Rend. Sem. Mat. Univ. Pol. Torino
Vol. 61, 3 (2003)
Splines and Radial Functions

C. Dagnino - V. Demichelis - E. Santi

ON OPTIMAL NODAL SPLINES AND THEIR APPLICATIONS

Abstract. We present a survey on optimal nodal splines and some their applications. Several approximation properties and the convergence rate, both in the univariate and bivariate case, are reported.

The application of such splines to numerical integration has been considered and a wide class of quadrature and cubature rules is presented for the evaluation of singular integrals, Cauchy principal value and Hadamard finite-part integrals. Convergence results and condition number are given.

Finally, a nodal spline collocation method, for the solution of Volterra integral equations of the second kind with weakly singular kernel, is also reported.

1. Introduction

It is well known that the polynomial spline approximation operators for real-valued functions are of great usefulness in the applications.

In their construction, it is desirable to obtain some nice properties as in particular:

1. the operator can be applied to a wide class of functions, including, for example, continuous or integrable functions;
2. they are local in the sense that can depend only on the values of f in a small neighbourhood of the evaluation point x;
3. the operators allow to approximate smooth functions f with an order of accuracy comparable to the best spline approximation. The key for obtaining operators with such property is to require that they reproduce appropriate class of polynomials.

The approximating splines obtained by applying the quasi-interpolatory operator defined in [24] satisfy the above properties and, recently, they have been widely used in the construction of integration formulas and in the numerical solution of integral and integro-differential equations, see, for instance, [3,4,7,10,13,22,27,23,28,30,32] and references therein.

This review paper is concerning the optimal nodal spline operators that, besides the properties 1., 2., 3., have the advantage of being interpolatory. These splines, introduced by DeVilliers and Rohwer [17,18] and studied in [12,14,16,19], have been utilized for constructing integration rules for the evaluation of weakly and strongly singular integrals also defined in the Hadamard finite part sense, in one or two dimensions and, more recently, for a collocation method producing the numerical solution of weakly singular Volterra integral equations.

In Section 2., after a brief outline of the construction of one-dimensional nodal spline operators, we shall present the tensor product of optimal nodal splines, recalling also some convergence results.

Section 3. is devoted to the application of the nodal spline operators in the approximation of different kind of 1D or 2D integrals and the main convergence results of the corresponding integration formulas are reported.

Finally, Section 4. deals with a collocation method, based on nodal splines, for the numerical solution of linear Volterra equation with weakly singular kernel.

2. Optimal nodal splines and their tensor product

2.1. One dimensional nodal splines

Let $J = [a, b]$ be a given finite interval of the real line $\mathbb{R}$, for a fixed integer $m \geq 3$ and $n \geq m - 1$, we define a partition Π_n of J by

$$\Pi_n : a = \tau_0 < \tau_1 < ... < \tau_n = b \ ,$$

generally called "primary partition". We insert $m - 2$ distinct points throughout $(\tau_\nu, \tau_{\nu+1})$, $\nu = 0, ..., n - 1$ obtaining a new partition of J

$$X_n : a = x_0 < x_1, < ... < x_{(m-1)n} = b,$$

where $x_{(m-1)i} = \tau_i$, $i = 0, ..., n$. Let

$$R_n = \max_{\substack{0 \leq k, j \leq n-1 \\ |k-j|=1}} \frac{\tau_{k+1} - \tau_k}{\tau_{j+1} - \tau_j} \ , \tag{1}$$

we say that the sequence of partitions $\{\Pi_n; n = m - 1, m, ...\}$ is locally uniform (l.u.) if, for all n, there exists a constant $A \geq 1$ such that $R_n \leq A$, i.e.

$$\frac{1}{A} \leq \frac{\tau_{k+1} - \tau_k}{\tau_{j+1} - \tau_j} \leq A \ , \quad k, j = 0, 1, ..., n - 1 \text{ and } |k - j| = 1 \ . \tag{2}$$

Since the convergence results of the nodal splines we shall consider are based on the local uniformity property of the primary partitions sequence and one of our objectives is the use of graded meshes, the following proposition shows that a sequence of primary graded partitions is l.u. [8]. For the definition of graded partitions see for example [2].

PROPOSITION 1. *Let $[a, b]$ be a finite interval. The sequence of partitions $\{\Pi_n\}$, obtained by using graded meshes of the form*

$$\tau_i = a + \left(\frac{i}{n}\right)^r (b - a) \ , \quad 0 \leq i \leq n \ ,$$

with grading exponent $r \in \mathbb{R}$ assumed ≥ 1, is l.u., i.e. it satisfies (2) with $A = 2^r - 1$.

