alpha_2}(f))\|_{L_p(\mathbb{T}^2)}.$$

We remark that $\|\Delta_{h_1}^{\alpha_1}(\Delta_{h_2}^{\alpha_2}(f))\|_{L_p(\mathbb{T}^2)} \leq C(\alpha_1, \alpha_2)\|f\|_{L_p(\mathbb{T}^2)}$, where $C(\alpha_1, \alpha_2) \leq 2^{\lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 2}$.

2.3 K-functional

First, let us recall the definition of the fractional integral and fractional derivative in the sense of Weyl of a function f defined on $\mathbb{T}$. If the Fourier series of a function $f \in L^1(\mathbb{T})$ is given by

$$\sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{\mathrm{i}nx}, \qquad c_0 = 0,$$

then the fractional integral or order $\rho > 0$ of f is defined by (see, e.g., [107, Ch. XII])

$$I^\alpha f(x) := \big(f * \psi_\rho\big)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\psi_\rho(x-t)\, \mathrm{d}t,$$

where

$$\psi_\alpha(x)=\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{\mathrm{e}^{inx}}{(\mathrm{i}n)^\rho}.$$

To define the fractional derivative or order $\rho>0$ of f, we put $n:=\lfloor\rho\rfloor+1$ and

$$f^{(\rho)}(x):=\frac{\mathrm{d}^n}{\mathrm{d}x^n}I^{n-\rho}f(x).$$

By $f^{(\rho_1,\rho_2)}$ we will denote the Weyl derivative of order $\rho_1\geq 0$ with respect to x and of order $\rho_2\geq 0$ with respect to y of the function $f\in L_1^0(\mathbb{T}^2)$.

Denote by $W_p^{(\alpha_1,0)}$ the Weyl class, i.e., the set of functions $f\in L_p^0(\mathbb{T}^2)$ such that $f^{(\alpha_1,0)}\in L_p^0(\mathbb{T}^2)$. Similarly, $W_p^{(0,\alpha_2)}$ is the set of functions $f\in L_p^0(\mathbb{T}^2)$ such that $f^{(0,\alpha_2)}\in L_p^0(\mathbb{T}^2)$. Moreover, $W_p^{(\alpha_1,\alpha_2)}$ is the set of functions $f\in L_p^0(\mathbb{T}^2)$ such that $f^{(\alpha_1,\alpha_2)}\in L_p^0(\mathbb{T}^2)$.

The mixed K-functional of a function $f\in L_p^0(\mathbb{T}^2)$ is given by

$$\begin{aligned}K(f,t_1,t_2,\alpha_1,\alpha_2,p) &= \inf_{g_1\in W_p^{(\alpha_1,0)},g_2\in W_p^{(0,\alpha_2)},g\in W_p^{(\alpha_1,\alpha_2)}}\Big[\|f-g_1-g_2-g\|_{L_p(\mathbb{T}^2)}\\ &+ t_1^{\alpha_1}\|g_1^{(\alpha_1,0)}\|_{L_p(\mathbb{T}^2)}+t_2^{\alpha_2}\|g_2^{(0,\alpha_2)}\|_{L_p(\mathbb{T}^2)}+t_1^{\alpha_1}t_2^{\alpha_2}\|g^{(\alpha_1,\alpha_2)}\|_{L_p(\mathbb{T}^2)}\Big].\end{aligned}$$

2.4 Special classes of functions

We define the function class $M_p, 1<p<\infty$, as the set of functions $f\in L_p^0(\mathbb{T}^2)$ such that the Fourier series of f is given by $\sum\limits_{\nu_1=1}^\infty\sum\limits_{\nu_2=1}^\infty a_{\nu_1,\nu_2}\cos\nu_1x\cos\nu_2y$, where

$$a_{\nu_1,\nu_2}-a_{\nu_1+1,\nu_2}-a_{\nu_1,\nu_2+1}+a_{\nu_1+1,\nu_2+1}\geq 0 \tag{2.1}$$

for any integers ν_1 and ν_2. Note that (2.1) implies

$$a_{n,m_1}\geq a_{n,m_2}\qquad m_1\leq m_2\quad\text{and}\quad a_{n_1,m}\geq a_{n_2,m}\qquad n_1\leq n_2. \tag{2.2}$$

We also define the function class $\Lambda_p, 1<p<\infty$, as the set of functions $f\in L_p^0(\mathbb{T}^2)$ such that the Fourier series of f is given by $\sum\limits_{\mu_1=0}^\infty\sum\limits_{\mu_2=0}^\infty\lambda_{\mu_1,\mu_2}\cos 2^{\mu_1}x\cos 2^{\mu_2}y$, where $\lambda_{\mu_1,\mu_2}\in\mathbb{R}$.

3 Auxiliary results

3.1 Jensen and Hardy inequalities

Lemma 3.1 *[51, Ch. 1] Let $a_k\geq 0, 0<\alpha\leq\beta<\infty$. Then*

$$\left(\sum_{k=1}^\infty a_k^\beta\right)^{1/\beta}\leq\left(\sum_{k=1}^\infty a_k^\alpha\right)^{1/\alpha}.$$

Lemma 3.2 *[41] Let $a_k \ge 0, b_k \ge 0$.*

(A). Suppose $\sum\limits_{k=1}^{n} a_k = a_n\gamma_n$. If $1 \le p < \infty$, then

$$\sum_{k=1}^{\infty} a_k \Big(\sum_{n=k}^{\infty} b_n\Big)^p \lesssim \sum_{k=1}^{\infty} a_k (b_k\gamma_k)^p.$$

If $0 < p \le 1$, then

$$\sum_{k=1}^{\infty} a_k \Big(\sum_{n=k}^{\infty} b_n\Big)^p \gtrsim \sum_{k=1}^{\infty} a_k (b_k\gamma_k)^p.$$

(B). Suppose $\sum\limits_{k=n}^{\infty} a_k = a_n\beta_n$. If $1 \le p < \infty$, then

$$\sum_{k=1}^{\infty} a_k \Big(\sum_{n=1}^{k} b_n\Big)^p \lesssim \sum_{k=1}^{\infty} a_k (b_k\beta_k)^p.$$

If $0 < p \le 1$, then

$$\sum_{k=1}^{\infty} a_k \Big(\sum_{n=1}^{k} b_n\Big)^p \gtrsim \sum_{k=1}^{\infty} a_k (b_k\beta_k)^p.$$

3.2 Results on angular approximation

Lemma 3.3 *[53] Let $f \in L_p^0(\mathbb{T}^2), 1 < p < \infty, n_i = 0, 1, 2, \dots, i = 1, 2$. Then*

$$\|f - s_{n_1,\infty}(f) - s_{\infty,n_2}(f) + s_{n_1,n_2}(f)\|_{L_p(\mathbb{T}^2)} \lesssim Y_{n_1,n_2}(f)_{L_p(\mathbb{T}^2)}.$$

Lemma 3.4 *[54] Let $f \in L_p^0(\mathbb{T}^2), 1 < p < q < \infty, \theta = \frac{1}{p} - \frac{1}{q}, N_i = 0, 1, 2, \dots, i = 1, 2$. Then*

$$Y_{2^{N_1}-1,2^{N_2}-1}(f)_{L_q(\mathbb{T}^2)} \lesssim \left\{ \sum_{\nu_1=N_1}^{\infty} \sum_{\nu_2=N_2}^{\infty} 2^{(\nu_1+\nu_2)\theta q} Y^q_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_{L_p(\mathbb{T}^2)} \right\}^{\frac{1}{q}}.$$

Note that similar results for functions on $\mathbb{R}^d$ can be found in [90].

3.3 Fourier coefficients of $L_p(\mathbb{T}^2)$-functions, Multipliers, and Littlewood-Paley theorem

Lemma 3.5 *(The Marcinkiewicz multiplier theorem, [51, Ch. 1]) Let the Fourier series of a function $f \in L_p^0(\mathbb{T}^2), 1 < p < \infty$, be*

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \big(a_{n_1,n_2} \cos n_1x \cos n_2y \quad + \quad b_{n_1,n_2} \sin n_1x \cos n_2y$$
$$+ \quad c_{n_1,n_2} \cos n_1x \sin n_2y \quad + \quad d_{n_1,n_2} \sin n_1x \sin n_2y\big) =: \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} A_{n_1,n_2}(x,y). \quad (3.1)$$

Let the number sequence $(\vartheta_{n_1,n_2})_{n_1,n_2=1}^{\infty}$ *satisfy*

$$|\vartheta_{n_1,n_2}|\le M,\quad \sum_{m_1=2^{n_1-1}+1}^{2^{n_1}}|\vartheta_{m_1,n_2}-\vartheta_{m_1+1,n_2}|\le M,\quad \sum_{m_2=2^{n_2-1}+1}^{2^{n_2}}|\vartheta_{n_1,m_2}-\vartheta_{n_1,m_2+1}|\le M$$

and

$$\sum_{m_1=2^{n_1-1}+1}^{2^{n_1}}\sum_{m_2=2^{n_2-1}+1}^{2^{n_2}}|\vartheta_{m_1,m_2}-\vartheta_{m_1+1,m_2}-\vartheta_{m_1,m_2+1}+\vartheta_{m_1+1,m_2+1}|\le M$$

for some finite M and any $n_i\in\mathbb{N}, i=1,2$. Then the trigonometric series $\sum\limits_{n_1=1}^{\infty}\sum\limits_{n_2=1}^{\infty}\vartheta_{n_1,n_2}A_{n_1,n_2}(x,y)$ *is the Fourier series of a function $\phi\in L_p^0(\mathbb{T}^2)$ and*

$$\|\phi\|_{L_p(\mathbb{T}^2)}\lesssim\|f\|_{L_p(\mathbb{T}^2)}.$$

Lemma 3.6 *(The Littlewood-Paley theorem, [51, Ch. 1]) Let the Fourier series of a function $f\in L_p^0(\mathbb{T}^2), 1<p<\infty$, be given by (3.1). Let $\Delta_{0,0}:=A_{1,1}(x,y)$,*

$$\Delta_{m_1,0}:=\sum_{\nu_1=2^{m_1-1}+1}^{2^{m_1}}A_{\nu_1,1}(x,y)\quad \textit{for}\ \ m_1\in\mathbb{N},\quad \Delta_{0,m_2}:=\sum_{\nu_1=2^{m_2-1}+1}^{2^{m_2}}A_{1,\nu_2}(x,y)\quad \textit{for}\ \ m_2\in\mathbb{N},$$

and

$$\Delta_{m_1,m_2}:=\sum_{\nu_1=2^{m_1-1}+1}^{2^{m_1}}\sum_{\nu_1=2^{m_2-1}+1}^{2^{m_2}}A_{\nu_1,\nu_2}(x,y)\quad \textit{for}\ \ m_1\in\mathbb{N}\quad \textit{and}\ \ m_2\in\mathbb{N}.$$

Then

$$\|f\|_{L_p(\mathbb{T}^2)}\asymp\left(\int_0^{2\pi}\int_0^{2\pi}\left(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}\Delta^2_{\nu_1,\nu_2}\right)^{p/2}\mathrm{d}x\,\mathrm{d}y\right)^{1/p}.$$

Lemma 3.7 *(The Hardy-Littlewood-Paley theorem, [29]) Let the Fourier series of a function $f\in L_1^0(\mathbb{T}^2)$ be given by (3.1).*
(A). Let $2\le p<\infty$ and

$$I:=\left(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}(|a_{n_1,n_2}|+|b_{n_1,n_2}|+|c_{n_1,n_2}|+|d_{n_1,n_2}|)^p(n_1n_2)^{p-2}\right)^{1/p}<\infty.$$

Then $f\in L_p^0(\mathbb{T}^2)$ and $\|f\|_{L_p(\mathbb{T}^2)}\lesssim I$.
(B). Let $f\in L_p^0(\mathbb{T}^2), 1<p\le 2$. Then $I\lesssim\|f\|_{L_p(\mathbb{T}^2)}$.

Lemma 3.8 *Let $f\in M_p, 1<p<\infty$, $r_i\ge 0, i=1,2$. Then*

$$\|f\|_{L_p(\mathbb{T}^2)}\asymp\left(\sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty}a^p_{\nu_1,\nu_2}(\nu_1\nu_2)^{p-2}\right)^{1/p}\tag{3.2}$$

and

$$\|f^{(r_1,r_2)}\|_{L_p(\mathbb{T}^2)}\asymp\left(\sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty}a^p_{\nu_1,\nu_2}\nu_1^{r_1p+p-2}\nu_2^{r_2p+p-2}\right)^{1/p}.\tag{3.3}$$

Proof. The proof of (3.2) is given in [46] (see also [30]).

Let us verify (3.3). If $2 \le p < \infty$, then the estimate from above in (3.3) follows from Lemma 3.7 (A). If $1 < p < 2$, then Lemmas 3.5 and 3.6 imply

$$I^p = \|f^{(r_1,r_2)}\|^p_{L_p(\mathbb{T}^2)} \asymp \int_0^{2\pi}\int_0^{2\pi} \Big(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} 2^{2(\nu_1 r_1+\nu_2 r_2)} \Delta^2_{\nu_1,\nu_2} \Big)^{p/2} \,\mathrm{d}x\,\mathrm{d}y.$$

Since $\frac{p}{2} < 1$, using Lemma 3.1, we get

$$I^p \lesssim \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} 2^{p(\nu_1 r_1+\nu_2 r_2)} \|\Delta_{\nu_1,\nu_2}\|_p^p.$$

In the paper [30] it was shown that $\|\Delta_{\nu_1,\nu_2}\|_p^p \lesssim 2^{(\nu_1+\nu_2)p-1} a^p_{\lfloor 2^{\nu_1-1}\rfloor+1,\lfloor 2^{\nu_2-1}\rfloor+1}$. Hence

$$I^p \lesssim \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} 2^{p(\nu_1 r_1+\nu_2 r_2)+(\nu_1+\nu_2)p-1} a^p_{\lfloor 2^{\nu_1-1}\rfloor+1,\lfloor 2^{\nu_2-1}\rfloor+1}$$

and by (2.2)

$$I^p \lesssim \sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty} a^p_{\nu_1,\nu_2} \nu_1^{(r_1+1)p-2} \nu_2^{(r_2+1)p-2}.$$

Thus, we have proved the part "$\lesssim$" in (3.3).

To show the estimate from below, if $1 < p \le 2$ then we simply use Lemma 3.7 (B). If $2 < p < \infty$, we use the inequality

$$\|f^{(r_1,r_2)}\|_p^p \gtrsim \sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty} (\nu_1\nu_2)^{-2} \Big(\sum_{\mu_1=\nu_1}^{\infty}\sum_{\mu_2=\nu_2}^{\infty} a_{\mu_1,\mu_2} \mu_1^{r_1} \mu_2^{r_2} \Big)^p$$

from the paper [52]. Therefore, by (2.2),

$$\|f^{(r_1,r_2)}\|_p^p \gtrsim \sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty} a^p_{\nu_1,\nu_2} \nu_1^{(r_1+1)p-2} \nu_2^{(r_2+1)p-2}.$$

$\square$

Lemma 3.9 *Let $f \in \Lambda_p, 1 < p < \infty$. Then*

$$\|f\|_{L_p(\mathbb{T}^2)} \asymp \left(\sum_{\mu_1=0}^{\infty}\sum_{\mu_2=0}^{\infty} \lambda^2_{\mu_1,\mu_2} \right)^{1/2}. \tag{3.4}$$

This lemma is well known in one dimension ([107, Ch. V, §8]) but we failed to find its multivariate version. For the sake of completeness we give a simple proof of this result.

Proof. Lemma 3.6 yields

$$I := \|f\|_{L_p(\mathbb{T}^2)} \asymp \left(\int_0^{2\pi}\int_0^{2\pi} \Big(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} \Delta^2_{\nu_1,\nu_2} \Big)^{p/2} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}.$$

Since $\Delta_{\nu_1,\nu_2} = \lambda_{\nu_1,\nu_2} \cos 2^{\nu_1}x \cos 2^{\nu_2}y$ for $f \in \Lambda_p$, we get

$$\begin{aligned} I &\asymp \left(\int_0^{2\pi}\int_0^{2\pi} \left(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} \lambda_{\nu_1,\nu_2}^2 \big(\cos 2^{\nu_1}x \cos 2^{\nu_2}y\big)^2 \right)^{p/2} \mathrm{d}x\,\mathrm{d}y \right)^{1/p} \qquad (3.5)\\ &\lesssim \left(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} \lambda_{\nu_1,\nu_2}^2 \right)^{1/2}. \end{aligned}$$

Let us now verify the estimate from below. If $1<p<2$, using Minkowski's inequality in (3.5), we have

$$I \gtrsim \left(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} \lambda_{\nu_1,\nu_2}^2 \left(\int_0^{2\pi}\int_0^{2\pi} \big|\cos 2^{\nu_1}x \cos 2^{\nu_2}y\big|^p \,\mathrm{d}x\,\mathrm{d}y \right)^{2/p} \right)^{1/2} \gtrsim \Big(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} \lambda_{\nu_1,\nu_2}^2 \Big)^{1/2}.$$

If $2 \le p < \infty$, then $I = \|f\|_{L_p(\mathbb{T}^2)} \gtrsim \|f\|_{L_2(\mathbb{T}^2)} \gtrsim \Big(\sum\limits_{\nu_1=0}^{\infty}\sum\limits_{\nu_2=0}^{\infty} \lambda_{\nu_1,\nu_2}^2 \Big)^{1/2}$. □

3.4 Auxiliary results for functions on $\mathbb{T}$

Below we collect several useful results for functions of one variable. As usual, $L_p(\mathbb{T})$ is the collection of 2π-periodic measurable functions f such that $\|f\|_{L_p(\mathbb{T})} = \left(\int\limits_0^{2\pi} |f(x)|^p \,\mathrm{d}x \right)^{1/p} < \infty$ and $L_p^0(\mathbb{T})$ is the collection of $f \in L_p(\mathbb{T})$ such that $\int\limits_0^{2\pi} f(x)\,\mathrm{d}x = 0$.

Let $s_n(f)$ be the n-th partial sum of the Fourier series $f \in L_p(\mathbb{T})$, i.e.,

$$s_n(f) = s_n(f,x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t) \frac{\sin{(n+\frac{1}{2})t}}{2\sin\frac{t}{2}} \,\mathrm{d}t.$$

Let also $f^{(\rho)}$ be the Weyl derivative of order $\rho > 0$ of the function f.

For $f \in L_p$ we define the difference of positive order α as follows

$$\Delta_h^\alpha(f) = \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\alpha}{\nu} f(x+(\alpha-\nu)h).$$

We let $\omega_\alpha(f,\delta)_{L_p(\mathbb{T})}$ denote the modulus of smoothness of f of positive order α ([11, 78, 92]), i.e.,

$$\omega_\alpha(f,\delta)_{L_p(\mathbb{T})} := \sup_{|h|\le\delta} \|\Delta_h^\alpha(f)\|_{L_p(\mathbb{T})}.$$

Lemma 3.10 *[11, 78] Let $f,g \in L_p^0(\mathbb{T})$, $1<p<\infty$, and $\alpha>0, \beta>0$. Then*

(a) $\triangle_h^\alpha(f+g) = \triangle_h^\alpha f + \triangle_h^\alpha g$;

(b) $\triangle_h^\alpha(\triangle_h^\beta f) = \triangle_h^{\alpha+\beta} f$;

(c) $\|\triangle_h^\alpha f\|_{L_p(\mathbb{T})} \lesssim \|f\|_{L_p(\mathbb{T})}$.

Lemma 3.11 *[11, 78] Let $1 < p < \infty$, $\alpha > 0$, and T_n be a trigonometric polynomial of degree at most n, $n \in \mathbb{N}$. Then*

(a) *we have for any $0 < |h| \leq \frac{\pi}{n}$*

$$\|\triangle_h^\alpha T_n\|_{L_p(\mathbb{T})} \lesssim n^{-\alpha} \|T_n^{(\alpha)}\|_{L_p(\mathbb{T})};$$

(b) *we have*

$$\|T_n^{(\alpha)}\|_{L_p(\mathbb{T})} \lesssim n^\alpha \|\triangle_{\frac{\pi}{n}}^\alpha T_n\|_{L_p(\mathbb{T})}.$$

Lemma 3.12 *[51] Let $f \in L_p^0(\mathbb{T})$, $1 < p < \infty$. Then*

$$\|s_n(f)\|_{L_p(\mathbb{T})} \lesssim \|f\|_{L_p(\mathbb{T})}, \qquad n \in \mathbb{N}.$$

Lemma 3.13 *[97] Let $f \in L_p^0(\mathbb{T})$, $1 < p < q < \infty$, $\theta := \frac{1}{p} - \frac{1}{q}$, $n = 0, 1, 2, \ldots$ Then*

$$\|f - s_{2^n}(f)\|_{L_q(\mathbb{T})} \lesssim \left\{ \sum_{\nu=n}^{\infty} 2^{\theta\nu q} \|f - s_{2^\nu}(f)\|_{L_p(\mathbb{T})}^q \right\}^{1/q}.$$

Lemma 3.14 *(The Hardy-Littlewood inequality for fractional integrals, [107]) Let $f \in L_p^0(\mathbb{T})$, $1 < p < q < \infty$, $\theta := \frac{1}{p} - \frac{1}{q}$, $\alpha > 0$. Then*

$$\|s_n^{(\alpha)}(f)\|_{L_q(\mathbb{T})} \lesssim \|s_n^{(\alpha+\theta)}(f)\|_{L_p(\mathbb{T})}, \qquad n \in \mathbb{N}.$$

4 Basic properties of the mixed moduli of smoothness

We collect the main properties of the mixed moduli of smoothness of $L_p(\mathbb{T}^2)$-functions, $1 < p < \infty$, in the following result.