Now, after introducing two integers [16]

$$i_0 = \begin{cases} \frac{1}{2}(m+1) & m \quad \text{odd} \\ \\ \frac{1}{2}m+1 & m \quad \text{even} \end{cases} \qquad \text{and} \quad i_1 = (m+1) - i_0$$

and two integer functions

$$p_\nu = \begin{cases} 0 & \nu = 0, 1, ..., i_1 - 2 \\ \nu - i_1 + 1 & \nu = i_1 - 1, ..., n - i_0 \\ n - (m-1) & \nu = n - i_0 + 1, ..., n - 1 \end{cases}$$

$$q_\nu = \begin{cases} m - 1 & \nu = 0, 1, ..., i_1 - 2 \\ \nu + i_0 & \nu = i_1 - 1, ..., n - i_0 \\ n & \nu = n - i_0 + 1, ..., n - 1 \end{cases}$$

consider the set $\{w_i(x); i = 0, 1, ..., n\}$ of functions defined as follows [17-19]

$$(3) \qquad w_i(x) = \begin{cases} l_i(x) & x \in [\tau_0, \tau_{i_1-1}], & i \leq m-1 \\ s_i(x) & x \in (\tau_{i_1-1}, \tau_{n-i_0+1}), & n \geq m \\ \bar{l}_i(x) & x \in [\tau_{n-i_0+1}, \tau_n], & i \geq n-(m-1) \end{cases}$$

where

$$l_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{m-1} \frac{x - \tau_k}{\tau_i - \tau_k}$$

$$\bar{l}_i(x) = \prod_{\substack{k=0 \\ k \neq n-i}}^{m-1} \frac{x - \tau_{n-k}}{\tau_i - \tau_{n-k}}$$

$$s_i(x) = \sum_{r=0}^{m-2} \sum_{j=j_0}^{j_1} \alpha_{i,r,j} B_{(m-1)(i+j)+r}(x)$$

with $j_0 = \max\{-i_0, i_1 - 2 - i\}$, $j_1 = \min\{-i_0 + m - 1, n - i_0 - i\}$. The coefficients $\alpha_{i,r,j}$ are given in [19] and the B-spline sequence is constructed from the set of the normalized B-splines for $i = (m-1)(i_1-2), (m-1)(i_1-2)+1, ..., (m-1)(n-i_0+1)$. Then, the following locality property holds [17]

$$(4) \qquad s_i(x) = 0 \quad , \quad x \notin [\tau_{i-i_0}, \tau_{i+i_1}].$$

Each $w_i(x)$ is nodal with respect to Π_n, in the sense that

$$w_i(\tau_j) = \delta_{i,j} \quad , \quad i, j = 0, 1, ..., n\ .$$

Therefore, being $\det[w_i(\tau_j)] \neq 0$, the functions $w_i(x), i = 0, 1, \ldots, n$, are linearly independent. Let $\mathbf{S}_{\Pi_n} = \text{span}\{w_i(x); i = 0, 1, ..., n\}$, it is proved in [18] that, for all $s \in \mathbf{S}_{\Pi_n}$, one has $s \in \mathbf{C}^{m-2}(J)$.

For all $g \in \mathbf{B}(J)$, where $\mathbf{B}(J)$ is the set of real-valued functions on J, we consider the spline operator $W_n : \mathbf{B}(J) \to \mathbf{S}\Pi_n$, so defined

$$W_n g = \sum_{i=0}^{n} g(\tau_i) w_i(x) \quad , \quad x \in J \ .$$

By (4), for $0 \leq \nu < n$ we can write:

$$W_n g = \sum_{i=p_\nu}^{q_\nu} g(\tau_i) w_i(x), \quad x \in [\tau_\nu, \tau_{\nu+1}] \ . \tag{5}$$

Moreover $W_n p = p$, for all $p \in \mathbb{P}_m$, where $\mathbb{P}_m$ denotes the set of polynomials of order m (degree $\leq m-1$), and $W_n g(\tau_i) = g(\tau_i)$, for $i = 0, 1, ..., n$, i.e. W_n is an interpolatory operator [17,18].

Using the results in [17-19] we deduce that, for l.u. $\{\Pi_n\}$, W_n is a bounded projection operator in S_{Π_n}. In fact, it is easy to show that

$$W_n s = s \quad , \quad \text{for all } s \in \mathbf{S}_{\Pi_n}$$

and, if we denote:

$$||W_n|| = \sup\{||W_n h||_\infty : h \in \mathbf{C}(J), ||h||_\infty < 1\},$$

with $||h||_\infty = \max_{x \in I} |h(x)|$, considering that

$$||W_n|| \leq (m+1) \left[\sum_{\lambda=1}^{m-1} (R_n)^\lambda \right]^{m-1} \quad ,$$

where R_n is defined in (1), from (2), if $\{\Pi_n\}$ is l.u., we obtain $||W_n|| < \infty$.

We remark that if we assume the $(m-2)$ points equally spaced throughout $(\tau_\nu, \tau_{\nu+1})$, $\nu = 0, 1, \ldots, n-1$, then the local uniformity constant of $\{X_n\}$ will be equal to that of $\{\Pi_n\}$.

Finally for all $g \in \mathbf{C}^{s-1}(J)$, with $1 \leq s \leq m$, we introduce the following quantity

$$E_{\nu s} = \begin{cases} D^\nu (g - W_n g) & , \quad 0 \leq \nu < s \\ D^\nu W_n g & , \quad s \leq \nu < m. \end{cases}$$

If $\{X_n\}$ is l.u., for $0 \leq \nu \leq s-1$ there results [14,19]

$$||E_{\nu s}||_\infty = O\big(H_n^{s-\nu-1} \omega(D^{s-1} g; H_n; J)\big) \tag{6}$$

where

$$H_n = \max_{0 \leq i \leq n-1} (\tau_{i+1} - \tau_i) \tag{7}$$

and for all $f \in \mathbf{C}(J)$, $\quad \omega(f;\delta;J) = \max_{\substack{x,\, x+h \in J \\ 0<h\leq\delta}} |f(x+h) - f(x)|.$

For $s \leq \nu < m$ in [14] a bound for $|E_{\nu s}|$ is given.

Furthermore, for every $t \subseteq J$ and $g \in \mathbf{C}^{\nu}(J)$, $0 \leq \nu < m-1$ [27]

$$\omega(D^{\nu} W_n g; t; J) = O\big(\omega(D^{\nu} g; t; J)\big).$$

2.2. Tensor product of optimal nodal splines

Let D be the $\mathbb{R}^2$ subset defined by $[a,b] \times [\tilde{a}, \tilde{b}]$. We consider partitions Π_n and X_n on which we construct the spline functions of order m $\{w_i(x), i = 0,\ldots,n\}$ defined in (3).

Then we consider similar partitions of $[\tilde{a}, \tilde{b}]$, $\tilde{\Pi}_{\tilde{n}}$ and $\tilde{X}_{\tilde{n}}$ and we construct the corresponding functions of order $\tilde{m}$ $\{\tilde{w}_{\tilde{i}}(\tilde{x}), \tilde{i} = 0,\ldots,\tilde{n}\}$.

Now we may generate a set of bivariate splines

$$w_{i,\tilde{i}}(x,\tilde{x}) = w_i(x) w_{\tilde{i}}(\tilde{x})$$

tensor product of the (3) ones.

Let $\mathbf{B}(D)$ denote the set of bounded real-valued functions on D. Then, for any $f \in \mathbf{B}(D)$ we may define the following spline interpolating operator for $(x,\tilde{x}) \in [\tau_j, \tau_{j+1}] \times [\tilde{\tau}_{\tilde{j}}, \tilde{\tau}_{\tilde{j}+1}]$,

$$W^*_{n\tilde{n}} f(x,\tilde{x}) = \sum_{i=p_j}^{q_j} \sum_{\tilde{i}=\tilde{p}_{\tilde{j}}}^{\tilde{q}_j} w_{i,\tilde{i}}(x,\tilde{x}) f(\tau_i, \tilde{\tau}_{\tilde{i}}), \tag{8}$$

with $j = 0, 1, \ldots, n-1$ and $\tilde{j} = 0, 1, \ldots, \tilde{n}-1$.

In order to obtain the maximal order polynomial reproduction, we can assume $m = \tilde{m}$, i.e. we use splines of the same order on both axes. We list in the following the main properties of $W^*_{n\tilde{n}}$.

(a) $W^*_{n\tilde{n}}$ is local, in the sense that $W^*_{n\tilde{n}} f(x,\tilde{x})$ depends only on the values of f in a small neighbourhood of $(x,\tilde{x})$;

(b) $W^*_{n\tilde{n}}$ interpolates f at the primary knots, i.e. $W^*_{n\tilde{n}} f(\tau_i, \tilde{\tau}_{\tilde{i}}) = f(\tau_i, \tilde{\tau}_{\tilde{i}})$;

(c) $W^*_{n\tilde{n}}$ has the optimal order polynomial reproduction property, that means $W^*_{n\tilde{n}} p = p$, for all $p \in \mathbb{P}^2_m$, where $\mathbb{P}^2_m$ is the set of bivariate polynomials of total order m.