Theorem 4.1 *Let $f, g \in L_p(\mathbb{T}^2)$, $1 < p < \infty$, $\alpha_i > 0$, $i = 1, 2$. Then*

(1) $$\omega_{\alpha_1,\alpha_2}(f, \delta_1, 0)_{L_p(\mathbb{T}^2)} = \omega_{\alpha_1,\alpha_2}(f, 0, \delta_2)_{L_p(\mathbb{T}^2)} = \omega_{\alpha_1,\alpha_2}(f, 0, 0)_{L_p(\mathbb{T}^2)} = 0;$$

(2) $$\omega_{\alpha_1,\alpha_2}(f + g, \delta_1, \delta_2)_{L_p(\mathbb{T}^2)} \lesssim \omega_{\alpha_1,\alpha_2}(f, \delta_1, \delta_2)_{L_p(\mathbb{T}^2)} + \omega_{\alpha_1,\alpha_2}(g, \delta_1, \delta_2)_{L_p(\mathbb{T}^2)};$$

(3) $$\omega_{\alpha_1,\alpha_2}(f, \delta_1, \delta_2)_{L_p(\mathbb{T}^2)} \lesssim \omega_{\alpha_1,\alpha_2}(f, t_1, t_2)_{L_p(\mathbb{T}^2)}$$

for $0 < \delta_i \leq t_i, i = 1, 2$;

$$\frac{\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}}{\delta_1^{\alpha_1}\delta_2^{\alpha_2}} \lesssim \frac{\omega_{\alpha_1,\alpha_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}}{t_1^{\alpha_1}t_2^{\alpha_2}} \tag{4}$$

for $0 < t_i \le \delta_i \le 1, i = 1,2$;

$$\omega_{\alpha_1,\alpha_2}(f,\lambda_1\delta_1,\lambda_2\delta_2)_{L_p(\mathbb{T}^2)} \lesssim \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \tag{5}$$

for $\lambda_i > 1,\ i = 1,2$;

$$\omega_{\beta_1,\beta_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \lesssim \omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \tag{6}$$

for $0 < \alpha_i < \beta_i,\ i = 1,2$;

$$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \lesssim \delta_1^{\alpha_1}\delta_2^{\alpha_2}\int_{\delta_1}^1\int_{\delta_2}^1 \frac{\omega_{\beta_1,\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}}{t_1^{\alpha_1}t_2^{\alpha_2}}\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2} \tag{7}$$

for $0 < \alpha_i < \beta_i,\ 0 < \delta_i \le \frac{1}{2},\ \ i = 1,2$ *(Marchaud's inequality);*

$$\frac{\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}}{\delta_1^{\alpha_1}\delta_2^{\alpha_2}} \lesssim \frac{\omega_{\beta_1,\beta_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}}{\delta_1^{\beta_1}\delta_2^{\beta_2}} \tag{8}$$

for $0 < \alpha_i < \beta_i,\ i = 1,2$;

$$\omega_{\beta_1+r_1,\beta_2+r_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \lesssim \delta_1^{r_1}\delta_2^{r_2}\omega_{\beta_1,\beta_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \tag{9}$$

for $\beta_i, r_i > 0, i = 1,2$;

$$\omega_{\beta_1,\beta_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \lesssim \int\limits_0^{\delta_1}\int\limits_0^{\delta_2} t_1^{-r_1}t_2^{-r_2}\omega_{\beta_1+r_1,\beta_2+r_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2} \tag{10}$$

for $\beta_i, r_i > 0,\ \ i = 1,2.$

Remark 4.1 *Note that sharp versions of inequalities given in (6)–(10) can be found in Sections 8 and 10 below.*

Proof of Theorem 4.1. Properties (1), (2), and (3) follow from Lemma 3.10 (a),

$$\sum_{\nu=0}^{\infty}(-1)^{\nu}\binom{\alpha}{\nu}=0,$$

and the definition of modulus of smoothness.

To prove (4), we write

$$\begin{aligned}\frac{\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}}{\delta_1^{\alpha_1}\delta_2^{\alpha_2}} &\asymp \frac{K(f,\delta_1,\delta_2,\alpha_1,\alpha_2,p)}{\delta_1^{\alpha_1}\delta_2^{\alpha_2}}\\ &\lesssim \frac{K(f,t_1,t_2,\alpha_1,\alpha_2,p)}{t_1^{\alpha_1}t_2^{\alpha_2}} \asymp \frac{\omega_{\alpha_1,\alpha_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}}{t_1^{\alpha_1}t_2^{\alpha_2}},\end{aligned}$$

where the equivalence between the modulus of smoothness and the K-functional is given by Theorem 6.1 below.

Property (4) yields (5). Note that Lemma 3.10 (b), (c) implies $\|\triangle_h^{\beta} f\|_{L_p(\mathbb{T})} \lesssim \|\triangle_h^{\alpha} f\|_{L_p(\mathbb{T})}$ for $0<\alpha<\beta$. Then property (6) follows.

The Marchaud inequality (7) can be easily shown using the direct and inverse approximation inequalities given below in this section by Theorem 4.2. Property (8) is a simple consequence of (4) and (7). Finally, properties (9) and (10) follow directly from Theorem 8.1 below. □

Theorem 4.1 was partially proved in the papers [73, 31]. For the one-dimensional case see [11, 75, 78].

4.1 Jackson and Bernstein-Stechkin type inequalities

The direct and inverse type results for periodic functions on $\mathbb{T}^2$ using the mixed modulus of smoothness are given by the following result.

Theorem 4.2 *Let $f\in L_p^0(\mathbb{T}^2)$, $1<p<\infty$, $n_1,n_2=0,1,2,\dots$, $\alpha_1,\alpha_2>0$. Then*

$$\begin{aligned}Y_{n_1,n_2}(f)_{L_p(\mathbb{T}^2)} &\lesssim \omega_{\alpha_1,\alpha_2}\Big(f,\frac{\pi}{n_1+1},\frac{\pi}{n_2+1}\Big)_{L_p(\mathbb{T}^2)} \qquad (4.1)\\ &\lesssim \frac{1}{(n_1+1)^{\alpha_1}(n_2+1)^{\alpha_2}}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1-1}\nu_2^{\alpha_2-1}Y_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathbb{T}^2)}.\end{aligned}$$

Theorem 4.2 was proved for integers α_1,α_2 in the paper [53]. In the general case, Theorem 4.2 follows from Theorem 9.1 which is sharp versions of Jackson and Bernstein-Stechkin inequalities. For non-mixed moduli of smoothness, see [51, Ch. 5] and [86, Chs. V-VI].

Remark 4.2 *Note that the results of Theorem 4.1 and Theorem 4.2 also hold in $L_p(\mathbb{T}^2)$, $p=1,\infty$; see [62] for Theorem 4.1 (1)-(8) and Theorem 4.2. Moreover, the Jackson inequality (4.1) is true in $L_p(\mathbb{T}^2)$, $0<p<1$; see [66].*

5 Constructive characteristic of the mixed moduli of smoothness

Theorem 5.1 *Let $f \in L_p^0(\mathbb{T}^2)$, $1<p<\infty$, $\alpha_i>0$, $n_i \in \mathbb{N}$, $i=1,2$. Then*

$$\begin{aligned}\omega_{\alpha_1,\alpha_2}\Big(f,\frac{\pi}{n_1},\frac{\pi}{n_2}\Big)_{L_p(\mathbb{T}^2)} &\asymp n_1^{-\alpha_1}n_2^{-\alpha_2}||s_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f)||_{L_p(\mathbb{T}^2)} + n_1^{-\alpha_1}||s_{n_1,\infty}^{(\alpha_1,0)}(f-s_{\infty,n_2}(f))||_{L_p(\mathbb{T}^2)}\\ &+ n_2^{-\alpha_2}||s_{\infty,n_2}^{(0,\alpha_2)}(f-s_{n_1,\infty}(f))||_{L_p(\mathbb{T}^2)}\\ &+ ||f-s_{n_1,\infty}(f)-s_{\infty,n_2}(f)+s_{n_1,n_2}(f)||_{L_p(\mathbb{T}^2)}.\end{aligned}$$

Proof. Using properties of the norm we get, for any h_i and $n_i \in \mathbb{N}, i=1,2$,

$$\begin{aligned}\|\triangle_{h_1}^{\alpha_1}(\triangle_{h_2}^{\alpha_2}(f))\|_{L_p(\mathbb{T}^2)} &\le \|\triangle_{h_1}^{\alpha_1}(\triangle_{h_2}^{\alpha_2}(f-s_{n_1,\infty}(f)-s_{\infty,n_2}(f)+s_{n_1,n_2}(f)))\|_{L_p(\mathbb{T}^2)}\\ &+ \|\triangle_{h_1}^{\alpha_1}(\triangle_{h_2}^{\alpha_2}(s_{n_1,\infty}(f-s_{\infty,n_2})))\|_{L_p(\mathbb{T}^2)}\\ &+ \|\triangle_{h_1}^{\alpha_1}(\triangle_{h_2}^{\alpha_2}(s_{\infty,n_2}(f-s_{n_1,\infty})))\|_{L_p(\mathbb{T}^2)}\\ &+ \|\triangle_{h_1}^{\alpha_1}(\triangle_{h_2}^{\alpha_2}(s_{n_1,n_2}(f)))\|_{L_p(\mathbb{T}^2)} =: I_1+I_2+I_3+I_4.\end{aligned}$$

First we estimate I_1 from above. Denote $\varphi(x,y) := f-s_{n_1,\infty}(f)-s_{\infty,n_2}(f)+s_{n_1,n_2}(f)$. By Lemma 3.10 (c), we have for a.e. y

$$\Bigg(\int_0^{2\pi}|\triangle_{h_1}^{\alpha_1}(\triangle_{h_2}^{\alpha_2}(\varphi))|^p\,\mathrm{d}x\Bigg)^{1/p} \lesssim \Bigg(\int_0^{2\pi}|\triangle_{h_2}^{\alpha_2}(\varphi)|^p\,\mathrm{d}x\Bigg)^{1/p}.$$

Then

$$\int_0^{2\pi}\int_0^{2\pi}|\triangle_{h_1}^{\alpha_1}(\triangle_{h_2}^{\alpha_2}(\varphi))|^p\,\mathrm{d}x\,\mathrm{d}y \lesssim \int_0^{2\pi}\int_0^{2\pi}|\triangle_{h_2}^{\alpha_2}(\varphi)|^p\,\mathrm{d}x\,\mathrm{d}y.$$

Therefore, $I_1 \lesssim \|\triangle_{h_2}^{\alpha_2}(\varphi)\|_{L_p(\mathbb{T}^2)} =: I_5$. Using Lemma 3.10 (c), we have for a.e. x

$$\Bigg(\int_0^{2\pi}|\triangle_{h_2}^{\alpha_2}(\varphi)|^p\,\mathrm{d}y\Bigg)^{1/p} \lesssim \Bigg(\int_0^{2\pi}|\varphi|^p\,\mathrm{d}y\Bigg)^{1/p}.$$

Then

$$\int_0^{2\pi}\int_0^{2\pi}|\triangle_{h_2}^{\alpha_2}(\varphi)|^p\,\mathrm{d}y\,\mathrm{d}x \lesssim \int_0^{2\pi}\int_0^{2\pi}|\varphi|^p\,\mathrm{d}y\,\mathrm{d}x$$

and $I_5 \lesssim \|\varphi\|_{L_p(\mathbb{T}^2)}$. Thus, $I_1 \lesssim \|f-s_{n_1,\infty}(f)-s_{\infty,n_2}(f)+s_{n_1,n_2}(f)\|_{L_p(\mathbb{T}^2)}$.

Similarly, to estimate I_2 from above, we denote $\psi := f-s_{\infty,n_2}(f)$. By Lemma 3.10 (c), for a.e. x,

$$\Bigg(\int_0^{2\pi}|\triangle_{h_1}^{\alpha_1}(\triangle_{h_2}^{\alpha_2}(s_{n_1,\infty}(\psi)))|^p\,\mathrm{d}y\Bigg)^{1/p} \lesssim \Bigg(\int_0^{2\pi}|\triangle_{h_1}^{\alpha_1}(s_{n_1,\infty}(\psi))|^p\,\mathrm{d}y\Bigg)^{1/p}.$$

Hence,

$$\int_0^{2\pi}\int_0^{2\pi}|\triangle_{h_1}^{\alpha_1}(\triangle_{h_2}^{\alpha_2}(s_{n_1,\infty}(\psi)))|^p\,\mathrm{d}y\,\mathrm{d}x \lesssim \int_0^{2\pi}\int_0^{2\pi}|\triangle_{h_1}^{\alpha_1}(s_{n_1,\infty}(\psi))|^p\,\mathrm{d}y\,\mathrm{d}x$$

and $I_2 \lesssim \|\triangle_{h_1}^{\alpha_1}(s_{n_1,\infty}(\psi))\|_{L_p(\mathbb{T}^2)} =: I_6$. Now by using Lemma 3.11 (a), we get for a.e. y and any $h_1 \in (0, \frac{\pi}{n_1})$

$$\left(\int_0^{2\pi} |\triangle_{h_1}^{\alpha_1}(s_{n_1,\infty}(\psi))|^p \,\mathrm{d}x\right)^{1/p} \lesssim n_1^{-\alpha_1} \left(\int_0^{2\pi} |s_{n_1,\infty}^{(\alpha_1,0)}(\psi)|^p \,\mathrm{d}x\right)^{1/p}.$$

Then we get

$$\int_0^{2\pi}\int_0^{2\pi} |\triangle_{h_1}^{\alpha_1}(s_{n_1,\infty}(\psi))|^p \,\mathrm{d}x\,\mathrm{d}y \lesssim n_1^{-\alpha_1} \int_0^{2\pi}\int_0^{2\pi} |s_{n_1,\infty}^{(\alpha_1,0)}(\psi)|^p \,\mathrm{d}x\,\mathrm{d}y.$$

Hence, $I_6 \lesssim n_1^{-\alpha_1}\|s_{n_1,\infty}^{(\alpha_1,0)}(\psi)\|_{L_p(\mathbb{T}^2)}$. We have shown that, for $0 < |h_1| < \frac{\pi}{n_1}$,

$$I_2 \lesssim n_1^{-\alpha_1}\|s_{n_1,\infty}^{(\alpha_1,0)}(f - s_{\infty,n_2}(f))\|_{L_p(\mathbb{T}^2)}.$$

Similarly, we obtain

$$\begin{aligned} I_3 &\lesssim n_2^{-\alpha_2}\|s_{\infty,n_2}^{(0,\alpha_2)}(f - s_{n_1,\infty}(f))\|_{L_p(\mathbb{T}^2)}, \\ I_4 &\lesssim n_1^{-\alpha_1}n_2^{-\alpha_2}\|s_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f)\|_{L_p(\mathbb{T}^2)}, \end{aligned}$$

for $0 < |h_1| < \frac{\pi}{n_1}$, $0 < |h_2| < \frac{\pi}{n_2}$.

Finally,

$$\begin{aligned} \omega_{\alpha_1,\alpha_2}(f, \frac{\pi}{n_1}, \frac{\pi}{n_2})_{L_p(\mathbb{T}^2)} &\lesssim ||f - s_{n_1,\infty}(f) - s_{\infty,n_2}(f) + s_{n_1,n_2}(f)||_{L_p(\mathbb{T}^2)} \\ &+ n_1^{-\alpha_1}||s_{n_1,\infty}^{(\alpha_1,0)}(f - s_{\infty,n_2}(f))||_{L_p(\mathbb{T}^2)} + n_2^{-\alpha_2}||s_{\infty,n_2}^{(0,\alpha_2)}(f - s_{n_1,\infty}(f))||_{L_p(\mathbb{T}^2)} \\ &+ n_1^{-\alpha_1}n_2^{-\alpha_2}||s_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f)||_{L_p(\mathbb{T}^2)}, \end{aligned}$$

and the upper estimate follows.

To prove the estimate from below, we use Lemma 3.3, Theorem 4.2, and properties of the mixed modulus of integer order

$$\begin{aligned} A_1 &:= ||f - s_{n_1,\infty}(f) - s_{\infty,n_2}(f) + s_{n_1,n_2}(f)||_{L_p(\mathbb{T}^2)} \lesssim Y_{n_1,n_2}(f)_{L_p(\mathbb{T}^2)} \\ &\lesssim \omega_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}(f, \frac{\pi}{n_1+1}, \frac{\pi}{n_2+1})_{L_p(\mathbb{T}^2)} \lesssim \omega_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}(f, \frac{\pi}{n_1}, \frac{\pi}{n_2})_{L_p(\mathbb{T}^2)}. \end{aligned}$$

By Lemma 3.10 (b), we get

$$A_1 \leq \sup_{|h_i|\leq\frac{\pi}{n_i}, i=1,2} \|\triangle_{h_1}^{\lfloor\alpha_1\rfloor+1-\alpha_1}(\triangle_{h_2}^{\lfloor\alpha_2\rfloor+1-\alpha_2}(\triangle_{h_1}^{\alpha_1}(\triangle_{h_2}^{\alpha_2}(f))))\|_{L_p(\mathbb{T}^2)}.$$

Using Lemma 3.10 (c),

$$A_1 \leq \sup_{|h_i|\leq\frac{\pi}{n_i}, i=1,2} \|\triangle_{h_1}^{\alpha_1}(\triangle_{h_2}^{\alpha_2}(f))\|_{L_p(\mathbb{T}^2)} = \omega_{\alpha_1,\alpha_2}(f, \frac{\pi}{n_1}, \frac{\pi}{n_2})_{L_p(\mathbb{T}^2)}.$$

Now let us estimate

$$A_2 := \|s_{n_1,\infty}^{(\alpha_1,0)}(f - s_{\infty,n_2}(f))\|_{L_p(\mathbb{T}^2)}.$$

Defining $\gamma(x,y) := f(x,y) - s_{\infty,n_2}(f)$ and using Lemma 3.11 (b), we have for a.e. y

$$\left(\int_0^{2\pi} |s_{n_1,\infty}(\gamma)|^p \,\mathrm{d}x\right)^{1/p} \lesssim n_1^{\alpha_1} \left(\int_0^{2\pi} |\triangle^{\alpha_1}_{\frac{\pi}{n_1}}(s_{n_1,\infty}(\gamma))|^p \,\mathrm{d}x\right)^{1/p}.$$

Hence,

$$\int_0^{2\pi}\int_0^{2\pi} |s_{n_1,\infty}(\gamma)|^p \,\mathrm{d}x\,\mathrm{d}y \lesssim n_1^{\alpha_1 p} \int_0^{2\pi}\int_0^{2\pi} |\triangle^{\alpha_1}_{\frac{\pi}{n_1}}(s_{n_1,\infty}(\gamma))|^p \,\mathrm{d}x\,\mathrm{d}y,$$

and

$$A_2 \lesssim n_1^{\alpha_1} \|s_{n_1,\infty}(\triangle^{\alpha_1}_{\frac{\pi}{n_1}}(\gamma))\|_{L_p(\mathbb{T}^2)}.$$

Lemma 3.12 implies

$$A_2 \lesssim n_1^{\alpha_1} \|\triangle^{\alpha_1}_{\frac{\pi}{n_1}}(f - s_{\infty,n_2}(f))\|_{L_p(\mathbb{T}^2)}.$$

Defining $\triangle^{\alpha_1}_{\frac{\pi}{n_1}}(f) =: F$, we get $A_2 \lesssim n_1^{\alpha_1}\|F - s_{\infty,n_2}(F)\|_{L_p(\mathbb{T}^2)}$. Since $s_{0,\infty}(F) = s_{0,n_2}(F) = 0$, then

$$A_2 \lesssim n_1^{\alpha_1} \|F - s_{0,\infty}(F) - s_{\infty,n_2}(F) + s_{0,n_2}(F)\|_{L_p(\mathbb{T}^2)}.$$

Therefore by Lemma 3.3, Theorem 4.2, and the properties of the mixed moduli of smoothness

$$A_2 \lesssim n_1^{\alpha_1} \omega_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}(F, \pi, \frac{\pi}{n_2+1})_{L_p(\mathbb{T}^2)} \lesssim n_1^{\alpha_1} \omega_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}(F, \pi, \frac{\pi}{n_2})_{L_p(\mathbb{T}^2)}.$$

Using Lemma 3.10 (b), we have

$$A_2 \lesssim n_1^{\alpha_1} \sup_{|h_1|\le\pi, |h_2|\le\frac{\pi}{n_2}} \|\triangle_{h_1}^{\lfloor\alpha_1\rfloor+1}(\triangle_{h_2}^{\lfloor\alpha_2\rfloor+1-\alpha_2}(\triangle_{h_2}^{\alpha_2}(F)))\|_{L_p(\mathbb{T}^2)}.$$

Now Lemma 3.10 (c) yields

$$A_2 \lesssim n_1^{\alpha_1} \sup_{|h_2|\le\frac{\pi}{n_2}} \|\triangle_{h_2}^{\alpha_2}(F)\|_{L_p(\mathbb{T}^2)} = n_1^{\alpha_1} \sup_{|h_2|\le\frac{\pi}{n_2}} \|\triangle_{h_2}^{\alpha_2}(\triangle^{\alpha_1}_{\frac{\pi}{n_1}}(f))\|_{L_p(\mathbb{T}^2)} \lesssim n_1^{\alpha_1} \omega_{\alpha_1,\alpha_2}(f, \frac{\pi}{n_1}, \frac{\pi}{n_2})_{L_p(\mathbb{T}^2)}.$$

Similarly, one can show that

$$A_3 := \|s^{(0,\alpha_2)}_{\infty,n_2}(f - s_{n_1,\infty}(f))\|_{L_p(\mathbb{T}^2)} \lesssim n_2^{\alpha_2} \omega_{\alpha_1,\alpha_2}(f, \frac{\pi}{n_1}, \frac{\pi}{n_2})_{L_p(\mathbb{T}^2)}$$

and

$$A_4 := \|s^{(\alpha_1,\alpha_2)}_{n_1,n_2}(f)\|_{L_p(\mathbb{T}^2)} \lesssim n_1^{\alpha_1} n_2^{\alpha_2} \omega_{\alpha_1,\alpha_2}(f, \frac{\pi}{n_1}, \frac{\pi}{n_2})_{L_p(\mathbb{T}^2)}.$$

Finally,

$$||f - s_{n_1,\infty}(f) - s_{\infty,n_2}(f) + s_{n_1,n_2}(f)||_{L_p(\mathbb{T}^2)} + n_1^{-\alpha_1}||s^{(\alpha_1,0)}_{n_1,\infty}(f - s_{\infty,n_2}(f))||_{L_p(\mathbb{T}^2)} +$$

$$n_2^{-\alpha_2}||s^{(0,\alpha_2)}_{\infty,n_2}(f - s_{n_1,\infty}(f))||_{L_p(\mathbb{T}^2)} + n_1^{-\alpha_1} n_2^{-\alpha_2}||s^{(\alpha_1,\alpha_2)}_{n_1,n_2}(f)||_{L_p(\mathbb{T}^2)} \lesssim \omega_{\alpha_1,\alpha_2}(f, \frac{\pi}{n_1}, \frac{\pi}{n_2})_{L_p(\mathbb{T}^2)},$$

i.e., the required estimate from below. □

Theorem 5.1 was stated in the papers [56, 73] without proof. This statement is called the realization result; in dimension one, see [24] for the moduli of smoothness of integer order and [74] for the fractional case. For the non-mixed moduli of smoothness of functions on $\mathbb{R}^d$ see, e.g., [25, (5.3)].

6 The mixed moduli of smoothness and the K-functionals

Theorem 6.1 *Let $f \in L_p^0(\mathbb{T}^2)$, $1 < p < \infty$, $\alpha_i > 0, 0 < \delta_i \le \pi, i = 1, 2$. Then*

$$\omega_{\alpha_1,\alpha_2}(f, \delta_1, \delta_2)_{L_p(\mathbb{T}^2)} \asymp K(f, \delta_1, \delta_2, \alpha_1, \alpha_2, p). \tag{6.1}$$

Proof. For any $\delta_i \in (0, \pi]$ take integers n_i such that $\frac{\pi}{n_i+1} < \delta_i \le \frac{\pi}{n_i}, i = 1, 2$. If $f \in L_p^0(\mathbb{T}^2)$, then

$$(s_{n_1+1,\infty}(f) - s_{n_1+1,n_2+1}(f)) \in W_p^{(\alpha_1,0)}; \quad (s_{\infty,n_2+1}(f) - s_{n_1+1,n_2+1}(f)) \in W_p^{(0,\alpha_2)};$$

$$s_{n_1+1,n_2+1}(f) \in W_p^{(\alpha_1,\alpha_2)}.$$

Then it is clear that

$$\begin{aligned}
&K(f, \delta_1, \delta_2, \alpha_1, \alpha_2, p) \\
&\le \|f - (s_{n_1+1,\infty}(f) - s_{n_1+1,n_2+1}(f)) - (s_{\infty,n_2+1}(f) - s_{n_1+1,n_2+1}(f)) - s_{n_1+1,n_2+1}(f)\|_{L_p(\mathbb{T}^2)} \\
&+ \delta_1^{\alpha_1} \|s_{n_1+1,\infty}^{(\alpha_1,0)}(f) - s_{n_1+1,n_2+1}^{(\alpha_1,0)}(f)\|_{L_p(\mathbb{T}^2)} + \delta_2^{\alpha_2} \|s_{\infty,n_2+1}^{(0,\alpha_2)}(f) - s_{n_1+1,n_2+1}^{(0,\alpha_2)}(f)\|_{L_p(\mathbb{T}^2)} \\
&\qquad\qquad\qquad\qquad + \delta_1^{\alpha_1}\delta_2^{\alpha_2} \|s_{n_1+1,n_2+1}^{(\alpha_1,\alpha_2)}(f)\|_{L_p(\mathbb{T}^2)} \\
&\lesssim \|f - s_{n_1+1,\infty}(f) - s_{\infty,n_2+1}(f) + s_{n_1,n_2}(f)\|_{L_p(\mathbb{T}^2)} + n_1^{-\alpha_1} \|s_{n_1+1,\infty}^{(\alpha_1,0)}(f - s_{\infty,n_2+1}(f))\|_{L_p(\mathbb{T}^2)} \\
&\qquad + n_2^{-\alpha_2} \|s_{\infty,n_2+1}^{(0,\alpha_2)}(f - s_{n_1+1,\infty}(f))\|_{L_p(\mathbb{T}^2)} + n_1^{-\alpha_1} n_2^{-\alpha_2} \|s_{n_1+1,n_2+1}^{(\alpha_1,\alpha_2)}(f)\|_{L_p(\mathbb{T}^2)}.
\end{aligned}$$

By Theorem 5.1, the last expression is bounded by $\omega_{\alpha_1,\alpha_2}\big(f, \frac{\pi}{n_1+1}, \frac{\pi}{n_2+1}\big)_{L_p(\mathbb{T}^2)}$ and therefore

$$K(f, \delta_1, \delta_2, \alpha_1, \alpha_2, p) \lesssim \omega_{\alpha_1,\alpha_2}\big(f, \delta_1, \delta_2\big)_{L_p(\mathbb{T}^2)}, \tag{6.2}$$

and the part "$\gtrsim$" in estimate (6.1) follows.

Let us prove the part "$\lesssim$" in estimate (6.1). Take any functions $g_1 \in W_p^{(\alpha_1,0)}$, $g_2 \in W_p^{(0,\alpha_2)}$, and $g \in W_p^{(\alpha_1,\alpha_2)}$. Then using Theorem 4.1 (2), we get

$$\begin{aligned}
\omega_{\alpha_1,\alpha_2}(f, \delta_1, \delta_2)_{L_p(\mathbb{T}^2)} \;\lesssim\;& \omega_{\alpha_1,\alpha_2}(f - g_1 - g_2 - g, \delta_1, \delta_2)_{L_p(\mathbb{T}^2)} + \omega_{\alpha_1,\alpha_2}(g_1, \delta_1, \delta_2)_{L_p(\mathbb{T}^2)} \\
+\;& \omega_{\alpha_1,\alpha_2}(g_2, \delta_1, \delta_2)_{L_p(\mathbb{T}^2)} + \omega_{\alpha_1,\alpha_2}(g, \delta_1, \delta_2)_{L_p(\mathbb{T}^2)} =: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

By Lemma 3.10, we have $J_1 \lesssim \|f - g_1 - g_2 - g\|_{L_p(\mathbb{T}^2)}$.

To estimate J_2, for any $\delta_i \in (0, \pi]$ we take integers n_i such that $\frac{\pi}{2^{n_i+1}} < \delta_i \le \frac{\pi}{2^{n_i}}$, $i = 1, 2$. We consider $B_2 := \omega_{\alpha_1,\alpha_2}\big(g_1, \frac{\pi}{2^{n_1}}, \frac{\pi}{2^{n_2}}\big)_{L_p(\mathbb{T}^2)}$. Lemma 3.10 yields

$$\begin{aligned}
B_2 \;\lesssim\;& \omega_{\alpha_1,\alpha_2}\big(g_1 - s_{2^{n_1},\infty}(g_1), \frac{\pi}{2^{n_1}}, \frac{\pi}{2^{n_2}}\big)_{L_p(\mathbb{T}^2)} + \omega_{\alpha_1,\alpha_2}\big(s_{2^{n_1},\infty}(g_1), \frac{\pi}{2^{n_1}}, \frac{\pi}{2^{n_2}}\big)_{L_p(\mathbb{T}^2)} \\
\lesssim\;& \|g_1 - s_{2^{n_1},\infty}(g_1)\|_{L_p(\mathbb{T}^2)} + \sup_{|h_1| \le \frac{\pi}{2^{n_1}}} \|\Delta_{h_1}^{\alpha_1}(s_{2^{n_1},\infty}(g_1))\|_{L_p(\mathbb{T}^2)} =: J_{21} + J_{22}.
\end{aligned}$$

Using Lemma 3.11 (a) and Lemma 3.12, we get for a.e. y and $0 < h_1 \le \frac{\pi}{2^{n_1}}$

$$\left(\int_0^{2\pi} \big|\Delta_{h_1}^{\alpha_1} s_{2^{n_1},\infty}(g_1)\big|^p \,\mathrm{d}x\right)^{1/p} \lesssim 2^{-n_1\alpha_1} \left(\int_0^{2\pi} \big|s_{2^{n_1},\infty}^{(\alpha_1,0)}(g_1)\big|^p \,\mathrm{d}x\right)^{1/p} \lesssim 2^{-n_1\alpha_1} \left(\int_0^{2\pi} \big|g_1^{(\alpha_1,0)}\big|^p \,\mathrm{d}x\right)^{1/p}.$$

Then the inequality

$$\int_0^{2\pi}\int_0^{2\pi} |\Delta_{h_1}^{\alpha_1} s_{2^{n_1},\infty}(g_1)|^p \,\mathrm{d}x\,\mathrm{d}y \lesssim 2^{-n_1\alpha_1 p}\int_0^{2\pi}\int_0^{2\pi} |g_1^{(\alpha_1,0)}|^p \,\mathrm{d}x\,\mathrm{d}y$$

implies $J_{22} \lesssim 2^{-n_1\alpha_1}\|g_1^{(\alpha_1,0)}\|_{L_p(\mathbb{T}^2)}$. Since $g_1 \in L_p^0(\mathbb{T}^2)$, then Lemmas 3.5 and 3.6 give

$$J_{21} \lesssim \Bigg(\int_0^{2\pi}\int_0^{2\pi}\Big(\sum_{\nu_1=n_1}^{\infty}\sum_{\nu_2=0}^{\infty}\Delta_{\nu_1,\nu_2}^2\Big)^{p/2}\mathrm{d}x\,\mathrm{d}y\Bigg)^{1/p} \lesssim 2^{-n_1\alpha_1}\Bigg(\int_0^{2\pi}\int_0^{2\pi}\Big(\sum_{\nu_1=n_1}^{\infty}\sum_{\nu_2=0}^{\infty}2^{2\nu_1\alpha_1}\Delta_{\nu_1,\nu_2}^2\Big)^{p/2}\mathrm{d}x\,\mathrm{d}y\Bigg)^{1/p}.$$

Using again Lemmas 3.5 and 3.6, and further Lemma 3.12, we get

$$\begin{aligned} J_{21} &\lesssim 2^{-n_1\alpha_1}\|g_1^{(\alpha_1,0)} - s_{2^{n_1}-1,\infty}(g_1^{(\alpha_1,0)})\|_{L_p(\mathbb{T}^2)} \\ &\lesssim 2^{-n_1\alpha_1}\Big(\|g_1^{(\alpha_1,0)}\|_{L_p(\mathbb{T}^2)} + \|s_{2^{n_1}-1,\infty}(g_1^{(\alpha_1,0)})\|_{L_p(\mathbb{T}^2)}\Big) \lesssim 2^{-n_1\alpha_1}\|g_1^{(\alpha_1,0)}\|_{L_p(\mathbb{T}^2)}. \end{aligned}$$

The estimates for J_{21} and J_{22} imply

$$B_2 \lesssim 2^{-n_1\alpha_1}\|g_1^{(\alpha_1,0)}\|_{L_p(\mathbb{T}^2)}.$$

Using properties of moduli of smoothness, we get $\omega_{\alpha_1,\alpha_2}\big(g_1,\delta_1,\delta_2\big)_{L_p(\mathbb{T}^2)} \lesssim \omega_{\alpha_1,\alpha_2}\big(g_1,\frac{\pi}{2^{n_1}},\frac{\pi}{2^{n_2}}\big)_{L_p(\mathbb{T}^2)}$ and

$$J_2 \lesssim 2^{-n_1\alpha_1}\|g_1^{(\alpha_1,0)}\|_{L_p(\mathbb{T}^2)}.$$

Similarly,

$$J_3 \lesssim 2^{-n_2\alpha_2}\|g_2^{(0,\alpha_2)}\|_{L_p(\mathbb{T}^2)} \qquad \text{and} \qquad J_4 \lesssim 2^{-n_1\alpha_1-n_2\alpha_2}\|g^{(\alpha_1,\alpha_2)}\|_{L_p(\mathbb{T}^2)}.$$

Finally, combining estimates for J_1, J_2, J_3, and J_4, we get

$$\begin{aligned} \omega_{\alpha_1,\alpha_2}\big(g_1,\delta_1,\delta_2\big)_{L_p(\mathbb{T}^2)} &\lesssim \|f-g_1-g_2-g\|_{L_p(\mathbb{T}^2)} + \delta_1^{\alpha_1}\|g_1^{(\alpha_1,0)}\|_{L_p(\mathbb{T}^2)} \\ &+ \delta_2^{\alpha_2}\|g_2^{(0,\alpha_2)}\|_{L_p(\mathbb{T}^2)} + \delta_1^{\alpha_1}\delta_2^{\alpha_2}\|g^{(\alpha_1,\alpha_2)}\|_{L_p(\mathbb{T}^2)}. \end{aligned}$$

Since the last inequality holds for any $g_1 \in W_p^{(\alpha_1,0)}$, $g_2 \in W_p^{(0,\alpha_2)}$, and $g \in W_p^{(\alpha_1,\alpha_2)}$, we get

$$\omega_{\alpha_1,\alpha_2}\big(g_1,\delta_1,\delta_2\big)_{L_p(\mathbb{T}^2)} \lesssim K(f,\delta_1,\delta_2,\alpha_1,\alpha_2,p) \tag{6.3}$$

and therefore the proof of the part "$\lesssim$" in estimate (6.1) follows. □

In the case of integers α_1 and α_2 Theorem 6.1 was proved in the paper [65] for $1 \le p \le \infty$, and in the paper [13] for $p = \infty$ using different methods. In the one-dimensional case and in the multivariate case for non-mixed moduli of smoothness, the equivalence between the moduli of smoothness and the corresponding K-functionals was proved in [37] (see also [6, p. 339]).

7 The mixed moduli of smoothness of L_p-functions and their Fourier coefficients

The classical Riemann-Lebesgue lemma states that the Fourier coefficients of an $L_p(\mathbb{T}^2)$ function tend to 0 as $|n| \to \infty$. Its quantitative version is written as follows:

$$\rho_{n_1,n_2} \lesssim \omega_{\alpha_1,\alpha_2}\Big(f, \frac{1}{n_1}, \frac{1}{n_2}\Big)_{L_p(\mathbb{T}^2)},$$

where

$$\rho_{n_1,n_2} = |a_{n_1,n_2}| + |b_{n_1,n_2}| + |c_{n_1,n_2}| + |d_{n_1,n_2}|$$

and the Fourier series of $f \in L_p^0(\mathbb{T}^2), 1 < p < \infty$, is given by (3.1). We extend this estimate by writing the following two-sided inequalities using weighted tail-type sums of the Fourier series.

Theorem 7.1 *Let $f \in L_p^0$, $1 < p < \infty$, and the Fourier series of f be given by (3.1). Let $\tau := \max(2,p)$, $\theta := \min(2,p)$, $\alpha_1 > 0, \alpha_2 > 0$, and $n_1 \in \mathbb{N}, n_2 \in \mathbb{N}$. Then*

$$\mathbb{I}(\theta) \lesssim \omega_{\alpha_1,\alpha_2}\Big(f, \frac{1}{n_1}, \frac{1}{n_2}\Big)_{L_p(\mathbb{T}^2)} \lesssim \mathbb{I}(\tau), \tag{7.1}$$

where

$$\mathbb{I}(s) := \frac{1}{n_1^{\alpha_1}}\frac{1}{n_2^{\alpha_2}}\Bigg\{\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} \rho^s_{\nu_1,\nu_2}\nu_1^{(\alpha_1+1)s-2}\nu_2^{(\alpha_2+1)s-2}\Bigg\}^{1/s} + \frac{1}{n_1^{\alpha_1}}\Bigg\{\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty} \rho^s_{\nu_1,\nu_2}\nu_1^{(\alpha_1+1)s-2}\nu_2^{s-2}\Bigg\}^{1/s}$$

$$+\frac{1}{n_2^{\alpha_2}}\Bigg\{\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=1}^{n_2} \rho^s_{\nu_1,\nu_2}\nu_1^{s-2}\nu_2^{(\alpha_2+1)s-2}\Bigg\}^{1/s} + \Bigg\{\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty} \rho^s_{\nu_1,\nu_2}(\nu_1\nu_2)^{s-2}\Bigg\}^{1/s}$$

for $1 < s < \infty$.

The next two theorems provide sharper results than (7.1) for special classes of functions defined in Section 2.4. In particular, these results show that the parameters $\tau = \max(2,p)$ and $\theta = \min(2,p)$ in (7.1) cannot be extended.

Theorem 7.2 *Let $f \in M_p, 1 < p < \infty, \alpha_1 > 0, \alpha_2 > 0, n_1 \in \mathbb{N}, n_2 \in \mathbb{N}$. Then*

$$\omega_{\alpha_1,\alpha_2}\Big(f, \frac{1}{n_1}, \frac{1}{n_2}\Big)_{L_p(\mathbb{T}^2)} \asymp \frac{1}{n_1^{\alpha_1}n_2^{\alpha_2}}\Bigg\{\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} a^p_{\nu_1,\nu_2}\nu_1^{(\alpha_1+1)p-2}\nu_2^{(\alpha_2+1)p-2}\Bigg\}^{1/p}$$

$$+\frac{1}{n_1^{\alpha_1}}\Bigg\{\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty} a^p_{\nu_1,\nu_2}\nu_1^{(\alpha_1+1)p-2}\nu_2^{p-2}\Bigg\}^{1/p} + \frac{1}{n_2^{\alpha_2}}\Bigg\{\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=1}^{n_2} a^p_{\nu_1,\nu_2}\nu_1^{p-2}\nu_2^{(\alpha_2+1)p-2}\Bigg\}^{1/p}$$

$$+\Bigg\{\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty} a^p_{\nu_1,\nu_2}(\nu_1\nu_2)^{p-2}\Bigg\}^{1/p}.$$

Theorem 7.3 *Let $f\in\Lambda_p, 1<p<\infty, \alpha_1>0, \alpha_2>0, n_i=0,1,2,\dots, i=1,2$. Then*

$$\omega_{\alpha_1,\alpha_2}\Big(f,\frac{1}{2^{n_1}},\frac{1}{2^{n_2}}\Big)_p \asymp \frac{1}{2^{n_1\alpha_1+n_2\alpha_2}}\left\{\sum_{\mu_1=0}^{n_1}\sum_{\mu_2=0}^{n_2}\lambda^2_{\mu_1,\mu_2}2^{2(\mu_1\alpha_1+\mu_2\alpha_2)}\right\}^{1/2}$$

$$+\frac{1}{2^{n_1\alpha_1}}\left\{\sum_{\mu_1=0}^{n_1}\sum_{\mu_2=n_2+1}^{\infty}\lambda^2_{\mu_1,\mu_2}2^{2\mu_1\alpha_1}\right\}^{\frac12}+\frac{1}{2^{n_2\alpha_2}}\left\{\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=0}^{n_2}\lambda^2_{\mu_1,\mu_2}2^{2\mu_2\alpha_2}\right\}^{1/2}$$

$$+\left\{\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=n_2+1}^{\infty}\lambda^2_{\mu_1,\mu_2}\right\}^{1/2}.$$

Proof of Theorem 7.1. By Theorem 5.1, we have

$$I:=\omega_{\alpha_1,\alpha_2}\Big(f,\frac{1}{n_1},\frac{1}{n_2}\Big)_{L_p(\mathbb{T}^2)}\asymp n_1^{-\alpha_1}n_2^{-\alpha_2}||s^{(\alpha_1,\alpha_2)}_{n_1,n_2}(f)||_{L_p(\mathbb{T}^2)}$$

$$+n_1^{-\alpha_1}||s^{(\alpha_1,0)}_{n_1,\infty}(f-s_{\infty,n_2}(f))||_{L_p(\mathbb{T}^2)}+n_2^{-\alpha_2}||s^{(0,\alpha_2)}_{\infty,n_2}(f-s_{n_1,\infty}(s))||_{L_p(\mathbb{T}^2)}+$$

$$+\|(f-s_{n_1,\infty}(f)-s_{\infty,n_2}(f)+s_{n_1,n_2}(f))\|_{L_p(\mathbb{T}^2)}.$$

Let first $2\le p<\infty$. Then taking into account Lemmas 3.5 and 3.7 (A), we get

$$\begin{aligned} I \quad &\lesssim \quad n_1^{-\alpha_1}n_2^{-\alpha_2}\Big(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\rho^p_{\nu_1,\nu_2}\nu_1^{(\alpha_1+1)p-2}\nu_2^{(\alpha_2+1)p-2}\Big)^{1/p}\\ &+ \quad n_1^{-\alpha_1}\Big(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty}\rho^p_{\nu_1,\nu_2}\nu_1^{(\alpha_1+1)p-2}\nu_2^{p-2}\Big)^{1/p}\\ &+ \quad n_2^{-\alpha_2}\Big(\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=1}^{n_2}\rho^p_{\nu_1,\nu_2}\nu_1^{p-2}\nu_2^{(\alpha_2+1)p-2}\Big)^{1/p}\\ &+ \quad \Big(\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty}\rho^p_{\nu_1,\nu_2}(\nu_1\nu_2)^{p-2}\Big)^{1/p}. \end{aligned}$$

Therefore, for $p\ge 2$ we show the estimate from above in Theorem 7.1.

If $1<p<2$, then Hölder's inequality gives

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)}\lesssim\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_2(\mathbb{T}^2)}.$$

Thus, the estimate from above is obtained.