For $f \in \mathbf{C}^{s-1}(D)$, $1 \leq s < m$ we introduce the following quantity

$$E_{\nu\tilde{\nu}s} = \begin{cases} D^{\nu,\tilde{\nu}}(f - W^*_{n\tilde{n}} f) & \text{if} \quad 0 \leq \nu + \tilde{\nu} < s \\ D^{\nu,\tilde{\nu}} W^*_{n\tilde{n}} f & \text{if} \quad s \leq \nu + \tilde{\nu} < m \end{cases}$$

where $D^{\nu,\tilde{\nu}}$ is the usual partial derivative operator.

Now we say that a collection of product partitions $\{X_n \times \tilde{X}_{\tilde{n}}\}$ of D is quasi uniform (q.u.) if there exists a positive constant σ such that

$$\frac{\Delta}{\hat{\delta}}, \frac{\Delta}{\tilde{\delta}}, \frac{\tilde{\Delta}}{\hat{\delta}}, \frac{\tilde{\Delta}}{\tilde{\delta}} \leq \sigma \ ,$$

where $\Delta = \max_{1\leq i\leq n(m-1)}(x_i - x_{i-1})$, $\hat{\delta} = \min_{1\leq i\leq n(m-1)}(x_i - x_{i-1})$ and $\tilde{\Delta} = \max_{1\leq \tilde{i}\leq \tilde{n}(m-1)}(\tilde{x}_{\tilde{i}} - \tilde{x}_{\tilde{i}-1})$, $\tilde{\delta} = \min_{1\leq \tilde{i}\leq \tilde{n}(m-1)}(\tilde{x}_{\tilde{i}} - \tilde{x}_{\tilde{i}-1})$.

We set

$$H^* = H_n + \tilde{H}_{\tilde{n}} \qquad \text{and} \qquad \Delta^* = \Delta + \tilde{\Delta} \tag{9}$$

where H_n is defined in (7) and likewise $\tilde{H}_{\tilde{n}}$.

Assuming that $f \in \mathbf{C}^{s-1}(D)$ with $1 \leq s < m$ and that $\{W_{n\tilde{n}} f\}$ is a q.u. sequence of nodal splines, then for $\nu, \tilde{\nu}$ such that $0 \leq \nu + \tilde{\nu} \leq s - 1$

$$||E_{\nu\tilde{\nu}s}||_\infty = O\big(H^{*s-\nu-\tilde{\nu}-1}\omega(D^{s-1}f; H^*; D)\big).$$

In [9] local bounds of $|E_{\nu\tilde{\nu}s}|$ are derived and local and global bounds of $|E_{\nu\tilde{\nu}s}|$, $s \leq \nu + \tilde{\nu} < m$, are also given.

Furthermore, for $f \in \mathbf{C}^p(D)$, $0 \leq p < m - 1$, and for a q.u. sequence of nodal splines $\{W^*_{n\tilde{n}} f\}$, there results for any non empty subset T of D

$$\omega(D^p W^*_{n\tilde{n}} f; T; D) = O\big(\omega(D^p f; T; D)\big) \ .$$

In the following we shall consider l.u. partitions in the one dimensional case and q.u. partitions in the 2D one and we shall suppose always that the norm of the partitions converges to zero as $n \to \infty$ or $n, \tilde{n} \to \infty$.

3. Numerical integration based on nodal spline operators

This section will deal with the numerical evaluation of some singular one-dimensional integrals and of certain 2D singular integrals.

3.1. Product integration of singular integrands

Consider integrals of the form

$$J(kf) = \int_I k(x) f(x) dx \tag{10}$$

where $kf \in \mathbf{L}_1(I)$, but f is unbounded in $I = [-1, 1]$.

In [26] product integration have been proposed, by substituting f by a sequence of interpolatory nodal splines $\{W_n f\}$ defined in (5), under different hypotheses on f.

By using (6) with $\nu = 0$, the author gets, firstly, the convergence of the quadrature sum $J(kW_n f)$, i.e.:

$$J(kW_n f) \to J(kf) \text{ as } \quad n \to \infty \tag{11}$$

by supposing $f \in \mathbf{C}(I)$, $k \in \mathbf{L}_1(I)$ and $H_n \to 0$ as $n \to \infty$.