To prove the estimate from below, if $1 < p \le 2$, we use Lemmas 3.5 and 3.7 (B):

$$\begin{aligned} I \gtrsim\ & n_1^{-\alpha_1} n_2^{-\alpha_2} \Big(\sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} \rho^p_{\nu_1,\nu_2} \nu_1^{(\alpha_1+1)p-2} \nu_2^{(\alpha_2+1)p-2} \Big)^{1/p} \\ +\ & n_1^{-\alpha_1} \Big(\sum_{\nu_1=1}^{n_1} \sum_{\nu_2=n_2+1}^{\infty} \rho^p_{\nu_1,\nu_2} \nu_1^{(\alpha_1+1)p-2} \nu_2^{p-2} \Big)^{1/p} \\ +\ & n_2^{-\alpha_2} \Big(\sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=1}^{n_2} \rho^p_{\nu_1,\nu_2} \nu_1^{p-2} \nu_2^{(\alpha_2+1)p-2} \Big)^{1/p} \\ +\ & \Big(\sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=n_2+1}^{\infty} \rho^p_{\nu_1,\nu_2} (\nu_1\nu_2)^{p-2} \Big)^{1/p}. \end{aligned}$$

Let now $2 < p < \infty$. Applying Hölder's inequality once again, we get

$$\omega_{\alpha_1,\alpha_2}\left(f, \frac{1}{n_1}, \frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)} \gtrsim \omega_{\alpha_1,\alpha_2}\left(f, \frac{1}{n_1}, \frac{1}{n_2}\right)_{L_2(\mathbb{T}^2)}.$$

$\square$

Proof of Theorem 7.2. Let us denote

$$\begin{aligned} I_1 &:= n_1^{-\alpha_1} n_2^{-\alpha_2} \| s^{(\alpha_1,\alpha_2)}_{n_1,n_2}(f) \|_{L_p(\mathbb{T}^2)}, \quad I_2 := n_1^{-\alpha_1} \| s^{(\alpha_1,0)}_{n_1,\infty}(f - s_{\infty,n_2}(f)) \|_{L_p(\mathbb{T}^2)}, \\ I_3 &:= n_2^{-\alpha_2} \| s^{(0,\alpha_2)}_{\infty,n_2}(f - s_{n_1,\infty}) \|_{L_p(\mathbb{T}^2)}, \quad I_4 := \| f - s_{n_1,\infty}(f) - s_{\infty,n_2}(f) + s_{n_1,n_2} \|_{L_p(\mathbb{T}^2)} \end{aligned}$$

and

$$\begin{aligned} A_1 &:= n_1^{-\alpha_1} n_2^{-\alpha_2} \Big(\sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} a^p_{\nu_1,\nu_2} \nu_1^{(\alpha_1+1)p-2} \nu_2^{(\alpha_2+1)p-2} \Big)^{1/p}, \\ A_2 &:= n_1^{-\alpha_1} \Big(\sum_{\nu_1=1}^{n_1} \sum_{\nu_2=n_2+1}^{\infty} a^p_{\nu_1,\nu_2} \nu_1^{(\alpha_1+1)p-2} \nu_2^{p-2} \Big)^{1/p}, \\ A_3 &:= n_2^{-\alpha_2} \Big(\sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=1}^{n_2} a^p_{\nu_1,\nu_2} \nu_1^{p-2} \nu_2^{(\alpha_2+1)p-2} \Big)^{1/p}, \\ A_4 &:= \Big(\sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=n_2+1}^{\infty} a^p_{\nu_1,\nu_2} (\nu_1\nu_2)^{p-2} \Big)^{1/p}. \end{aligned}$$

Let us establish the interrelation between I_1, I_2, I_3, I_4 and A_1, A_2, A_3, A_4. By Lemma 3.8, we have

$$I_1 \asymp A_1. \tag{7.2}$$

To estimate I_2 and A_2, we use

$$\eta_1(x,y) := \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{\infty} a^*_{\nu_1,\nu_2} \cos \nu_1 x \cos \nu_2 y \tag{7.3}$$

and

$$\eta_2(x,y) := \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} a^*_{\nu_1,\nu_2}\cos\nu_1 x\cos\nu_2 y, \tag{7.4}$$

where

$$a^*_{\nu_1,\nu_2} := \begin{cases} a_{\nu_1,\nu_2} \text{ for } 1\le\nu_1\le n_1, \nu_2>n_2 \\ a_{\nu_1,n_2} \text{ for } 1\le\nu_1\le n_1, 1\le\nu_2\le n_2. \end{cases}$$

Then

$$s^{(\alpha_1,0)}_{n_1,\infty}(f-s_{\infty,n_2}(f)) = (\eta_1-\eta_2)^{(\alpha_1,0)}. \tag{7.5}$$

Let us first estimate I_2. It is clear that

$$I_2 \lesssim n_1^{-\alpha_1}\left\{\|\eta_1^{(\alpha_1,0)}\|_{L_p(\mathbb{T}^2)} + \|\eta_2^{(\alpha_2,0)}\|_{L_p(\mathbb{T}^2)}\right\}.$$

Now Lemma 3.8 implies that

$$\begin{aligned} I_2 &\le n_1^{-\alpha_1}\Big(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{\infty}(a^*_{\nu_1,\nu_2})^p\nu_1^{(\alpha_1+1)p-2}\nu_2^{p-2}\Big)^{1/p} \\ &+ n_1^{-\alpha_1}\Big(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}(a^*_{\nu_1,\nu_2})^p\nu_1^{(\alpha_1+1)p-2}\nu_2^{p-2}\Big)^{1/p} \\ &\lesssim n_1^{-\alpha_1}\Big(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty}a^p_{\nu_1,\nu_2}\nu_1^{(\alpha_1+1)p-2}\nu_2^{p-2}\Big)^{1/p} \\ &+ n_1^{-\alpha_1}\Big(\sum_{\nu_1=1}^{n_1}a^p_{\nu_1,n_2}\nu_1^{(\alpha_1+1)p-2}n_2^{p-1}\Big)^{1/p} \\ &\lesssim A_2 + n_1^{-\alpha_1}n_2^{-\alpha_2}\Big(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=\lfloor\frac{n_2}{2}\rfloor+1}^{n_2}a^p_{\nu_1,\nu_2}\nu_1^{(\alpha_1+1)p-2}\nu_2^{(\alpha_2+1)p-2}\Big)^{1/p} \lesssim A_2+A_1. \end{aligned} \tag{7.6}$$

Estimating A_2 from above, we write

$$A_2 \lesssim n_1^{-\alpha_1}\Big(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{\infty}(a^*_{\nu_1,\nu_2})^p\nu_1^{(\alpha_1+1)p-2}\nu_2^{p-2}\Big)^{1/p}.$$

Lemma 3.8 and formulas (7.3)-(7.4) imply

$$A_2 \lesssim n_1^{-\alpha_1}\|\eta_1^{(\alpha_1,0)}\|_{L_p(\mathbb{T}^2)} \lesssim I_2 + n_1^{-\alpha_1}\|\eta_2^{(\alpha_1,0)}\|_{L_p(\mathbb{T}^2)}$$

and

$$\begin{aligned} A_2 &\lesssim I_2 + n_1^{-\alpha_1}\Big(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}(a^*_{\nu_1,\nu_2})^p\nu_1^{(\alpha_1+1)p-2}\nu_2^{p-2}\Big)^{1/p} \\ &\lesssim I_2 + n_1^{-\alpha_1}\Big(\sum_{\nu_1=1}^{n_1}a^p_{\nu_1,n_2}\nu_1^{(\alpha_1+1)p-2}n_2^{p-1}\Big)^{1/p} \lesssim I_2+A_1. \end{aligned}$$

Now we use (7.2) to get
$$A_2 \lesssim I_2 + I_1. \tag{7.7}$$
Similarly, we get
$$I_3 \lesssim A_1 + A_3 \tag{7.8}$$
and
$$A_3 \lesssim I_3 + I_1. \tag{7.9}$$
Thus, it is shown that
$$A_1 + A_2 + A_3 \asymp I_1 + I_2 + I_3. \tag{7.10}$$

Now we consider $B := f - s_{n_1,\infty} - s_{\infty,n_1} + s_{n_1,n_2}$ and

$$\varphi_1(x,y) := \sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty} b_{\nu_1,\nu_2}\cos\nu_1 x\cos\nu_2 y,$$

$$\varphi_2(x,y) := \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty} b_{\nu_1,\nu_2}\cos\nu_1 x\cos\nu_2 y,$$

$$\varphi_3(x,y) := \sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=1}^{n_2} b_{\nu_1,\nu_2}\cos\nu_1 x\cos\nu_2 y,$$

$$\varphi_4(x,y) := \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} b_{\nu_1,\nu_2}\cos\nu_1 x\cos\nu_2 y,$$

where
$$b_{\nu_1,\nu_2} := \begin{cases} a_{\nu_1,\nu_2} & \text{for} \quad \nu_1 > n_1, \nu_2 > n_2, \\ a_{\nu_1,n_2} & \text{for} \quad \nu_1 > n_1, 1 \le \nu_2 \le n_2, \\ a_{n_1,\nu_2} & \text{for} \quad 1 \le \nu_1 \le n_1, \nu_2 > n_2, \\ a_{n_1,n_2} & \text{for} \quad 1 \le \nu_1 \le n_1, 1 \le \nu_2 \le n_2. \end{cases}$$
Then
$$B = \varphi_1 - \varphi_2 - \varphi_3 + \varphi_4. \tag{7.11}$$

We first estimate

$$I_4 = \|B\|_{L_p(\mathbb{T}^2)} \lesssim \|\varphi_1\|_{L_p(\mathbb{T}^2)} + \|\varphi_2\|_{L_p(\mathbb{T}^2)} + \|\varphi_3\|_{L_p(\mathbb{T}^2)} + \|\varphi_4\|_{L_p(\mathbb{T}^2)}.$$

Lemma 3.8 yields

$$\|\varphi_1\|_{L_p(\mathbb{T}^2)} \asymp \sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty} b^p_{\nu_1,\nu_2}(\nu_1\nu_2)^{p-2} \asymp \sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty} a^p_{\nu_1,\nu_2}(\nu_1\nu_2)^{p-2}$$

$$+ \sum_{\nu_1=n_1+1}^{\infty} a^p_{\nu_1,n_2}\nu_1^{p-2}n_2^{p-1} + \sum_{\nu_2=n_2+1}^{\infty} a^p_{n_1,\nu_2}\nu_2^{p-2}n_1^{p-1} + a^p_{n_1,n_2}(n_1n_2)^{p-1} =: J_1 + J_2 + J_3 + J_4.$$

To estimate $\|\varphi_2\|_{L_p(\mathbb{T}^2)}$, we consider

$$\varphi_{21}(x,y) := \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{\infty} b^*_{\nu_1,\nu_2}\cos\nu_1 x\cos\nu_2 y,$$

$$\varphi_{22}(x,y) := \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} b^*_{\nu_1,\nu_2}\cos\nu_1 x\cos\nu_2 y,$$

where

$$b^*_{\nu_1,\nu_2} := \begin{cases} b_{\nu_1,\nu_2} \text{ for } 1\le \nu_1\le n_1, \nu_2 > n_2,\\ b_{\nu_1,n_2} \text{ for } 1\le \nu_1\le n_1, 1\le\nu_2\le n_2.\end{cases}$$

Then $\|\varphi_2\|_{L_p(\mathbb{T}^2)} \lesssim \|\varphi_{21}\|_{L_p(\mathbb{T}^2)} + \|\varphi_{22}\|_{L_p(\mathbb{T}^2)}$. Using Lemma 3.8, we have

$$\begin{aligned}\|\varphi_{21}\|_{L_p(\mathbb{T}^2)} &\asymp \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{\infty}(b^*_{\nu_1,\nu_2})^p(\nu_1\nu_2)^{p-2} \lesssim \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty} b^p_{\nu_1,\nu_2}(\nu_1\nu_2)^{p-2}\\ &+ \sum_{\nu_1=1}^{n_1} b^p_{\nu_1,n_2}\nu_1^{p-2}n_2^{p-1} \lesssim \sum_{\nu_2=n_2+1}^{\infty} a^p_{n_1,\nu_2}\nu_2^{p-2}n_1^{p-1} + a^p_{n_1,n_2}(n_1n_2)^{p-1} =: J_3+J_4,\\ \|\varphi_{22}\|_{L_p(\mathbb{T}^2)} &\asymp \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}(b^*_{\nu_1,\nu_2})^p(\nu_1\nu_2)^{p-2} \lesssim a^p_{n_1,n_2}(n_1n_2)^{p-1} \lesssim J_4.\end{aligned}$$

Therefore, $\|\varphi_2\|^p_{L_p(\mathbb{T}^2)} \lesssim J_3+J_4$. Similarly, $\|\varphi_3\|^p_{L_p(\mathbb{T}^2)} \lesssim J_2+J_4$ and $\|\varphi_4\|^p_{L_p(\mathbb{T}^2)} \lesssim J_4$. Combining these estimates, we get

$$\begin{aligned} I_4 &\lesssim \left\{\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty} a^p_{\nu_1,\nu_2}(\nu_1\nu_2)^{p-2}\right\}^{1/p} + \left\{\sum_{\nu_1=n_1+1}^{\infty} a^p_{\nu_1,n_2}\nu_1^{p-2}\right\}^{1/p} n_2^{1-\frac1p}\\ &+ \left\{\sum_{\nu_2=n_2+1}^{\infty} a^p_{n_1,\nu_2}\nu_2^{p-2}\right\}^{1/p} n_1^{1-\frac1p} + a_{n_1,n_2}(n_1n_2)^{1-\frac1p}.\end{aligned}$$

Note that the latter inequality holds also if $n_1=0$ and/or $n_2=0$. It is easy to verify that the last estimate implies

$$I_4 \lesssim A_1+A_2+A_3+A_4. \tag{7.12}$$

Let us estimate A_4 from above. It is clear that

$$A_4 \lesssim \left\{\sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty} b^p_{\nu_1,\nu_2}(\nu_1\nu_2)^{p-2}\right\}^{1/p}.$$

Lemma 3.8 yields $A_4 \lesssim \|\varphi_1\|_{L_p(\mathbb{T}^2)}$ and, by (7.11), we get $A_4 \lesssim \|B\|_{L_p(\mathbb{T}^2)} + \|\varphi_2\|_{L_p(\mathbb{T}^2)} + \|\varphi_3\|_{L_p(\mathbb{T}^2)} + \|\varphi_4\|_{L_p(\mathbb{T}^2)}$. Then

$$\begin{aligned} A_4 &\le I_4+J_2+J_3+J_4 \le I_4 + \left\{\sum_{\nu_1=1}^{\infty} a^p_{\nu_1,n_2}\nu_1^{p-2}\right\}^{1/p} n_2^{1-\frac1p}\\ &+ \left\{\sum_{\nu_2=n_2+1}^{\infty} a^p_{n_1,\nu_2}\nu_2^{p-2}\right\}^{1/p} n_1^{1-\frac1p} + a_{n_1,n_2}(n_1n_2)^{1-\frac1p} \lesssim I_4+A_2+A_3+A_1.\end{aligned}$$

Using estimates (7.10) and (7.12), we have

$$A_4 \leq I_1 + I_2 + I_3 + I_4. \tag{7.13}$$

This, (7.10) and (7.12) give us

$$I_1 + I_2 + I_3 + I_4 \asymp A_1 + A_2 + A_3 + A_4.$$

Since Theorem 5.1 implies

$$\omega_{\alpha_1,\alpha_2}\left(f, \frac{1}{n_1}, \frac{1}{n_2}\right)^p_{L_p(\mathbb{T}^2)} \asymp I_1 + I_2 + I_3 + I_4,$$

the proof of Theorem 7.2 is now complete. $\square$

Proof of Theorem 7.3. Theorem 5.1 implies

$$\begin{aligned}
I &:= \omega_{\alpha_1,\alpha_2}\left(f, \frac{1}{2^{n_1}}, \frac{1}{2^{n_2}}\right)^p_{L_p(\mathbb{T}^2)} \asymp 2^{-n_1\alpha_1} 2^{-n_2\alpha_2} \|s^{(\alpha_1,\alpha_2)}_{2^{n_1},2^{n_2}}\|_{L_p(\mathbb{T}^2)} \\
&+ 2^{-n_1\alpha_1}\|s^{(\alpha_1,0)}_{2^{n_1},\infty}(f - s_{\infty,2^{n_2}}(f))\|_{L_p(\mathbb{T}^2)} + 2^{-n_2\alpha_2}\|s^{(0,\alpha_2)}_{\infty,2^{n_2}}(f - s_{2^{n_1},\infty}(f))\|_{L_p(\mathbb{T}^2)} \\
&+ \|f - s_{2^{n_1},\infty}(f) - s_{\infty,2^{n_2}}(f) + s_{2^{n_1},2^{n_2}}(f)\|_{L_p(\mathbb{T}^2)}.
\end{aligned}$$

Lemmas 3.5 and 3.9 yield

$$\begin{aligned}
I &\asymp 2^{-n_1\alpha_1 - n_2\alpha_2}\left\{\sum_{\mu_1=0}^{n_1}\sum_{\mu_2=0}^{n_2} \lambda^2_{\mu_1,\mu_2} 2^{2(\alpha_1\mu_1+\alpha_2\mu_2)}\right\}^{1/2} + 2^{-n_1\alpha_1}\left\{\sum_{\mu_1=0}^{n_1}\sum_{\mu_2=n_1+1}^{\infty} \lambda^2_{\mu_1,\mu_2} 2^{2\alpha_1\mu_1}\right\}^{1/2} \\
&+ 2^{-n_2\alpha_2}\left\{\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=0}^{n_2} \lambda^2_{\mu_1,\mu_2} 2^{2\alpha_2\mu_2}\right\}^{1/2} + \left\{\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=n_2+1}^{\infty} \lambda^2_{\mu_1,\mu_2}\right\}^{1/2},
\end{aligned}$$

i.e., the required equivalence. $\square$

Theorem 7.1 is stated in the paper [31], while Theorems 7.2-7.3 can be found in the papers [31, 56, 57]. The one-dimensional version of Theorem 7.2 for functions with general monotone coefficients is proved in [33]. Theorem 7.3 is given in [7] in the case of integer order moduli.

8 The mixed moduli of smoothness of L_p-functions and their derivatives

Let $1 < p < \infty$. We start this section with two well-known relations for the one-dimensional modulus of smoothness: for f, $f^{(k)} \in L_p(\mathbb{T})$, we have the estimate (see [19, p. 46])

$$\omega_{r+k}(f,\delta)_{L_p(\mathbb{T})} \lesssim \delta^k \omega_r(f^{(k)},\delta)_{L_p(\mathbb{T})}, \qquad \text{where} \qquad r,k \in \mathbb{N}$$

and its weak inverse (see [19, p. 178])

$$\omega_r(f^{(k)},\delta)_{L_p(\mathbb{T})} \lesssim \int_0^\delta \frac{\omega_{r+k}(f,u)_{L_p(\mathbb{T})}}{u^{k+1}}\,du, \qquad \text{where} \qquad r,k \in \mathbb{N}.$$

The analogues of these inequalities for the mixed moduli of smoothness are given by

$$\begin{aligned}\delta_1^{-r_1}\delta_2^{-r_2}\omega_{\beta_1+r_1,\beta_2+r_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} &\lesssim \omega_{\beta_1,\beta_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}\\ &\lesssim \int_0^{\delta_1}\int_0^{\delta_2} t_1^{-r_1}t_2^{-r_2}\omega_{\beta_1+r_1,\beta_2+r_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2},\end{aligned}$$

where $\beta_j, r_j>0, \quad j=1,2$ (see Theorem 4.1; properties (9) and (10)).

Now we provide a generalization of these estimates (see [58]). Note that this idea goes back to the papers by Besov [8] and Marcinkiewicz [45]. We also remark that the right-hand side inequality was done for non-mixed moduli in [26, 91]. Moreover, a similar estimate for mixed moduli of smoothness of functions defined on $\mathbb{R}^d$ was proved in [89].

Theorem 8.1 *Let $f\in L_p^0(\mathbb{T}^2), 1<p<\infty$, $\theta:=\min(2,p), \tau:=\max(2,p)$, $\beta_1,\beta_2,r_1,r_2>0$, and $\delta_1,\delta_2\in(0,\frac12)$. If*

$$\left(\int_0^1\int_0^1 t_1^{-r_1\theta-1}t_2^{-r_2\theta-1}\omega^\theta_{r_1+\beta_1,r_2+\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\,\mathrm{d}t_1\,\mathrm{d}t_2\right)^{1/\theta}<\infty,$$

then f has a mixed derivative $f^{(r_1,r_2)}\in L_p(\mathbb{T}^2)$ in the sense of Weyl and

$$\omega_{\beta_1,\beta_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}\lesssim\left\{\int_0^{\delta_1}\int_0^{\delta_2} t_1^{-r_1\theta-1}t_2^{-r_2\theta-1}\omega^\theta_{r_1+\beta_1,r_2+\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\,\mathrm{d}t_1\,\mathrm{d}t_2\right\}^{1/\theta}. \tag{8.1}$$

If $f\in L_p(\mathbb{T}^2)$ has a mixed derivative $f^{(r_1,r_2)}\in L_p(\mathbb{T}^2)$ in the sense of Weyl, then

$$\left\{\int_0^{\delta_1}\int_0^{\delta_2} t_1^{-r_1\tau-1}t_2^{-r_2\tau-1}\omega^\tau_{r_1+\beta_1,r_2+\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\,\mathrm{d}t_1\,\mathrm{d}t_2\right\}^{1/\tau}\lesssim\omega_{\beta_1,\beta_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}.$$

Now we deal with the function classes M_p and Λ_p defined in Section 2.4.

Theorem 8.2 *Let $f\in M_p, 1<p<\infty$, $\beta_1,\beta_2,r_1,r_2>0$, and $\delta_1,\delta_2\in(0,\frac12)$. Then*

$$\omega_{\beta_1,\beta_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}\asymp\left\{\int_0^{\delta_1}\int_0^{\delta_2} t_1^{-r_1p-1}t_2^{-r_2p-1}\omega^p_{r_1+\beta_1,r_2+\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\,\mathrm{d}t_1\,\mathrm{d}t_2\right\}^{1/p}.$$

Theorem 8.3 *Let $f\in\Lambda_p, 1<p<\infty$, $\beta_1,\beta_2,r_1,r_2>0$, and $\delta_1,\delta_2\in(0,\frac12)$. Then*

$$\omega_{\beta_1,\beta_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}\asymp\left\{\int_0^{\delta_1}\int_0^{\delta_2} t_1^{-2r_1-1}t_2^{-2r_2-1}\omega^2_{r_1+\beta_1,r_2+\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\,\mathrm{d}t_1\,\mathrm{d}t_2\right\}^{1/2}.$$

Proof of Theorem 8.1. We choose integers n_1, n_2 such that $2^{-n_i} < \delta_i \le 2^{-n_i+1}$, $i = 1, 2$. By Lemmas 3.5, 3.6, and Theorem 5.1, we have

$$\begin{aligned}
\omega_{\beta_1,\beta_2}(f^{(r_1,r_2)}, \delta_1, \delta_2)_p &\lesssim \left\| \sum_{\nu_1=2^{n_1}+1}^{\infty} \sum_{\nu_2=n_2+1}^{\infty} \nu_1^{r_1} \nu_2^{r_2} A_{\nu_1\nu_2}(x_1, x_2) \right\|_p \\
&+ 2^{-n_1\beta_1} \left\| \sum_{\nu_1=1}^{2^{n_1}} \sum_{\nu_2=2^{n_2}+1}^{\infty} \nu_1^{r_1+\beta_1} \nu_2^{r_2} A_{\nu_1\nu_2}(x_1, x_2) \right\|_p \\
&+ 2^{-n_2\beta_2} \left\| \sum_{\nu_1=2^{n_1}+1}^{\infty} \sum_{\nu_2=1}^{2^{n_2}} \nu_1^{r_1} \nu_2^{r_2+\beta_2} A_{\nu_1\nu_2}(x_1, x_2) \right\|_p \\
&+ 2^{-n_1\beta_1} 2^{-n_2\beta_2} \left\| \sum_{\nu_1=1}^{2^{n_1}} \sum_{\nu_2=1}^{2^{n_2}} \nu_1^{r_1+\beta_1} \nu_2^{r_2+\beta_2} A_{\nu_1\nu_2}(x_1, x_2) \right\|_p \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

where $A_{\nu_1\nu_2}$ is given by (3.1). Estimating I_2 as follows

$$I_2 \lesssim 2^{-n_1\beta_1} \left\{ \int_0^{2\pi}\int_0^{2\pi} \left[\sum_{\nu_1=0}^{n_1} \sum_{\nu_2=n_2+1}^{\infty} 2^{2\nu_1(r_1+\beta_1)+2\nu_2 r_2} \triangle^2_{\nu_1\nu_2} \right]^{p/2} \mathrm{d}x_1\, \mathrm{d}x_2 \right\}^{1/p},$$

we use

$$2^{2\nu_j r_j} \asymp \left(\sum_{\xi_j=n_j+1}^{\nu_j} 2^{\xi_j r_j \theta} \right)^{2/\theta}, \quad j = 1, 2, \tag{8.2}$$

and Minkowski's inequality. Hence,

$$I_2 \lesssim 2^{-n_1\beta_1} \left\{ \int_0^{2\pi}\int_0^{2\pi} \left[\sum_{\xi_2=n_2+1}^{\infty} 2^{\xi_2 r_2\theta} \left(\sum_{\nu_2=\xi_2}^{\infty} \sum_{\nu_1=0}^{n_1} 2^{2\nu_1(r_1+\beta_1)} \triangle^2_{\nu_1\nu_2} \right)^{\theta/2} \right]^{p/\theta} \mathrm{d}x_1\, \mathrm{d}x_2 \right\}^{1/p}.$$

Applying again Minkowski's inequality for sums and integrals, we have

$$I_2 \lesssim \left\{ 2^{-n_1\beta_1\theta} \sum_{\xi_2=n_2+1}^{\infty} 2^{\xi_2 r_2\theta} \left(\int_0^{2\pi}\int_0^{2\pi} \left\langle \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=\xi_2}^{\infty} 2^{2\nu_1(r_1+\beta_1)} \triangle^2_{\nu_1\nu_2} \right\rangle^{p/2} \mathrm{d}x_1\, \mathrm{d}x_2 \right)^{\theta/p} \right\}^{1/\theta}.$$

Using now

$$2^{-n_j\beta_j\theta} \asymp \sum_{\xi_j=n_j+1}^{\infty} 2^{-\xi_j\beta_j\theta}, \quad j = 1, 2, \tag{8.3}$$

as well as Lemmas 3.5, 3.6, and Theorem 5.1, we get

$$\begin{aligned} I_2 &\lesssim \left\{ \sum_{\xi_1=n_1}^{\infty} 2^{-\xi_1\beta_1\theta} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_2 r_2\theta} \left\| \sum_{\nu_1=1}^{2^{\xi_1}} \sum_{\nu_2=2^{\xi_2}+1}^{\infty} \nu_1^{r_1+\beta_1} A_{\nu_1\nu_2}(x_1,x_2) \right\|_p^{\theta} \right\}^{1/\theta} \\ &\lesssim \left\{ \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\theta} 2^{\xi_2 r_2\theta} \omega^{\theta}_{r_1+\beta_1, r_2+\beta_2}\left(f, \frac{1}{2^{\xi_1}}, \frac{1}{2^{\xi_2}}\right)_p \right\}^{1/\theta}. \end{aligned}$$

For the estimate of I_3 we use the same reasoning:

$$\begin{aligned} I_3 &\lesssim \left\{ \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\theta} 2^{-\xi_2\beta_2\theta} \left\| \sum_{\nu_1=2^{\xi_1}+1}^{\infty} \sum_{\nu_2=1}^{2^{\xi_2}} \nu_2^{r_2+\beta_2} A_{\nu_1\nu_2}(x_1,x_2) \right\|_p^{\theta} \right\}^{1/\theta} \\ &\lesssim \left\{ \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\theta} 2^{\xi_2 r_2\theta} \omega^{\theta}_{r_1+\beta_1, r_2+\beta_2}\left(f, \frac{1}{2^{\xi_1}}, \frac{1}{2^{\xi_2}}\right)_p \right\}^{1/\theta}. \end{aligned}$$

Now we estimate I_1:

$$\begin{aligned} I_1 &\lesssim \left\{ \int_0^{2\pi}\int_0^{2\pi} \left[\sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=n_2+1}^{\infty} 2^{2\nu_1 r_1+2\nu_2 r_2} \triangle^2_{\nu_1\nu_2} \right]^{p/2} \mathrm{d}x_1\, \mathrm{d}x_2 \right\}^{1/p} \\ &\lesssim \left\{ \sum_{\xi_2=n_2}^{\infty} 2^{\xi_2 r_2\theta} \left(\int_0^{2\pi}\int_0^{2\pi} \left\langle \sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=\xi_2+1}^{\infty} 2^{2\nu_1 r_1} \triangle^2_{\nu_1\nu_2} \right\rangle^{p/2} \mathrm{d}x_1\, \mathrm{d}x_2 \right)^{\theta/p} \right\}^{1/\theta} \\ &\lesssim \left\{ \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\theta} 2^{\xi_2 r_2\theta} \left(\int_0^{2\pi}\int_0^{2\pi} \left\langle \sum_{\nu_1=\xi_1+1}^{\infty} \sum_{\nu_2=\xi_2+1}^{\infty} \triangle^2_{\nu_1\nu_2} \right\rangle^{p/2} \mathrm{d}x_1\, \mathrm{d}x_2 \right)^{\theta/p} \right\}^{1/\theta} \\ &\lesssim \left\{ \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\theta} 2^{\xi_2 r_2\theta} \left\| \sum_{\nu_1=2^{\xi_1}+1}^{\infty} \sum_{\nu_2=2^{\xi_2}+1}^{\infty} A_{\nu_1\nu_2}(x_1,x_2) \right\|_p^{\theta} \right\}^{1/\theta} \\ &\lesssim \left\{ \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\theta} 2^{\xi_2 r_2\theta} \omega^{\theta}_{r_1+\beta_1, r_2+\beta_2}\left(f, \frac{1}{2^{\xi_1}}, \frac{1}{2^{\xi_2}}\right)_p \right\}^{1/\theta}. \end{aligned}$$

To estimate I_4, we use the expression (8.3) twice:

$$
\begin{aligned}
I_4 &\lesssim 2^{-n_1\beta_1}2^{-n_2\beta_2}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\nu_1=0}^{n_1}\sum_{\nu_2=0}^{n_2}2^{2\nu_1(r_1+\beta_1)+2\nu_2(r_2+\beta_2)}\triangle^2_{\nu_1\nu_2}\right]^{p/2}\,\mathrm{d}x_1\,\mathrm{d}x_2\right\}^{1/p}\\
&\lesssim \left\{\sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty}2^{-\xi_1\beta_1\theta}2^{-\xi_2\beta_2\theta}\left\|\sum_{\nu_1=1}^{2^{\xi_1}}\sum_{\nu_2=1}^{2^{\xi_2}}\nu_1^{r_1+\beta_1}\nu_2^{r_2+\beta_2}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\theta}\right\}^{1/\theta}\\
&\lesssim \left\{\sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty}2^{\xi_1r_1\theta}2^{\xi_2r_2\theta}\omega^{\theta}_{r_1+\beta_1,r_2+\beta_2}\left(f,\frac{1}{2^{\xi_1}},\frac{1}{2^{\xi_2}}\right)_p\right\}^{1/\theta}.
\end{aligned}
$$

Collecting estimates for $I_j, j=1,2,3,4,$ we finally get

$$
\omega_{\beta_1,\beta_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_p \lesssim \left\{\sum_{\xi_1=n_1+1}^{\infty}\sum_{\xi_2=n_2+1}^{\infty}2^{\xi_1r_1\theta}2^{\xi_2r_2\theta}\omega^{\theta}_{r_1+\beta_1,r_2+\beta_2}\left(f,\frac{1}{2^{\xi_1}},\frac{1}{2^{\xi_2}}\right)_p\right\}^{1/\theta}
$$

and (8.1) follows.

Now let us prove the reverse inequality. We denote

$$
\begin{aligned}
K &:= \int_0^{\delta_1}\int_0^{\delta_2}t_1^{-r_1\tau-1}t_2^{-r_2\tau-1}\omega^{\tau}_{r_1+\beta_1,r_2+\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\,\mathrm{d}t_1\,\mathrm{d}t_2\\
&\lesssim \sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty}2^{\xi_1r_1\tau}2^{\xi_2r_2\tau}\omega^{\tau}_{r_1+\beta_1,r_2+\beta_2}\left(f,\frac{1}{2^{\xi_1}},\frac{1}{2^{\xi_2}}\right)_p.
\end{aligned}
$$

Using Lemmas 3.5, 3.6, and Theorem 5.1,

$$
\begin{aligned}
K &\lesssim \sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty}2^{\xi_1r_1\tau}2^{\xi_2r_2\tau}\left\|\sum_{\nu_1=2^{\xi_1}+1}^{\infty}\sum_{\nu_2=2^{\xi_2}+1}^{\infty}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\tau}\\
&+ \sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty}2^{\xi_1r_1\tau}2^{-\xi_2\beta_2\tau}\left\|\sum_{\nu_1=2^{\xi_1}+1}^{\infty}\sum_{\nu_2=1}^{2^{\xi_2}}\nu_2^{r_2+\beta_2}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\tau}\\
&+ \sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty}2^{-\xi_1\beta_1\tau}2^{\xi_2r_2\tau}\left\|\sum_{\nu_1=1}^{2^{\xi_1}}\sum_{\nu_2=2^{\xi_2}+1}^{\infty}\nu_1^{r_1+\beta_1}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\tau}\\
&+ \sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty}2^{-\xi_1\beta_1\tau}2^{-\xi_2\beta_2\tau}\left\|\sum_{\nu_1=1}^{2^{\xi_1}}\sum_{\nu_2=1}^{2^{\xi_2}}\nu_1^{r_1+\beta_1}\nu_2^{r_2+\beta_2}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\tau}\\
&=: K_1+K_2+K_3+K_4.
\end{aligned}
$$

Let us estimate K_2.

$$\begin{aligned}
K_2 &\lesssim \sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\tau}2^{-\xi_2\beta_2\tau}\left\|\sum_{\nu_1=2^{\xi_1}+1}^{\infty}\sum_{\nu_2=1}^{2^{n_2}}\nu_2^{r_2+\beta_2}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\tau}+\\
&+ \sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\tau}2^{-\xi_2\beta_2\tau}\left\|\sum_{\nu_1=2^{\xi_1}+1}^{\infty}\sum_{\nu_2=2^{n_2}+1}^{2^{\xi_2}}\nu_2^{r_2+\beta_2}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\tau}\\
&=: K_{21}+K_{22}.
\end{aligned}$$

Using (8.3) and Lemmas 3.5, 3.6, we get

$$K_{21}\lesssim 2^{-n_2\beta_2\tau}\sum_{\xi_1=n_1}^{\infty}2^{\xi_1 r_1\tau}\left(\int_0^{2\pi}\int_0^{2\pi}\left\langle\sum_{\nu_1=\xi_1+1}^{\infty}\sum_{\nu_2=0}^{n_2}2^{2\nu_2(r_2+\beta_2)}\triangle_{\nu_1\nu_2}^2\right\rangle^{p/2}\mathrm{d}x_1\,\mathrm{d}x_2\right)^{\tau/p}.$$

Using Minkowski's inequality, (8.2), Lemmas 3.5, 3.6, and Theorem 5.1, we have

$$\begin{aligned}
K_{21} &\lesssim 2^{-n_2\beta_2\tau}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\xi_1=n_1}^{\infty}2^{\xi_1 r_1\tau}\left\langle\sum_{\nu_1=\xi_1+1}^{\infty}\sum_{\nu_2=0}^{n_2}2^{2\nu_2(r_2+\beta_2)}\triangle_{\nu_1\nu_2}^2\right\rangle^{\tau/2}\right]^{p/\tau}\mathrm{d}x_1\,\mathrm{d}x_2\right\}^{\tau/p}\\
&\lesssim 2^{-n_2\beta_2\tau}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\nu_1=n_1+1}^{\infty}\left\langle\sum_{\xi_1=n_1}^{\nu_1}2^{\xi_1 r_1\tau}\right\rangle^{2/\tau}\sum_{\nu_2=0}^{n_2}2^{2\nu_2(r_2+\beta_2)}\triangle_{\nu_1\nu_2}^2\right]^{p/2}\mathrm{d}x_1\,\mathrm{d}x_2\right\}^{\tau/p}\\
&\lesssim 2^{-n_2\beta_2\tau}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=0}^{n_2}2^{2\nu_1 r_1}2^{2\nu_2(r_2+\beta_2)}\triangle_{\nu_1\nu_2}^2\right]^{p/2}\mathrm{d}x_1\,\mathrm{d}x_2\right\}^{\tau/p}\\
&\lesssim 2^{-n_2\beta_2\tau}\left\|\sum_{\nu_1=2^{n_1}+1}^{\infty}\sum_{\nu_2=1}^{2^{n_2}}\nu_1^{r_1}\nu_2^{r_2+\beta_2}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\tau}\\
&\lesssim \omega_{\beta_1,\beta_2}^{\tau}(f^{(r_1,r_2)},\delta_1,\delta_2)_p.
\end{aligned}$$

Estimating K_{22} as above, we get:

$$\begin{aligned}
K_{22} &\lesssim \sum_{\xi_2=n_2}^{\infty}2^{-\xi_2\beta_2\tau}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\xi_2}2^{2\nu_1 r_1}2^{2\nu_2(r_2+\beta_2)}\triangle_{\nu_1\nu_2}^2\right]^{p/2}\mathrm{d}x_1\,\mathrm{d}x_2\right\}^{\tau/p}\\
&\lesssim \left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\xi_2=n_2}^{\infty}2^{-\xi_2\beta_2\tau}\left\langle\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\xi_2}2^{2\nu_1 r_1}2^{2\nu_2(r_2+\beta_2)}\triangle_{\nu_1\nu_2}^2\right\rangle^{\tau/2}\right]^{p/\tau}\mathrm{d}x_1\,\mathrm{d}x_2\right\}^{\tau/p}.
\end{aligned}$$

Further,

$$
\begin{aligned}
K_{22} &\lesssim \left\{ \int_0^{2\pi}\int_0^{2\pi} \left[\sum_{\nu_2=n_2+1}^{\infty} 2^{2\nu_2(r_2+\beta_2)} \left\langle \sum_{\xi_2=\nu_2}^{\infty} 2^{-\xi_2\beta_2\tau} \right\rangle^{2/\tau} \sum_{\nu_1=n_1+1}^{\infty} 2^{2\nu_1 r_1} \triangle^2_{\nu_1\nu_2} \right]^{p/2} \mathrm{d}x_1\,\mathrm{d}x_2 \right\}^{\tau/p} \\
&\lesssim \left\| \sum_{\nu_1=2^{n_1}+1}^{\infty} \sum_{\nu_2=2^{n_2}+1}^{\infty} \nu_1^{r_1}\nu_2^{r_2} A_{\nu_1\nu_2}(x_1,x_2) \right\|_p^{\tau} \\
&\lesssim \omega^{\tau}_{\beta_1,\beta_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_p.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
K_3 &\lesssim 2^{-n_1\beta_1\tau} \left\| \sum_{\nu_1=1}^{2^{n_1}} \sum_{\nu_2=2^{n_2}+1}^{\infty} \nu_1^{r_1+\beta_1}\nu_2^{r_2} A_{\nu_1\nu_2}(x_1,x_2) \right\|_p^{\tau} + \left\| \sum_{\nu_1=2^{n_1}+1}^{\infty} \sum_{\nu_2=2^{n_2}+1}^{\infty} \nu_1^{r_1}\nu_2^{r_2} A_{\nu_1\nu_2}(x_1,x_2) \right\|_p^{\tau} \\
&\lesssim \omega^{\tau}_{\beta_1,\beta_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_p.
\end{aligned}
$$

Finally, we obtain the estimates for K_1 and K_4:

$$
\begin{aligned}
K_1 &\lesssim \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\tau} 2^{\xi_2 r_2\tau} \left\{ \int_0^{2\pi}\int_0^{2\pi} \left[\sum_{\nu_1=\xi_1+1}^{\infty} \sum_{\nu_2=\xi_2+1}^{\infty} \triangle^2_{\nu_1\nu_2} \right]^{p/2} \mathrm{d}x_1\,\mathrm{d}x_2 \right\}^{\tau/p} \\
&\lesssim \left\| \sum_{\nu_1=2^{n_1}+1}^{\infty} \sum_{\nu_2=2^{n_2}+1}^{\infty} \nu_1^{r_1}\nu_2^{r_2} A_{\nu_1\nu_2}(x_1,x_2) \right\|_p^{\tau} \lesssim \omega^{\tau}_{\beta_1,\beta_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_p
\end{aligned}
$$

and

$$
\begin{aligned}
K_4 &\lesssim \left\| \sum_{\nu_1=2^{n_1}+1}^{\infty} \sum_{\nu_2=2^{n_2}+1}^{\infty} \nu_1^{r_1}\nu_2^{r_2} A_{\nu_1\nu_2}(x_1,x_2) \right\|_p^{\tau} + 2^{-n_2\beta_2\tau} \left\| \sum_{\nu_1=2^{n_1}+1}^{\infty} \sum_{\nu_2=1}^{2^{n_2}} \nu_1^{r_1}\nu_2^{r_2+\beta_2} A_{\nu_1\nu_2}(x_1,x_2) \right\|_p^{\tau} \\
&+ 2^{-n_1\beta_1\tau} \left\| \sum_{\nu_1=1}^{2^{n_1}} \sum_{\nu_2=2^{n_2}+1}^{\infty} \nu_1^{r_1+\beta_1}\nu_2^{r_2} A_{\nu_1\nu_2}(x_1,x_2) \right\|_p^{\tau} \\
&+ 2^{-n_1\beta_1\tau-n_2\beta_2\tau} \left\| \sum_{\nu_1=1}^{2^{n_1}} \sum_{\nu_2=1}^{2^{n_2}} \nu_1^{r_1+\beta_1}\nu_2^{r_2+\beta_2} A_{\nu_1\nu_2}(x_1,x_2) \right\|_p^{\tau} \lesssim \omega^{\tau}_{\beta_1,\beta_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_p.
\end{aligned}
$$

□

Proof of Theorems 8.2 and 8.3. These proofs are similar to the proof of Theorem 8.1. We use the following two statements which follow from Lemmas 3.8, 3.9 and Theorems 7.2, 7.3.

Let $f \in M_p, 1<p<\infty, \beta_i, r_i>0, i=1,2,\ n_1, n_2 \in \mathbb{N}$. Then

$$\begin{aligned}\omega_{\beta_1,\beta_2}\Big(f^{(r_1,r_2)}, \frac{1}{n_1}, \frac{1}{n_2}\Big)_{L_p(\mathbb{T}^2)} \asymp\ & \frac{1}{n_1^{\beta_1} n_2^{\beta_2}}\left\{\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2}^p \nu_1^{(\beta_1+r_1+1)p-2}\nu_2^{(\beta_2+r_2+1)p-2}\right\}^{1/p}\\ &+ \frac{1}{n_1^{\beta_1}}\left\{\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^p \nu_1^{(\beta_1+r_1+1)p-2}\nu_2^{(r_2+1)p-2}\right\}^{1/p}\\ &+ \frac{1}{n_2^{\beta_2}}\left\{\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2}^p \nu_1^{(r_1+1)p-2}\nu_2^{(\beta_2+r_2+1)p-2}\right\}^{1/p}\\ &+ \left\{\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^p \nu_1^{(r_1+1)p-2}\nu_2^{(r_2+1)p-2}\right\}^{1/p}.\end{aligned}$$

Let $f \in \Lambda_p, 1<p<\infty, \beta_i, r_i>0, n_i \in \mathbb{N}, i=1,2$. Then

$$\begin{aligned}\omega_{\beta_1,\beta_2}\Big(f^{(r_1,r_2)}, \frac{1}{2^{n_1}}, \frac{1}{2^{n_2}}\Big)_{p} \asymp\ & \frac{1}{2^{n_1\beta_1+n_2\beta_2}}\left\{\sum_{\mu_1=0}^{n_1}\sum_{\mu_2=0}^{n_2} \lambda_{\mu_1,\mu_2}^2 2^{2(\mu_1(\beta_1+r_1)+\mu_2(\beta_2+r_2))}\right\}^{1/2}\\ &+ \frac{1}{2^{n_1\beta_1}}\left\{\sum_{\mu_1=0}^{n_1}\sum_{\mu_2=n_2+1}^{\infty} \lambda_{\mu_1,\mu_2}^2 2^{2(\mu_1(\beta_1+r_1)+\mu_2 r_2)}\right\}^{1/2}\\ &+ \frac{1}{2^{n_2\beta_2}}\left\{\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=0}^{n_2} \lambda_{\mu_1,\mu_2}^2 2^{2(\mu_1 r_1+\mu_2(\beta_2+r_2))}\right\}^{1/2}\\ &+ \left\{\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=n_2+1}^{\infty} \lambda_{\mu_1,\mu_2}^2 2^{2\mu_1 r_1+2\mu_2 r_2}\right\}^{1/2}.\end{aligned}$$

$\square$

Note that Theorems 8.1-8.3 deal with the most important case when $\omega_\beta(f^{(\mathbf{r})}, \delta)_p$ is estimated in terms of $\omega_{\mathbf{r}+\beta}(f, \mathbf{t})_p$. However, to completely solve the general problem on the interrelation between

$$\omega_\beta(f^{(\mathbf{l})}, \delta)_p \qquad \text{and} \qquad \omega_{\mathbf{r}+\beta}(f, \mathbf{t})_p,$$

we have to consider two more cases:

(i). $\quad \mathbf{l} = \mathbf{r} - \alpha, \quad \mathbf{0} < \alpha < \mathbf{r};$

(ii). $\quad \mathbf{l} = \mathbf{r} + \alpha, \quad \mathbf{0} < \alpha < \beta.$

The results covering these cases are provided below:

Theorem 8.4 (i). *Let $f \in L_p^0(\mathbb{T}^2)$, $1<p<\infty$, $\theta := \min(2,p)$, and $\tau := \max(2,p)$. Let $\beta_1, \beta_2, > 0$, $0<\alpha_1<r_1$, $0<\alpha_2<r_2$, and $\delta_1, \delta_2 \in (0, \frac{1}{2})$.*
A). *If*

$$\int_0^1\int_0^1 t_1^{-(r_1-\alpha_1)\theta-1} t_2^{-(r_2-\alpha_2)\theta-1} \omega_{r_1+\beta_1, r_2+\beta_2}^\theta(f, t_1, t_2)_{L_p(\mathbb{T}^2)}\, \mathrm{d}t_1\, \mathrm{d}t_2 < \infty,$$

then f has a mixed derivative $f^{(r_1-\alpha_1,r_2-\alpha_2)} \in L_p^0(\mathbb{T}^2)$ in the sense of Weyl and

$$\omega_{\beta_1,\beta_2}(f^{(r_1-\alpha_1,r_2-\alpha_2)},\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \lesssim J(\theta),$$

where

$$J(s) := \left\{ \int_0^1 \int_0^1 t_1^{-(r_1-\alpha_1)s} \min\left(1, \frac{\delta_1}{t_1}\right)^{\beta_1 s} t_2^{-(r_2-\alpha_2)s} \min\left(1, \frac{\delta_2}{t_2}\right)^{\beta_2 s} \omega^s_{r_1+\beta_1,r_2+\beta_2}(f,t_1,t_2)_p \, \frac{\mathrm{d}t_1}{t_1} \, \frac{\mathrm{d}t_2}{t_2} \right\}^{1/s}.$$

B). *If $f \in L_p(\mathbb{T}^2)$ has a mixed derivative $f^{(r_1-\alpha_1,r_2-\alpha_2)} \in L_p^0(\mathbb{T}^2)$ in the sense of Weyl, then*

$$\omega_{\beta_1,\beta_2}(f^{(r_1-\alpha_1,r_2-\alpha_2)},\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \gtrsim J(\tau).$$

(ii). *Let $f \in L_p^0(\mathbb{T}^2)$, $1 < p < \infty$, $\theta := \min(2,p)$, and $\tau := \max(2,p)$. Let $r_1, r_2 > 0$, $0 < \alpha_1 < \beta_1$, $0 < \alpha_2 < \beta_2$, and $\delta_1, \delta_2 \in (0, \frac{1}{2})$.*
A). *If*

$$\int_0^1 \int_0^1 t_1^{-(r_1+\alpha_1)\theta-1} t_2^{-(r_2+\alpha_2)\theta-1} \omega^\theta_{r_1+\beta_1,r_2+\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)} \, \mathrm{d}t_1 \, \mathrm{d}t_2 < \infty, \tag{8.4}$$

then f has a mixed derivative $f^{(r_1+\alpha_1,r_2+\alpha_2)} \in L_p^0(\mathbb{T}^2)$ in the sense of Weyl and

$$D(f^{(r_1+\alpha_1,r_2+\alpha_2)},\tau) \lesssim E(f,\theta),$$

where

$$D(f^{(r_1+\alpha_1,r_2+\alpha_2)},s) := \left\{ \int_{\delta_1}^1 \int_{\delta_2}^1 \left(\frac{\delta_1}{t_1}\right)^{(\beta_1-\alpha_1)s} \left(\frac{\delta_2}{t_2}\right)^{(\beta_2-\alpha_2)s} \omega^s_{\beta_1,\beta_2}(f^{(r_1+\alpha_1,r_2+\alpha_2)},t_1,t_2)_{L_p(\mathbb{T}^2)} \, \frac{\mathrm{d}t_1}{t_1} \, \frac{\mathrm{d}t_2}{t_2} \right\}^{1/s}$$

and

$$E(f,s) := \left\{ \int_0^{\delta_1} \int_0^{\delta_2} t_1^{-(r_1+\alpha_1)s-1} t_2^{-(r_2+\alpha_2)s-1} \omega^\theta_{r_1+\beta_1,r_2+\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)} \, \mathrm{d}t_1 \, \mathrm{d}t_2 \right\}^{1/s}.$$

B). *If $f \in L_p(\mathbb{T}^2)$ has a mixed derivative $f^{(r_1+\alpha_1,r_2+\alpha_2)} \in L_p^0(\mathbb{T}^2)$ in the sense of Weyl, then*

$$E(f,\tau) \lesssim D(f^{(r_1+\alpha_1,r_2+\alpha_2)},\theta).$$

The proof can be found in [58]; see also [59]. As in Theorems 8.2 and 8.3, the sharpness of this result can be shown by considering function classes M_p and Λ_p defined in Section 2.4.

Moreover, in part (ii), $D(f^{(r_1+\alpha_1,r_2+\alpha_2)},\theta)$ could not be replaced by $\omega_{\beta_1,\beta_2}(f^{(r_1+\alpha_1,r_2+\alpha_2)},\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}$. Indeed, we consider

$$f_1(x,y) := \sum_{n_1=1}^\infty \sum_{n_2=1}^\infty \frac{n_1^{-(r_1+\beta_1+1-\frac{1}{p})}}{\ln^{A_1+\frac{1}{p}}(2n_1)} \, \frac{n_2^{-(r_2+\beta_2+1-\frac{1}{p})}}{\ln^{A_2+\frac{1}{p}}(2n_2)} \cos n_1 x \, \cos n_2 y,$$

where $p \in (1,\infty)$, $r_i, \beta_i, A_i > 0\,(i = 1,2)$. Then, by Lemma 3.8, we have $f \in L_p^0(\mathbb{T}^2)$ and, by Theorem 7.2,

$$\omega_{r_1+\beta_1,r_2+\beta_2}(f_1,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \asymp \delta_1^{r_1+\beta_1}\delta_2^{r_2+\beta_2}.$$

Hence condition (8.4) holds and $f_1^{(r_1+\alpha_1,r_2+\alpha_2)} \in L_p^0(\mathbb{T}^2)$. Moreover, one can easily check that

$$D(f_1^{(r_1+\alpha_1,r_2+\alpha_2)},p) \asymp E(f_1,p) \asymp \delta_1^{\beta_1-\alpha_1}\delta_2^{\beta_2-\alpha_2}.$$

On the other hand,

$$\omega_{\beta_1,\beta_2}(f_1^{(r_1+\alpha_1,r_2+\alpha_2)},\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \asymp \delta_1^{\beta_1-\alpha_1}\delta_2^{\beta_2-\alpha_2}\big|\ln\delta_1\big|^{-A_1}\big|\ln\delta_2\big|^{-A_2}$$

and thus $D(f_1^{(r_1+\alpha_1,r_2+\alpha_2)},p)$ and $\omega_{\beta_1,\beta_2}(f_1^{(r_1+\alpha_1,r_2+\alpha_2)})_{L_p(\mathbb{T}^2)}$ indeed have different orders of magnitude.

9 The mixed moduli of smoothness of L_p-functions and their angular approximation

As we already mentioned, Jackson and Bernstein-Stechkin type results are given by the following inequalities:

$$\begin{aligned} Y_{n_1,n_2}(f)_{L_p(\mathbb{T}^2)} &\lesssim \omega_{k_1,k_2}(f,\frac{\pi}{n_1+1},\frac{\pi}{n_2+1})_{L_p(\mathbb{T}^2)} \\ &\lesssim \frac{1}{(n_1+1)^{k_1}(n_2+1)^{k_2}}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{k_1-1}\nu_2^{k_2-1}Y_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathbb{T}^2)}. \end{aligned}$$

The next theorem provides sharper estimates (sharp Jackson and sharp inverse inequality).

Theorem 9.1 *Let $f \in L_p^0(\mathbb{T}^2), 1 < p < \infty$, $\sigma := \max(2,p), \theta := \min(2,p)$, $\alpha_i > 0$, $n_i \in \mathbb{N}$, $i = 1,2$. Then*

$$\begin{aligned} \frac{1}{n_1^{\alpha_1}}\frac{1}{n_2^{\alpha_2}}\left\{\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1\sigma-1}\nu_2^{\alpha_2\sigma-1}Y^{\sigma}_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathbb{T}^2)}\right\}^{1/\sigma} &\lesssim \omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)} \\ &\lesssim \frac{1}{n_1^{\alpha_1}}\frac{1}{n_2^{\alpha_2}}\left\{\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1\theta-1}\nu_2^{\alpha_2\theta-1}Y^{\theta}_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathbb{T}^2)}\right\}^{1/\theta}. \end{aligned}$$

The sharpness of this result follows from considering special function classes, see Section 2.4.

Theorem 9.2 *Let $f \in M_p, 1 < p < \infty, \alpha_i > 0, n_i \in \mathbb{N}, i = 1,2$. Then*

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)} \asymp \frac{1}{n_1^{\alpha_1}}\frac{1}{n_2^{\alpha_2}}\left\{\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}Y^{p}_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathbb{T}^2)}\right\}^{1/p}.$$

Theorem 9.3 *Let $f \in \Lambda_p, 1 < p < \infty, \alpha_i > 0, n_i \in \mathbb{N}, i = 1,2$. Then*

$$\omega_{\alpha_1,\alpha_2}\left(f, \frac{1}{n_1}, \frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)} \asymp \frac{1}{n_1^{\alpha_1}} \frac{1}{n_2^{\alpha_2}} \left\{\sum_{\nu_1=1}^{n_1+1} \sum_{\nu_2=1}^{n_2+1} \nu_1^{2\alpha_1-1} \nu_2^{2\alpha_2-1} Y^2_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathbb{T}^2)}\right\}^{1/2}.$$

Proof of Theorem 9.1. Define

$$I^\sigma := \frac{1}{n^{\alpha_1\sigma} n^{\alpha_2\sigma}} \sum_{\nu_1=1}^{n_1+1} \sum_{\nu_2=1}^{n_2+1} \nu_1^{\alpha_1\sigma-1} \nu_2^{\alpha_2\sigma-1} Y^\sigma_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathbb{T}^2)}.$$

For given $n_i \in \mathbb{N}$ we find integers $m_i \geq 0$ such that $2^{m_i} \leq n_i < 2^{m_i+1}, i = 1,2$. Then using monotonicity properties of $Y_{\nu_1,\nu_2}(f)_{L_p(\mathbb{T}^2)}$ we get

$$I^\sigma \lesssim 2^{-(\alpha_1 m_1 + \alpha_2 m_2)\sigma} \sum_{\mu_1=0}^{m_1+1} \sum_{\mu_2=0}^{m_2+1} 2^{(\mu_1\alpha_1+\mu_2\alpha_2)\sigma} Y^\sigma_{\lfloor 2^{\mu_1-1}\rfloor, \lfloor 2^{\mu_2-1}\rfloor}(f)_{L_p(\mathbb{T}^2)}.$$

Using Lemmas 3.5 and 3.6, we have

$$\begin{aligned}
I^\sigma &\lesssim 2^{-(\alpha_1 m_1 + \alpha_2 m_2)\sigma} \sum_{\mu_1=0}^{m_1+1} \sum_{\mu_2=0}^{m_2+1} 2^{(\mu_1\alpha_1+\mu_2\alpha_2)\sigma} \left\{\int_0^{2\pi}\int_0^{2\pi} \left(\sum_{\nu_1=\mu_1}^{\infty} \sum_{\nu_2=\mu_2}^{\infty} \Delta^2_{\nu_1\nu_2}\right)^{p/2} \mathrm{d}x\,\mathrm{d}y\right\}^{\sigma/p} \\
&\lesssim \left\{\int_0^{2\pi}\int_0^{2\pi} \left(\sum_{\nu_1=m_1+1}^{\infty} \sum_{\nu_2=m_2+1}^{\infty} \Delta^2_{\nu_1\nu_2}\right)^{p/2} \mathrm{d}x\,\mathrm{d}y\right\}^{\sigma/p} \\
&+ 2^{-\alpha_1 m_1\sigma} \sum_{\mu_1=0}^{m_1} 2^{\alpha_1\mu_1\sigma} \left\{\int_0^{2\pi}\int_0^{2\pi} \left(\sum_{\nu_1=\mu_1}^{m_1} \sum_{\nu_2=m_2+1}^{\infty} \Delta^2_{\nu_1\nu_2}\right)^{p/2} \mathrm{d}x\,\mathrm{d}y\right\}^{\sigma/p} \\
&+ 2^{-\alpha_2 m_2\sigma} \sum_{\mu_2=0}^{m_2} 2^{\alpha_2\mu_2\sigma} \left\{\int_0^{2\pi}\int_0^{2\pi} \left(\sum_{\nu_1=m_1+1}^{\infty} \sum_{\nu_2=\mu_2}^{m_2} \Delta^2_{\nu_1\nu_2}\right)^{p/2} \mathrm{d}x\,\mathrm{d}y\right\}^{\sigma/p} \\
&+ 2^{(-\alpha_1 m_1 - \alpha_2 m_2)\sigma} \sum_{\mu_1=1}^{m_1+1} \sum_{\mu_2=1}^{m_2+1} 2^{(\mu_1\alpha_1+\mu_2\alpha_2)\sigma} \left\{\int_0^{2\pi}\int_0^{2\pi} \left(\sum_{\nu_1=\mu_1}^{m_1} \sum_{\nu_2=\mu_2}^{m_2} \Delta^2_{\nu_1\nu_2}\right)^{p/2} \mathrm{d}x\,\mathrm{d}y\right\}^{\sigma/p} \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By Lemma 3.6, we have

$$I_1 \lesssim \|f - s_{2^{m_1},\infty} - s_{\infty,2^{m_2}} + s_{2^{m_1},2^{m_2}}\|^\sigma_{L_p(\mathbb{T}^2)}.$$

Now we estimate

$$I_2 = 2^{-\alpha_1 m_1\sigma} \sum_{\mu_1=0}^{m_1} 2^{\alpha_1\mu_1\sigma} \left\{\int_0^{2\pi}\int_0^{2\pi} \left(\sum_{\nu_1=\mu_1}^{m_1} \sum_{\nu_2=m_2+1}^{\infty} \Delta^2_{\nu_1\nu_2}\right)^{p/2} \mathrm{d}x\,\mathrm{d}y\right\}^{\sigma/p}.$$

Minkowski's inequality ($\sigma/p \ge 1$) implies

$$I_2 \lesssim 2^{-\alpha_1 m_1 \sigma} \left\{ \int_0^{2\pi}\int_0^{2\pi} \left[\sum_{\mu_1=0}^{m_1} 2^{\alpha_1\mu_1\sigma} \left(\sum_{\nu_1=\mu_1}^{m_1} \sum_{\nu_2=m_2+1}^{\infty} \Delta^2_{\nu_1\nu_2} \right)^{\sigma/2} \right]^{p/\sigma} \mathrm{d}x\,\mathrm{d}y \right\}^{\sigma/p}.$$

Taking into account Lemma 3.2, we get

$$I_2 \lesssim 2^{-\alpha_1 m_1 \sigma} \left\{ \int_0^{2\pi}\int_0^{2\pi} \left[\sum_{\nu_1=0}^{m_1} 2^{\alpha_1\nu_1\sigma} \left(\sum_{\nu_2=m_2+1}^{\infty} \Delta^2_{\nu_1\nu_2} \right)^{\sigma/2} \right]^{p/\sigma} \mathrm{d}x\,\mathrm{d}y \right\}^{\sigma/p}.$$

Since $\sigma/2 \ge 1$, Lemma 3.1 yields

$$I_2 \lesssim 2^{-\alpha_1 m_1 \sigma} \left\{ \int_0^{2\pi}\int_0^{2\pi} \left[\sum_{\nu_1=0}^{m_1} \sum_{\nu_2=m_2+1}^{\infty} 2^{\alpha_1\nu_1 2} \Delta^2_{\nu_1\nu_2} \right]^{p/2} \mathrm{d}x\,\mathrm{d}y \right\}^{\sigma/p}.$$

Then Lemmas 3.5 and 3.6 imply

$$I_2 \lesssim 2^{-\alpha_1 m_1 \sigma} \| s^{(\alpha_1,0)}_{2^{m_1},\infty}(f - s_{\infty,2^{m_2}}(f)) \|^{\sigma}_{L_p(\mathbb{T}^2)}.$$

Similarly,

$$I_3 \lesssim 2^{-\alpha_2 m_2 \sigma} \| s^{(0,\alpha_2)}_{\infty,2^{m_2}}(f - s_{2^{m_1},\infty}(f)) \|^{\sigma}_{L_p(\mathbb{T}^2)}$$

and

$$I_4 \lesssim 2^{-(\alpha_1 m_1 + \alpha_2 m_2)\sigma} \| s^{(\alpha_1,\alpha_2)}_{2^{m_1},2^{m_2}}(f) \|^{\sigma}_{L_p(\mathbb{T}^2)}.$$

Therefore,

$$\begin{aligned} I^\sigma &\lesssim \| f - s_{2^{m_1},\infty} - s_{\infty,2^{m_2}} + s_{2^{m_1},2^{m_2}} \|^{\sigma}_{L_p(\mathbb{T}^2)} \\ &+ 2^{-\alpha_1 m_1} \| s^{(\alpha_1,0)}_{2^{m_1},\infty}(f - s_{\infty,2^{m_2}}(f)) \|^{\sigma}_{L_p(\mathbb{T}^2)} + 2^{-\alpha_2 m_2} \| s^{(0,\alpha_2)}_{\infty,2^{m_2}}(f - s_{2^{m_1},\infty}(f)) \|^{\sigma}_{L_p(\mathbb{T}^2)} \\ &+ 2^{-(\alpha_1 m_1 + \alpha_2 m_2)\sigma} \| s^{(\alpha_1,\alpha_2)}_{2^{m_1},2^{m_2}}(f) \|^{\sigma}_{L_p(\mathbb{T}^2)}. \end{aligned}$$

Applying Theorem 5.1, we get $I \lesssim \omega_{\alpha_1,\alpha_2}\left(f, \frac{1}{2^{m_1}}, \frac{1}{2^{m_2}}\right)_{L_p(\mathbb{T}^2)}$. Then using properties of the mixed modulus of smoothness,

$$I \lesssim \omega_{\alpha_1,\alpha_2}\left(f, \frac{1}{2^{m_1+1}}, \frac{1}{2^{m_2+1}}\right)_{L_p(\mathbb{T}^2)} \lesssim \omega_{\alpha_1,\alpha_2}\left(f, \frac{1}{n_1}, \frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)}.$$

Thus, the estimate from below in Theorem 9.1 is shown. Now we prove the estimate from above. First,

$$I_5 := \omega_{\alpha_1,\alpha_2}\left(f, \frac{1}{n_1}, \frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)} \lesssim \omega_{\alpha_1,\alpha_2}\left(f, \frac{1}{2^{m_1}}, \frac{1}{2^{m_2}}\right)_{L_p(\mathbb{T}^2)},$$

where $2^{m_i} \le n_i < 2^{m_i+1}, i = 1,2$. Theorem 5.1 gives

$$\begin{aligned} I_5^p &\lesssim 2^{-(\alpha_1 m_1 + \alpha_2 m_2)p} \| s^{(\alpha_1,\alpha_2)}_{2^{m_1},2^{m_2}}(f) \|^{p}_{L_p(\mathbb{T}^2)} + 2^{-\alpha_1 m_1 p} \| s^{(\alpha_1,0)}_{2^{m_1},\infty}(f - s_{\infty,2^{m_2}}(f)) \|^{p}_{L_p(\mathbb{T}^2)} \\ &+ 2^{-\alpha_2 m_2 p} \| s^{(0,\alpha_2)}_{\infty,2^{m_2}}(f - s_{2^{m_1},\infty}(f)) \|^{p}_{L_p(\mathbb{T}^2)} + \| f - s_{2^{m_1},\infty} - s_{\infty,2^{m_2}} + s_{2^{m_1},2^{m_2}} \|^{p}_{L_p(\mathbb{T}^2)} \\ &=: I_6 + I_7 + I_8 + I_9. \end{aligned}$$

Lemmas 3.5 and 3.6 imply

$$I_6 \lesssim 2^{-(m_1\alpha_1+m_2\alpha_2)p} \int_0^{2\pi}\int_0^{2\pi} \left(\sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} 2^{2(\mu_1\alpha_1+\mu_2\alpha_2)} \Delta^2_{\mu_1\mu_2} \right)^{p/2} \mathrm{d}x\,\mathrm{d}y.$$

If $2 \le p < \infty$, we use Minkowski's inequality and Lemma 3.3

$$\begin{aligned}
I_6 &\lesssim 2^{-(m_1\alpha_1+m_2\alpha_2)p} \left\{ \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} 2^{2(\mu_1\alpha_1+\mu_2\alpha_2)} \left(\int_0^{2\pi}\int_0^{2\pi} |\Delta_{\mu_1\mu_2}|^p \,\mathrm{d}x\,\mathrm{d}y \right)^{2/p} \right\}^{p/2} \\
&= 2^{(m_1\alpha_1+m_2\alpha_2)p} \left\{ \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} 2^{2(\mu_1\alpha_1+\mu_2\alpha_2)} \| s_{2^{\mu_1},2^{\mu_2}}(f) - s_{2^{\mu_1},\lfloor 2^{\mu_2-1}\rfloor}(f) - \right. \\
&- \left. s_{\lfloor 2^{\mu_1-1}\rfloor,2^{\mu_2}}(f) + s_{\lfloor 2^{\mu_1-1}\rfloor,\lfloor 2^{\mu_2-1}\rfloor}(f) \|^2_{L_p(\mathbb{T}^2)} \right\}^{p/2} \\
&\lesssim 2^{-(m_1\alpha_1+m_2\alpha_2)p} \left\{ \sum_{\mu_1=0}^{m_1+1} \sum_{\mu_2=0}^{m_2+1} 2^{2(\mu_1\alpha_1+\mu_2\alpha_2)} Y^2_{\lfloor 2^{\mu_1-1}\rfloor,\lfloor 2^{\mu_2-1}\rfloor}(f)_{L_p(\mathbb{T}^2)} \right\}^{p/2}.
\end{aligned}$$

If $1 < p < 2$, we use Lemma 3.1 and Lemma 3.3

$$\begin{aligned}
I_6 &\lesssim 2^{-(m_1\alpha_1+m_2\alpha_2)p} \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} 2^{p(\mu_1\alpha_1+\mu_2\alpha_2)} \int_0^{2\pi}\int_0^{2\pi} |\Delta_{\mu_1,\mu_2}|^p \,\mathrm{d}x\,\mathrm{d}y \\
&= 2^{(m_1\alpha_1+m_2\alpha_2)p} \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} 2^{p(\mu_1\alpha_1+\mu_2\alpha_2)} \| s_{2^{\mu_1},2^{\mu_2}}(f) - s_{2^{\mu_1},\lfloor 2^{\mu_2-1}\rfloor}(f) - \\
&- s_{\lfloor 2^{\mu_1-1}\rfloor,2^{\mu_2}}(f) + s_{\lfloor 2^{\mu_1-1}\rfloor,\lfloor 2^{\mu_2-1}\rfloor}(f) \|^p_{L_p(\mathbb{T}^2)} \\
&\lesssim 2^{-(m_1\alpha_1+m_2\alpha_2)p} \sum_{\mu_1=0}^{m_1+1} \sum_{\mu_2=0}^{m_2+1} 2^{p(\mu_1\alpha_1+\mu_2\alpha_2)} Y^p_{\lfloor 2^{\mu_1-1}\rfloor,\lfloor 2^{\mu_2-1}\rfloor}(f)_{L_p(\mathbb{T}^2)}.
\end{aligned}$$

Thus, we show

$$I_6 \lesssim 2^{-(m_1\alpha_1+m_2\alpha_2)p} \left\{ \sum_{\mu_1=0}^{m_1+1} \sum_{\mu_2=0}^{m_2+1} 2^{\theta(\mu_1\alpha_1+\mu_2\alpha_2)} Y^\theta_{\lfloor 2^{\mu_1-1}\rfloor,\lfloor 2^{\mu_2-1}\rfloor}(f)_{L_p(\mathbb{T}^2)} \right\}^{p/\theta}.$$

Now let us estimate I_7. Lemmas 3.5 and 3.6 imply

$$I_7 \lesssim 2^{-m_1\alpha_1 p} \int_0^{2\pi}\int_0^{2\pi} \left(\sum_{\mu_1=0}^{m_1} \sum_{\mu_2=m_2+1}^{\infty} 2^{2\mu_1\alpha_1} \Delta^2_{\mu_1,\mu_2} \right)^{p/2} \mathrm{d}x\,\mathrm{d}y.$$

Again, for $2 \le p < \infty$, Minkowski's inequality, Lemmas 3.6 and 3.3 give

$$\begin{aligned}
I_7 &\lesssim 2^{m_1\alpha_1 p}\left\{\sum_{\mu_1=0}^{m_1} 2^{2\mu_1\alpha_1}\left[\int_0^{2\pi}\int_0^{2\pi}\left(\sum_{\mu_2=m_2+1}^{\infty}\Delta_{\mu_1\mu_2}^2\right)^{p/2}\mathrm{d}x\,\mathrm{d}y\right]^{2/p}\right\}^{p/2}\\
&\lesssim 2^{m_1\alpha_1 p}\left\{\sum_{\mu_1=0}^{m_1} 2^{2\mu_1\alpha_1}\left[\int_0^{2\pi}\int_0^{2\pi}\left(\sum_{\nu_1=\mu_1}^{\infty}\sum_{\mu_2=m_2+1}^{\infty}\Delta_{\nu_1\mu_2}^2\right)^{p/2}\mathrm{d}x\,\mathrm{d}y\right]^{2/p}\right\}^{p/2}\\
&\lesssim 2^{m_1\alpha_1 p}\left\{\sum_{\mu_1=0}^{m_1+1} 2^{2\mu_1\alpha_1}Y^2_{\lfloor 2^{\mu_1-1}\rfloor,2^{m_2}}(f)_{L_p(\mathbb{T}^2)}\right\}^{p/2}\\
&\lesssim 2^{-(m_1\alpha_1+m_2\alpha_2)p}\left\{\sum_{\mu_1=0}^{m_1+1}\sum_{\mu_2=0}^{m_2+1} 2^{2(\mu_1\alpha_1+\mu_2\alpha_2)}Y^2_{\lfloor 2^{\mu_1-1}\rfloor,\lfloor 2^{\mu_2-1}\rfloor}(f)_{L_p(\mathbb{T}^2)}\right\}^{p/2}.
\end{aligned}$$

If $1<p\le 2$, Lemmas 3.1 and 3.3 imply

$$\begin{aligned}
I_7 &\lesssim 2^{-m_1\alpha_1 p}\int_0^{2\pi}\int_0^{2\pi}\sum_{\mu_1=0}^{m_1}\sum_{\mu_2=m_2+1}^{\infty}2^{\mu_1\alpha_1 p}|\Delta_{\mu_1\mu_2}|^p\,\mathrm{d}x\,\mathrm{d}y\\
&\lesssim 2^{-m_1\alpha_1 p}\sum_{\mu_1=0}^{m_1}2^{p\mu_1\alpha_1}\int_0^{2\pi}\int_0^{2\pi}\sum_{\nu_1=\mu_1}^{\infty}\sum_{\mu_2=m_2+1}^{\infty}|\Delta_{\nu_1\mu_2}|^p\,\mathrm{d}x\,\mathrm{d}y\\
&\lesssim 2^{-m_1\alpha_1 p}\sum_{\mu_1=0}^{m_1+1}2^{p\mu_1\alpha_1}Y^p_{\lfloor 2^{\mu_1-1}\rfloor,2^{m_2}}(f)_{L_p(\mathbb{T}^2)}\\
&\lesssim 2^{-(m_1\alpha_1+m_2\alpha_2)p}\sum_{\mu_1=0}^{m_1+1}\sum_{\mu_2=0}^{m_2+1}2^{p(\mu_1\alpha_1+\mu_2\alpha_2)}Y^p_{\lfloor 2^{\mu_1-1}\rfloor,\lfloor 2^{\mu_2-1}\rfloor}(f)_{L_p(\mathbb{T}^2)}.
\end{aligned}$$

Thus,

$$I_7 \lesssim 2^{-(m_1\alpha_1+m_2\alpha_2)p}\left\{\sum_{\mu_1=0}^{m_1+1}\sum_{\mu_2=0}^{m_2+1}2^{\theta(\mu_1\alpha_1+\mu_2\alpha_2)}Y^\theta_{\lfloor 2^{\mu_1-1}\rfloor,\lfloor 2^{\mu_2-1}\rfloor}(f)_{L_p(\mathbb{T}^2)}\right\}^{p/\theta}.$$

Similarly,

$$I_8 \lesssim 2^{-(m_1\alpha_1+m_2\alpha_2)p}\left\{\sum_{\mu_1=0}^{m_1+1}\sum_{\mu_2=0}^{m_2+1}2^{\theta(\mu_1\alpha_1+\mu_2\alpha_2)}Y^\theta_{\lfloor 2^{\mu_1-1}\rfloor,\lfloor 2^{\mu_2-1}\rfloor}(f)_{L_p(\mathbb{T}^2)}\right\}^{p/\theta}.$$

By Lemma 3.3, we get

$$I_9 \lesssim Y^p_{2^{m_1},2^{m_2}}(f)_{L_p(\mathbb{T}^2)}.$$

Combining these estimates, we have

$$I_5^p \lesssim 2^{-(m_1\alpha_1+m_2\alpha_2)p}\left\{\sum_{\mu_1=0}^{m_1+1}\sum_{\mu_2=0}^{m_2+1} 2^{\theta(\mu_1\alpha_1+\mu_2\alpha_2)} Y^{\theta}_{\lfloor 2^{\mu_1-1}\rfloor,\lfloor 2^{\mu_2-1}\rfloor}(f)_{L_p(\mathbb{T}^2)}\right\}^{p/\theta},$$

and using properties of $Y_{\nu_1,\nu_2}(f)_{L_p(\mathbb{T}^2)}$,

$$I_5 \lesssim \frac{1}{n_1^{\alpha_1}}\frac{1}{n_2^{\alpha_2}}\left\{\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1\theta-1}\nu_2^{\alpha_2\theta-1}Y^{\theta}_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathbb{T}^2)}\right\}^{1/\theta}.$$

Thus, the proof of Theorem 9.1 is complete. □

Proof of Theorem 9.2. Define

$$I^p := \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}Y^{p}_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathbb{T}^2)}.$$

By Theorem 4.2, we have

$$I^p \lesssim \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}\omega^p_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{\nu_1+1},\frac{1}{\nu_2+1}\right)_{L_p(\mathbb{T}^2)}.$$

Theorem 7.2 implies

$$\begin{aligned}
I^p \ \lesssim\ & \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\frac{\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}}{(\nu_1+1)^{(\lfloor\alpha_1\rfloor+1)p}(\nu_2+1)^{(\lfloor\alpha_2\rfloor+1)p}}\left[\sum_{\mu_1=1}^{\nu_1}\sum_{\mu_2=1}^{\nu_2}a^p_{\mu_1,\mu_2}\mu_1^{(\lfloor\alpha_1\rfloor+2)p-2}\mu_2^{(\lfloor\alpha_2\rfloor+2)p-2}\right]\\
+\ & \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\frac{\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}}{(\nu_1+1)^{(\lfloor\alpha_1\rfloor+1)p}}\left[\sum_{\mu_1=1}^{\nu_1}\sum_{\mu_2=\nu_2+1}^{\infty}a^p_{\mu_1,\mu_2}\mu_1^{(\lfloor\alpha_1\rfloor+2)p-2}\mu_2^{p-2}\right]\\
+\ & \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\frac{\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}}{(\nu_2+1)^{(\lfloor\alpha_2\rfloor+1)p}}\left[\sum_{\mu_1=\nu_1+1}^{\infty}\sum_{\mu_2=1}^{\nu_2}a^p_{\mu_1,\mu_2}\mu_2^{(\lfloor\alpha_1\rfloor+2)p-2}\mu_1^{p-2}\right]\\
+\ & \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}\left[\sum_{\mu_1=\nu_1+1}^{\infty}\sum_{\mu_2=\nu_2+1}^{\infty}a^p_{\mu_1,\mu_2}(\mu_1\mu_2)^{p-2}\right] =: A_1+A_2+A_3+A_4.
\end{aligned}$$

Changing the order of summation, we get

$$A_1 \lesssim \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\mu_1=1}^{n_1+1}\sum_{\mu_2=1}^{n_2+1}a^p_{\mu_1,\mu_2}\mu_1^{(\alpha_1+1)p-2}\mu_2^{(\alpha_2+1)p-2} =: B_1.$$

We proceed by estimating A_2.

$$\begin{aligned}
A_2 \ \lesssim\ & \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{(\alpha_1-\lfloor\alpha_1\rfloor-1)p-1}\nu_2^{\alpha_2 p-1}\sum_{\mu_1=1}^{\nu_1}\sum_{\mu_2=\nu_2}^{n_2}a^p_{\mu_1,\mu_2}\mu_1^{(\lfloor\alpha_1\rfloor+2)p-2}\mu_2^{p-2}\\
+\ & \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{(\alpha_1-\lfloor\alpha_1\rfloor-1)p-1}\nu_2^{\alpha_2 p-1}\sum_{\mu_1=1}^{\nu_1}\sum_{\mu_2=n_2+1}^{\infty}a^p_{\mu_1,\mu_2}\mu_1^{(\lfloor\alpha_1\rfloor+2)p-2}\mu_2^{p-2} =: A_{11}+A_{12}.
\end{aligned}$$

Changing the order of summation, since $\lfloor\alpha_1\rfloor+1>\alpha_1$ and $\alpha_2>0$, we get

$$A_{11}\lesssim \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\mu_1=1}^{n_1+1}\sum_{\mu_2=1}^{n_2+1} a^p_{\mu_1,\mu_2}\mu_1^{(\alpha_1+1)p-2}\mu_2^{(\alpha_2+1)p-2}=B_1.$$

It is clear that

$$A_{12}\lesssim \frac{1}{n_1^{\alpha_1 p}}\sum_{\nu_1=1}^{n_1+1}\nu_1^{(\alpha_1-\lfloor\alpha_1\rfloor-1)p-1}\sum_{\mu_1=1}^{\nu_1}\sum_{\mu_2=n_2+1}^{\infty} a^p_{\mu_1,\mu_2}\mu_1^{(\lfloor\alpha_1\rfloor+2)p-2}\mu_2^{p-2}.$$

Once again, changing the order of summation, we have

$$A_{12}\lesssim \frac{1}{n_1^{\alpha_1 p}}\sum_{\mu_1=1}^{n_1+1}\sum_{\mu_2=n_2+1}^{\infty} a^p_{\mu_1,\mu_2}\mu_1^{(\alpha_1+1)p-2}\mu_2^{p-2}=:B_2.$$

Thus, $A_2\lesssim B_1+B_2$.

Similarly, we get $A_3\lesssim B_1+B_3$ and $A_4\lesssim B_1+B_2+B_3+B_4$, where

$$B_3:=\frac{1}{n_2^{\alpha_2 p}}\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=1}^{n_2+1} a^p_{\mu_1,\mu_2}\mu_1^{p-2}\mu_2^{(\alpha_1+1)p-2}$$

and

$$B_4:=\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=n_2+1}^{\infty} a^p_{\mu_1,\mu_2}(\mu_1\mu_2)^{p-2}.$$

Combining the estimates for A_1, A_2 , A_3, and A_4,

$$\begin{aligned} I^p \ \lesssim\ & \frac{1}{n_1^{\alpha_1 p}n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} a^p_{\nu_1,\nu_2}\nu_1^{(\alpha_1+1)p-2}\nu_2^{(\alpha_2+1)p-2}+\frac{1}{n_1^{\alpha_1 p}}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty} a^p_{\nu_1,\nu_2}\nu_1^{(\alpha_1+1)p-2}\nu_2^{p-2}\\ +\ & \frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=1}^{n_2} a^p_{\nu_1,\nu_2}\nu_1^{p-2}\nu_2^{(\alpha_2+1)p-2}+\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty} a^p_{\nu_1,\nu_2}(\nu_1\nu_2)^{p-2}. \end{aligned}$$

Theorem 7.2 yields $I\lesssim \omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)}$, i.e., the estimate from below is obtained.

To get the estimate from above, by Theorem 7.2, we have

$$\begin{aligned} a^p_{n_1,n_2}(n_1n_2)^{p-1} \ &\lesssim\ \frac{1}{n_1^{(\alpha_1]+1)p}n_2^{(\lfloor\alpha_2\rfloor+1])p}}\sum_{\nu_1=\lfloor n_1/2\rfloor+1}^{n_1}\sum_{\nu_2=\lfloor n_2/2\rfloor+1}^{n_2} a^p_{\nu_1,\nu_2}\nu_1^{(\lfloor\alpha_1\rfloor+2)p-2}\nu_2^{(\lfloor\alpha_2\rfloor+2)p-2}\\ &\lesssim\ \omega^p_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)}. \end{aligned}$$

Moreover,

$$\begin{aligned} n_1^{p-1}\sum_{\nu_2=n_2+1}^{\infty} a^p_{n_1,\nu_2}\nu_2^{p-2} \ &\lesssim\ \frac{1}{n_1^{(\lfloor\alpha_1\rfloor+1)p}}\sum_{\nu_1=\lfloor\frac{n_1}{2}\rfloor+1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty} a^p_{\nu_1,\nu_2}\nu_1^{(\lfloor\alpha_1\rfloor+2)p-2}\nu_2^{p-2}\\ &\lesssim\ \omega^p_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)},\\ n_2^{p-1}\sum_{\nu_1=n_1+1}^{\infty} a^p_{\nu_1,n_2}\nu_1^{p-2} \ &\lesssim\ \omega^p_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)}, \end{aligned}$$

and

$$\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^p \nu_1^{p-2}\nu_2^{p-2} \lesssim \omega^p_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)}.$$

Again, using Theorem 7.2, we have

$$\begin{aligned}\omega^p_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)} &\lesssim \frac{1}{n_1^{\alpha_1 p}n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}(\nu_1^{p-1}\nu_2^{p-1}a^p_{\nu_1,\nu_2})\\ +\frac{1}{n_1^{\alpha_1 p}}\sum_{\nu_1=1}^{n_1}\nu_1^{\alpha_1 p-1}\Big(\nu_1^{p-1}\sum_{\nu_2=n_2+1}^{\infty}a^p_{\nu_1,\nu_2}\nu_2^{p-2}\Big) &+ \frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_2=1}^{n_2}\nu_2^{\alpha_2 p-1}\Big(\nu_2^{p-1}\sum_{\nu_1=n_1+1}^{\infty}a^p_{\nu_1,\nu_2}\nu_1^{p-2}\Big)\\ &+ \sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty}a^p_{\nu_1,\nu_2}(\nu_1\nu_2)^{p-2}.\end{aligned}$$

Then applying the above mentioned inequalities, we have

$$\begin{aligned}\omega^p_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)} &\lesssim \frac{1}{n_1^{\alpha_1 p}n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}\omega^p_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{\nu_1},\frac{1}{\nu_2}\right)_{L_p(\mathbb{T}^2)}\\ &+ \frac{1}{n_1^{\alpha_1 p}}\sum_{\nu_1=1}^{n_1}\nu_1^{\alpha_1 p-1}\omega^p_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{\nu_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)}\\ &+ \frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_2=1}^{n_2}\nu_2^{\alpha_2 p-1}\omega^p_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{n_1},\frac{1}{\nu_2}\right)_{L_p(\mathbb{T}^2)}\\ &+ \omega^p_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)}.\end{aligned}$$

Properties of the mixed moduli of smoothness imply

$$\omega^p_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)} \lesssim \frac{1}{n_1^{\alpha_1 p}n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}\omega^p_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{\nu_1},\frac{1}{\nu_2}\right)_{L_p(\mathbb{T}^2)}.$$

Then, by Theorem 4.2,

$$\begin{aligned}&\omega^p_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)}\\ &\lesssim \frac{1}{n_1^{\alpha_1 p}n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\frac{\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}}{\nu_1^{(\lfloor\alpha_1\rfloor+1)p}\nu_2^{(\lfloor\alpha_2\rfloor+1)p}}\left(\sum_{\mu_1=1}^{\nu_1}\sum_{\mu_2=1}^{\nu_2}\mu_1^{\lfloor\alpha_1\rfloor}\mu_2^{\lfloor\alpha_2\rfloor}Y_{\mu_1-1,\mu_2-1}(f)_{L_p(\mathbb{T}^2)}\right)^p.\end{aligned}$$

By Hardy's inequality (Lemma 3.2), we get

$$\omega^p_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)} \lesssim \frac{1}{n_1^{\alpha_1 p}n_2^{\alpha_2 p}}\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=1}^{n_2}\mu_1^{\alpha_1 p-1}\mu_2^{\alpha_2 p-1}Y^p_{\mu_1-1,\mu_2-1}(f)_{L_p(\mathbb{T}^2)}.$$

Thus, the proof of Theorem 9.2 is now complete. $\square$

Proof of Theorem 9.3. Here, it is enough to show that $\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{2^{m_1}},\frac{1}{2^{m_2}}\right)_{L_p(\mathbb{T}^2)} \asymp I$, where

$$I := 2^{-(m_1\alpha_1+m_2\alpha_2)}\left\{\sum_{\mu_1=1}^{m_1}\sum_{\mu_2=1}^{m_2} 2^{2(\mu_1\alpha_1+\mu_2\alpha_2)} Y^2_{2^{\mu_1}-1,2^{\mu_2}-1}(f)_{L_p(\mathbb{T}^2)}\right\}^{1/2}.$$

Lemmas 3.3 and 3.9 imply

$$\begin{aligned} I &\asymp 2^{-(m_1\alpha_1+m_2\alpha_2)} \\ &\quad \left\{\sum_{\mu_1=1}^{m_1}\sum_{\mu_2=1}^{m_2} 2^{2(\mu_1\alpha_1+\mu_2\alpha_2)}\|f-s_{2^{\mu_1}-1,\infty}(f)-s_{\infty,2^{\mu_2}-1}(f)+s_{2^{\mu_1}-1,2^{\mu_2}-1}(f)\|^2_{L_p(\mathbb{T}^2)}\right\}^{1/2} \\ &\asymp 2^{-(m_1\alpha_1+m_2\alpha_2)}\left\{\sum_{\mu_1=1}^{m_1}\sum_{\mu_2=1}^{m_2} 2^{2(\mu_1\alpha_1+\mu_2\alpha_2)}\sum_{\nu_1=\mu_1}^{\infty}\sum_{\nu_2=\mu_2}^{\infty}\lambda^2_{\mu_1,\mu_2}\right\}^{1/2} \\ &\asymp 2^{-(m_1\alpha_1+m_2\alpha_2)}\left\{\sum_{\mu_1=1}^{m_1}\sum_{\mu_2=1}^{m_2}\lambda^2_{\mu_1,\mu_2}2^{2(\mu_1\alpha_1+\mu_2\alpha_2)}\right\}^{1/2} + 2^{-m_1\alpha_1}\left\{\sum_{\mu_1=1}^{m_1}\sum_{\mu_2=m_2+1}^{\infty}\lambda^2_{\mu_1,\mu_2}2^{2\mu_1\alpha_1}\right\}^{1/2} \\ &+ 2^{-m_2\alpha_2}\left\{\sum_{\mu_1=m_1+1}^{\infty}\sum_{\mu_2=1}^{m_2}\lambda^2_{\mu_1,\mu_2}2^{2\mu_2\alpha_2}\right\}^{1/2} + \left\{\sum_{\mu_1=m_1+1}^{\infty}\sum_{\mu_2=m_2+1}^{\infty}\lambda^2_{\mu_1,\mu_2}\right\}^{1/2}. \end{aligned}$$

Using now Theorem 7.3, we finally get $I \asymp \omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{2^{m_1}},\frac{1}{2^{m_2}}\right)_{L_p(\mathbb{T}^2)}$. □

Theorem 9.1 was proved in the case of $\alpha_i \in \mathbb{N}$ in the paper [55]. For the proof of Theorem 9.2, see [31]. The one-dimensional sharp Jackson inequality was proved in [87]. The history of this topic and new results can be found in [14].

10 Interrelation between the mixed moduli of smoothness of different orders in L_p

We have stated above (see Theorem 4.1) that the following properties of mixed moduli are well known:

$$\begin{aligned} \omega_{\beta_1,\beta_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} &\lesssim \omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}; \\ \frac{\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}}{\delta_1^{\alpha_1}\delta_2^{\alpha_2}} &\lesssim \frac{\omega_{\beta_1,\beta_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}}{\delta_1^{\beta_1}\delta_2^{\beta_2}}; \\ \omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} &\lesssim \delta_1^{\alpha_1}\delta_2^{\alpha_2}\int_{\delta_1}^{1}\int_{\delta_2}^{1}\frac{\omega_{\beta_1,\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}}{t_1^{\alpha_1}t_2^{\alpha_2}}\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}, \end{aligned}$$

where $f \in L_p^0(\mathbb{T}^2)$, $1<p<\infty$, and $0<\alpha_i<\beta_i$, $\delta_i \in (0,\frac{1}{2})$, $i=1,2$.

The following theorem sharpens all these estimates.

Theorem 10.1 *Let $f \in L_p^0(\mathbb{T}^2)$, $1 < p < \infty$, $\tau := \max(2,p)$, $\theta := \min(2,p)$, $0 < \alpha_i < \beta_i$, $\delta_i \in (0, \frac{1}{2})$, $i = 1, 2$. Then*

$$\delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\{\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{-\alpha_2}\omega_{\beta_1,\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\right]^{\tau}\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right\}^{1/\tau} \lesssim \omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)}$$

$$\lesssim \delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\{\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{-\alpha_2}\omega_{\beta_1,\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\right]^{\theta}\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right\}^{1/\theta}.$$

These estimates are the best possible in the sense of order. This can be shown using the following two equivalence results for the function classes M_p and Λ_p defined in Section 2.4.

Theorem 10.2 *Let $f \in M_p, 1 < p < \infty$, $0 < \alpha_i < \beta_i$, $\delta_i \in (0, \frac{1}{2}), i = 1, 2$. Then*

$$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \asymp \delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\{\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{-\alpha_2}\omega_{\beta_1,\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\right]^{p}\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right\}^{1/p}.$$

Theorem 10.3 *Let $f \in \Lambda_p, 1 < p < \infty$, $0 < \alpha_i < \beta_i$, $\delta_i \in (0, \frac{1}{2}), i = 1, 2$. Then*

$$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} \asymp \delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\{\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{-\alpha_2}\omega_{\beta_1,\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\right]^{2}\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right\}^{1/2}.$$

Proof of Theorem 10.1. We consider

$$I := \delta_1^{\alpha_1}\delta_2^{\alpha_2}\left(\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{-\alpha_2}\omega_{\beta_1,\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\right]^{\tau}\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right)^{1/\tau}.$$

For given $\delta_i \in (0, \frac{1}{2})$, we take integers n_i such that $\frac{1}{n_i+1} \le \delta_i < \frac{1}{n_i}, i = 1, 2$. Then

$$I^\tau \lesssim n_1^{-\alpha_1\tau}n_2^{-\alpha_2\tau}\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=1}^{n_2}\mu_1^{\alpha_1\tau-1}\mu_2^{\alpha_2\tau-1}\omega_{\beta_1,\beta_2}^{\tau}\left(f,\frac{1}{\mu_1},\frac{1}{\mu_1}\right)_{L_p(\mathbb{T}^2)}.$$

Further, Theorem 9.1 implies

$$I^\tau \lesssim n_1^{-\alpha_1\tau}n_2^{-\alpha_2\tau}\sum_{\mu_1=1}^{n_1+1}\sum_{\mu_2=1}^{n_2+1}\mu_1^{\alpha_1\tau-\beta_1\tau-1}\mu_2^{\alpha_2\tau-\beta_2\tau-1}\left\{\sum_{\nu_1=1}^{\mu_1+1}\sum_{\nu_2=1}^{\mu_2+1}\nu_1^{\beta_1\theta-1}\nu_2^{\beta_2\theta-1}Y_{\nu_1-1,\nu_2-1}^{\theta}(f)_{L_p(\mathbb{T}^2)}\right\}^{\tau/\theta}.$$

Since $\tau/\theta \ge 1$, using Lemma 3.2, we get

$$I^\tau \lesssim n_1^{-\alpha_1\tau}n_2^{-\alpha_2\tau}\sum_{\mu_1=1}^{n_1+1}\sum_{\mu_2=1}^{n_2+1}\mu_1^{\alpha_1\tau-1}\mu_2^{\alpha_2\tau-1}Y_{\mu_1-1,\mu_2-1}^{\tau}(f)_{L_p(\mathbb{T}^2)}.$$

By Theorem 9.1, we have

$$I^\tau \lesssim \omega^\tau_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)},$$

which implies

$$I \lesssim \omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1+1},\frac{1}{n_2+1}\right)_{L_p(\mathbb{T}^2)} \le \omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_p.$$

The estimate "$\lesssim \omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_p$" is proved.

Let us verify the part "$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_p \lesssim$". For given $\delta_i\in(0,\frac12)$, we take integers n_i such that $\frac{1}{n_i+1}\le\delta_i<\frac{1}{n_i}$, $i=1,2$. Then, by Theorem 9.1, we get

$$\begin{aligned}\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} &\lesssim \omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)}\\ &\lesssim n_1^{-\alpha_1}n_2^{-\alpha_2}\left\{\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1\theta-1}\nu_2^{\alpha_2\theta-1}Y^\theta_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathbb{T}^2)}\right\}^{1/\theta}.\end{aligned}$$

Then Theorem 4.2 gives us

$$\begin{aligned}\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathbb{T}^2)} &\lesssim n_1^{-\alpha_1}n_2^{-\alpha_2}\left\{\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1\theta-1}\nu_2^{\alpha_2\theta-1}\omega^\theta_{\beta_1,\beta_2}\left(f,\frac{1}{\nu_1},\frac{1}{\nu_2}\right)_{L_p(\mathbb{T}^2)}\right\}^{1/\theta}\\ &\lesssim \delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\{\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{-\alpha_2}\omega_{\beta_1,\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\right]^\theta\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right\}^{1/\theta},\end{aligned}$$

which completes the proof of Theorem 10.1. $\square$

Proof of Theorem 10.2. Denoting

$$A := \delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\{\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{-\alpha_2}\omega_{\beta_1,\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\right]^p\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right\}^{1/p},$$

and choosing integers n_i such that $\frac{1}{n_i+1}\le\delta_i<\frac{1}{n_i}$, $i=1,2$, we have

$$A^p \asymp n_1^{-\alpha_1 p}n_2^{-\alpha_2 p}\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=1}^{n_2}\mu_1^{\alpha_1 p-1}\mu_2^{\alpha_2 p-1}\omega^p_{\beta_1,\beta_2}\big(f,\frac{1}{\mu_1},\frac{1}{\mu_2}\big)_{L_p(\mathbb{T}^2)}.$$

Then Theorem 9.2 yields

$$\begin{aligned}A^p &\asymp n_1^{-\alpha_1 p}n_2^{-\alpha_2 p}\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=1}^{n_2}\mu_1^{\alpha_1 p-\beta_1 p-1}\mu_2^{\alpha_2 p-\beta_2 p-1}\sum_{\nu_1=1}^{\mu_1+1}\sum_{\nu_2=1}^{\mu_2+1}\nu_1^{\beta_1 p-1}\nu_2^{\beta_2 p-1}Y^p_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathbb{T}^2)}\\ &\asymp n_1^{-\alpha_1 p}n_2^{-\alpha_2 p}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}Y^p_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathbb{T}^2)}.\end{aligned}$$

Finally, by Theorem 9.2, we have $A^p \asymp \omega_{\alpha_1,\alpha_2}\big(f, \frac{1}{n_1}, \frac{1}{n_2}\big)_{L_p(\mathbb{T}^2)}$, which completes the proof. $\square$

Proof of Theorem 10.3. As above, let

$$B := \delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\{\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{-\alpha_2}\omega_{\beta_1,\beta_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\right]^2\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right\}^{1/2}.$$

and $\frac{1}{2^{n_i+1}} \le \delta_i < \frac{1}{2^{n_i}}, i = 1, 2$. Then

$$B^2 \asymp 2^{-n_1\alpha_1}2^{-n_2\alpha_2}\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=1}^{n_2}2^{2\mu_1\alpha_1+2\mu_2\alpha_2}\omega^2_{\beta_1,\beta_2}\left(f,\frac{1}{2^{\mu_1}},\frac{1}{2^{\mu_2}}\right)_{L_p(\mathbb{T}^2)}.$$

Using now Theorem 9.3, we have

$$B^2 \asymp 2^{-n_1\alpha_1}2^{-n_2\alpha_2}\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=1}^{n_2}2^{2\mu_1\alpha_1+2\mu_2\alpha_2-2\mu_1\beta_1-2\mu_2\beta_2}\sum_{\nu_1=1}^{\mu_1+1}\sum_{\nu_2=1}^{\mu_2+1}2^{2\nu_1\beta_1+2\nu_2\beta_2}Y^2_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_{L_p(\mathbb{T}^2)}$$

$$\asymp 2^{-2(n_1\alpha_1+n_2\alpha_2)}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}2^{2(\nu_1\alpha_1+\nu_2\alpha_2)}Y^2_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_{L_p(\mathbb{T}^2)}.$$

By Theorem 9.3, we get $B \asymp \omega_{\alpha_1,\alpha_2}\left(f, \frac{1}{2^{n_1}}, \frac{1}{2^{n_2}}\right)_{L_p(\mathbb{T}^2)} \asymp \omega_{\alpha_1,\alpha_2}\Big(f, \delta_1, \delta_2\Big)_{L_p(\mathbb{T}^2)}$. $\square$

The one-dimensional version of Theorem 10.2 for functions with general monotone coefficients was stated in [83].

11 Interrelation between the mixed moduli of smoothness in various (L_p,L_q) metrics

For functions on $\mathbb{T}$, the classical Ul'yanov inequality ([97], see also [25]) states that

$$\omega_\alpha(f,\delta)_{L_q(\mathbb{T})} \lesssim \left(\int_0^\delta\left[t^{-\theta}\omega_\alpha(f,t)_{L_p(\mathbb{T})}\right]^q\frac{\mathrm{d}t}{t}\right)^{1/q},$$

where $f \in L_p(\mathbb{T}), 1 < p < q < \infty$, $\theta := \frac{1}{p} - \frac{1}{q}$, and $\alpha \in \mathbb{N}$. Very recently, this inequality was generalized using the fractional modulus of smoothness ($\alpha > 0$) as follows (see [60, 75, 91]):

$$\omega_\alpha(f,\delta)_{L_q(\mathbb{T})} \lesssim \left(\int_0^\delta\left[t^{-\theta}\omega_{\alpha+\theta}(f,t)_{L_p(\mathbb{T})}\right]^q\frac{\mathrm{d}t}{t}\right)^{1/q}.$$

Below, we present an analogue of the sharp Ul'yanov inequality for the mixed moduli of smoothness (see [61]).

Theorem 11.1 *Let $f \in L_p^0(\mathbb{T}^2), 1<p<q<\infty$, $\theta := \frac{1}{p}-\frac{1}{q}, \alpha_1>0, \alpha_2>0$. Then*

$$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_q(\mathbb{T}^2)} \lesssim \left(\int_0^{\delta_1}\int_0^{\delta_2}\Big[(t_1t_2)^{-\theta}\omega_{\alpha_1+\theta,\alpha_2+\theta}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\Big]^q \frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right)^{1/q}. \tag{11.1}$$

Proof. For given $\delta_i \in (0,1)$, there exist integers n_i such that $\frac{1}{2^{n_i+1}} \le \delta_i < \frac{1}{2^{n_i}}, i=1,2$. Then, by Theorem 5.1, we get

$$\begin{aligned}
I &:= \omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_q(\mathbb{T}^2)} \lesssim \omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{2^{n_1}},\frac{1}{2^{n_2}}\right)_{L_q(\mathbb{T}^2)} \\
&\lesssim \|f - s_{2^{n_1},\infty} - s_{\infty,2^{n_2}} + s_{2^{n_1},2^{n_2}}\|_{L_q(\mathbb{T}^2)} \\
&+ 2^{-\alpha_1 n_1}\|s^{(\alpha_1,0)}_{2^{n_1},\infty}(f - s_{\infty,2^{n_2}})\|_{L_q(\mathbb{T}^2)} + 2^{-\alpha_2 n_2}\|s^{(0,\alpha_2)}_{\infty,2^{n_2}}(f - s_{2^{n_1},\infty})\|_{L_q(\mathbb{T}^2)} \\
&+ 2^{-\alpha_1 n_1-\alpha_2 n_2}\|s^{(\alpha_1,\alpha_2)}_{2^{n_1},2^{n_2}}(f)\|_{L_q(\mathbb{T}^2)} =: I_1+I_2+I_3+I_4.
\end{aligned}$$

To estimate I_1, we use Lemmas 3.3 and 3.4

$$I_1 \lesssim \left\{\sum_{\nu_1=n_1}^{\infty}\sum_{\nu_2=n_2}^{\infty} 2^{(\nu_1+\nu_2)\theta q} Y^q_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_{L_p(\mathbb{T}^2)}\right\}^{1/q}.$$

Further, by Theorem 4.2, we get

$$I_1 \lesssim \left\{\sum_{\nu_1=n_1}^{\infty}\sum_{\nu_2=n_2}^{\infty} 2^{(\nu_1+\nu_2)\theta q}\omega^q_{\alpha_1+\theta,\alpha_2+\theta}\left(f,\frac{1}{2^{\nu_1}},\frac{1}{2^{\nu_2}}\right)_{L_p(\mathbb{T}^2)}\right\}^{1/q} =: J.$$

Let us now estimate I_2. Define $\varphi(x,y) := s^{(\alpha_1,0)}_{2^{n_1},\infty}(f)$. Then $I_2 = 2^{-\alpha_1 n_1}\|\varphi - s_{\infty,2^{n_2}}(\varphi)\|_{L_q(\mathbb{T}^2)}$. Lemma 3.13 implies, for a.e. x,

$$\int_0^{2\pi}|\varphi - s_{\infty,2^{n_2}}(\varphi)|^q\,\mathrm{d}y \lesssim \sum_{\nu_2=n_2}^{\infty} 2^{\nu_2\theta q}\left(\int_0^{2\pi}|\varphi - s_{\infty,2^{\nu_2}}(\varphi)|^p\,\mathrm{d}y\right)^{q/p}.$$

Therefore,

$$\int_0^{2\pi}\int_0^{2\pi}|\varphi - s_{\infty,2^{n_2}}(\varphi)|^q\,\mathrm{d}y\,\mathrm{d}x \lesssim \sum_{\nu_2=n_2}^{\infty} 2^{\nu_2\theta q}\int_0^{2\pi}\left(\int_0^{2\pi}|\varphi - s_{\infty,2^{\nu_2}}(\varphi)|^p\,\mathrm{d}y\right)^{q/p}\mathrm{d}x.$$

Using Minkowski's inequality, we have

$$\int_0^{2\pi}\left(\int_0^{2\pi}|\varphi - s_{\infty,2^{\nu_2}}(\varphi)|^p\,\mathrm{d}y\right)^{q/p}\mathrm{d}x \lesssim \left(\int_0^{2\pi}\left(\int_0^{2\pi}|\varphi - s_{\infty,2^{\nu_2}}(\varphi)|^q\,\mathrm{d}x\right)^{p/q}\mathrm{d}y\right)^{q/p}.$$

Since $\varphi - s_{\infty,2^{\nu_2}}(\varphi) = s^{(\alpha_1,0)}_{2^{n_1},\infty}(f - s_{\infty,2^{\nu_2}}(f))$, then Lemma 3.14 implies, for a.e. y,

$$\left(\int\limits_0^{2\pi} |s^{(\alpha_1,0)}_{2^{n_1},\infty}(f - s_{\infty,2^{\nu_2}}(f))|^q \,\mathrm{d}x\right)^{1/q} \lesssim \left(\int\limits_0^{2\pi} |s^{(\alpha_1+\theta,0)}_{2^{n_1},\infty}(f - s_{\infty,2^{\nu_2}}(f))|^p \,\mathrm{d}x\right)^{1/p}.$$

This gives

$$\int\limits_0^{2\pi}\left(\int\limits_0^{2\pi} |s^{(\alpha_1,0)}_{2^{n_1},\infty}(f - s_{\infty,2^{\nu_2}}(f))|^q \,\mathrm{d}x\right)^{p/q} \mathrm{d}y \lesssim \int\limits_0^{2\pi}\int\limits_0^{2\pi} |s^{(\alpha_1+\theta,0)}_{2^{n_1},\infty}(f - s_{\infty,2^{\nu_2}}(f))|^p \,\mathrm{d}x\,\mathrm{d}y.$$

Thus,

$$\begin{aligned} I_2 &\lesssim 2^{-\alpha_1 n_1}\left\{\sum_{\nu_2=n_2}^{\infty} 2^{\nu_2\theta q}\|s^{(\alpha_1+\theta,0)}_{2^{n_1},\infty}(f - s_{\infty,2^{\nu_2}}(f))\|^q_{L_p(\mathbb{T}^2)}\right\}^{1/q} \\ &= 2^{n_1\theta}\left\{\sum_{\nu_2=n_2}^{\infty} 2^{\nu_2\theta q}\Big[2^{-(\alpha_1+\theta)n_1}\|s^{(\alpha_1+\theta,0)}_{2^{n_1},\infty}(f - s_{\infty,2^{\nu_2}}(f))\|_{L_p(\mathbb{T}^2)}\Big]^q\right\}^{1/q}. \end{aligned}$$

Now using Theorem 5.1, we get

$$I_2 \lesssim \left\{2^{n_1\theta q}\sum_{\nu_2=n_2}^{\infty} 2^{\nu_2\theta q}\omega^q_{\alpha_1+\theta,\alpha_2+\theta}\left(f, \frac{1}{2^{n_1}}, \frac{1}{2^{\nu_2}}\right)_{L_p(\mathbb{T}^2)}\right\}^{1/q} \lesssim J.$$

Similarly, one can show that $I_3 \lesssim J$ and $I_4 \lesssim J$. Combining the estimates for I_1, I_2, I_3, and I_4, we get

$$I \lesssim \left\{\sum_{\nu_1=n_1}^{\infty}\sum_{\nu_2=n_2}^{\infty} 2^{(\nu_1+\nu_2)\theta q}\omega^q_{\alpha_1+\theta,\alpha_2+\theta}\left(f, \frac{1}{2^{\nu_1}}, \frac{1}{2^{\nu_2}}\right)_{L_p(\mathbb{T}^2)}\right\}^{1/q}.$$

Taking into account monotonicity properties of the mixed moduli of smoothness, we finally have

$$I \lesssim \left\{\int\limits_0^{2^{\frac{1}{n_1+1}}}\int\limits_0^{2^{\frac{1}{n_2+1}}} \Big[(t_1t_2)^{-\theta}\omega_{\alpha_1+\theta,\alpha_2+\theta}(f, t_1, t_2)_{L_p(\mathbb{T}^2)}\Big]^q \frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right\}^{1/q} \lesssim$$

$$\lesssim \left\{\int\limits_0^{\delta_1}\int\limits_0^{\delta_2} \Big[(t_1t_2)^{-\theta}\omega_{\alpha_1+\theta,\alpha_2+\theta}(f, t_1, t_2)_{L_p(\mathbb{T}^2)}\Big]^q \frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right\}^{1/q}.$$

□

11.1 Sharpness

Let us show that it is impossible to obtain the reverse part $\gtrsim$ of inequality (11.1), i.e., in general, (11.1) is not an equivalence. Let us construct a function $f_0(x,y) \in L_p^0(\mathbb{T}^2)$ such that the terms on the left- and right-hand side of (11.1) have different orders as functions of δ_1 and δ_2. Consider

$$f_0(x,y) := \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} a_{\nu_1,\nu_2} \cos 2^{\nu_1} x \cos 2^{\nu_2} y, \text{ where } a_{\nu_1,\nu_2} := \frac{(\nu_1+1)^{\beta_1}(\nu_2+1)^{\beta_2}}{2^{\alpha_1\nu_1+\alpha_2\nu_2}},$$

$1 < p < q < \infty$, $\theta := \frac{1}{p} - \frac{1}{q}$, $\beta_i > -\frac{1}{2}, \alpha_i > \theta$, $(i = 1, 2)$. By Theorem 7.3, we get

$$\omega_{\alpha_1,\alpha_2}\left(f_0, \frac{1}{2^{n_1}}, \frac{1}{2^{n_2}}\right)_{L_q(\mathbb{T}^2)} \asymp \frac{(n_1+1)^{\beta_1+\frac{1}{2}}(n_2+1)^{\beta_2+\frac{1}{2}}}{2^{\alpha_1\nu_1+\alpha_2\nu_2}}$$

and

$$\omega_{\alpha_1+\theta,\alpha_2+\theta}\left(f_0, \frac{1}{2^{n_1}}, \frac{1}{2^{n_2}}\right)_{L_p(\mathbb{T}^2)} \asymp \frac{(n_1+1)^{\beta_1}(n_2+1)^{\beta_2}}{2^{\alpha_1\nu_1+\alpha_2\nu_2}}.$$

Using these estimates, one can easily see that

$$\omega_{\alpha_1,\alpha_2}(f_0,\delta_1,\delta_2)_{L_q(\mathbb{T}^2)} \asymp \delta_1^{\alpha_1}\delta_2^{\alpha_2}\left(\ln\frac{2}{\delta_1}\right)^{\beta_1+\frac{1}{2}}\left(\ln\frac{2}{\delta_2}\right)^{\beta_2+\frac{1}{2}}$$

and

$$\left\{\int_0^{\delta_1}\int_0^{\delta_2}\left[(t_1t_2)^{-\theta}\omega_{\alpha_1+\theta,\alpha_2+\theta}\left(f_0,t_1,t_2\right)_{L_p(\mathbb{T}^2)}\right]^q\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right\}^{1/q} \asymp \delta_1^{\alpha_1-\theta}\delta_2^{\alpha_2-\theta}\left(\ln\frac{2}{\delta_1}\right)^{\beta_1}\left(\ln\frac{2}{\delta_2}\right)^{\beta_2}. \tag{11.2}$$

Thus, the inequality

$$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_q(\mathbb{T}^2)} \gtrsim \left(\int_0^{\delta_1}\int_0^{\delta_2}\left[(t_1t_2)^{-\theta}\omega_{\alpha_1+\theta,\alpha_2+\theta}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\right]^q\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right)^{1/q}$$

does not hold for f_0.

Finally, we would like to mention that inequality (11.1) improves the classical Ul'yanov inequality for the mixed moduli of smoothness given by ([54])

$$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_q(\mathbb{T}^2)} \lesssim \left(\int_0^{\delta_1}\int_0^{\delta_2}\left[(t_1t_2)^{-\theta}\omega_{\alpha_1,\alpha_2}(f,t_1,t_2)_{L_p(\mathbb{T}^2)}\right]^q\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right)^{1/q}. \tag{11.3}$$

Indeed, for f_0 we have

$$\omega_{\alpha_1,\alpha_2}(f_0,t_1,t_2)_{L_p(\mathbb{T}^2)} \asymp t_1^{\alpha_1}t_2^{\alpha_2}\left(\ln\frac{2}{t_1}\right)^{\beta_1+\frac{1}{2}}\left(\ln\frac{2}{t_2}\right)^{\beta_2+\frac{1}{2}}$$

and therefore for this function

$$\left\{\int_0^{\delta_1}\int_0^{\delta_2}\Big[(t_1t_2)^{-\theta}\omega_{\alpha_1,\alpha_2}(f_0,t_1,t_2)_{L_p(\mathbb{T}^2)}\Big]^q\frac{\mathrm{d}t_1}{t_1}\frac{\mathrm{d}t_2}{t_2}\right\}^{1/q}\asymp\delta_1^{\alpha_1-\theta}\delta_2^{\alpha_2-\theta}\left(\ln\frac{2}{\delta_1}\right)^{\beta_1+\frac12}\left(\ln\frac{2}{\delta_2}\right)^{\beta_2+\frac12}.$$

Thus, for the function f_0 the integrals in the right-hand sides of (11.1) and (11.3) have different orders of magnitude as functions of δ_1 and δ_2 (cf. (11.2) and (11.3)).

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