We recall that a computational procedure to generate the weights $\{v_i(k) = \int_I k(x) w_i(x) dx\}$ of the above quadrature is given in [6].

Moreover in [26] the case when $f \in \mathbf{P}C(I)$, $k \in L_1(I)$ is studied and the convergence of the quadrature rules sequence is proved.

We remark that in [11] the convergence (11) has been proved also for $f \in \mathbf{R}(I)$, the class of Riemann integrable functions on I and $k \in \mathbf{L}_1(I)$.

When the function f in (10) is singular in $z \in [-1, 1)$ in [25] the author defines the family of real valued functions $M_d(z; k)$:

$$\begin{array}{l} M_d(z; k) = \{f : f \in \mathbf{P}C(z, 1], \exists F : F = 0 \\ \text{ on } [-1, z],\ F \text{ is non negative, continuous} \\ \text{and nonincreasing on } (z, 1), kF \in \mathbb{L}_1(I) \text{ and } |f| \le F \text{ on } I\} \end{array} . \tag{12}$$

He supposes that k satisfies one of the following conditions A, B:

(A) There exists $\delta > 0 : |k(x)| \le K(x)$, $\forall x \in (z, z + \delta]$, K is positive nonincreasing in that interval and KF, F defined in (12), is a $\mathbf{L}_1$ function in I.

(B) Given $q_0 \in (0, 1)$, $\exists \delta, T$, positive numbers (possibly depending on q_0), such that

$$\int_c^{c+h} |k(x)| dx \le hT|k(c + qh)|$$

$\forall q \in [q_0, 1]$, $\forall c$ and h satisfying $z \le c < c + h \le z + \delta$. Besides $|k(x) f(x)| \le G(x)$, $\forall x \in (z, z + \delta]$, where G is a positive non increasing $\mathbf{L}_1$ function in that interval.

The following theorem can be proved.

THEOREM 1. *Assume that $f \in M_d(-1; k)$ and k satisfies (A) or (B). If the sequence of partitions $\{\Pi_n\}$ is l.u. and the norm converges to zero as $n \to \infty$, then (11) holds.*

As consequence of that theorem if $z = -1$ the singularity can be ignored, provided k satisfies (A) or (B).

In the case when z is an interior singularity, it must, in general, be avoided, i.e. we must define a new integration rule

$$J^*(kW_n f) = \sum_{i=J}^{n} v_i(k) f(\tau_i)$$

where J is the smallest integer such that $z \leq \tau_{J-\lambda}$, where $\tau_{J-\lambda}$ is the left bound of the support of $s_J(x)$ and, if we assume that n is so large that $J \geq m$, then $w_i = s_i$ and $v_i(k)$ is given by:

$$v_i(k) = \int_{\tau_{i-\lambda}}^{\tau_{i+\mu}} k(x)s_i(x)dx ,$$

with $\lambda = i_0$ and $\mu = i_1$.

Therefore, assuming that $f \in M_d(z; k)$, $z > -1$, and k satisfying (A) or (B). If $\{\Pi_n\}$ is locally uniform and the norm tends to zero as $n \to \infty$, then

$$J^*(kW_n f) \to J(Kf) \quad \text{as} \quad n \to \infty .$$

If one wishes to use $J(kW_n f)$ rather than $J^*(kW_n f)$ then k must be restricted in $[-1, z)$ as well as in $(z, 1]$, for satisfying one of the following conditions $(\hat{A})$ or $(\hat{B})$.

$(\hat{\mathbf{A}})$: (A) holds and, in addition, $|k_z(x)| \leq K(x)$ in $(z, z + \delta]$, where $k_z \in L_1(2z - 1, 2z + 1)$ is defined by $k_z(z + y) = k(z - y)$.

$(\hat{\mathbf{B}})$: (B) holds and so does (B) with k replaced by k_z.

THEOREM 2. *Let $f \in M_d(z; k)$, $z > -1$. Assume that k satsfies $(\hat{A})$ or $(\hat{B})$ and that $\{\Pi_n\}$ is l.u. and the norm converges to zero as $n \to \infty$.*

Define

$$\hat{J}(kW_n f) = J(kW_n f) - v_\rho f(\tau_\rho)$$

where τ_ρ is the value of $\tau_i \geq z$ closest to z. Then

$$\hat{J}(kW_n f) \to J(kf) \quad \text{as} \quad n \to \infty .$$

In particular, if $\tau_\rho = z$ then (11) holds. If z is such that for all n, $\tau_\rho - z > C(\tau_\rho - \tau_{\rho-1})$, then (11) holds.

3.2. Cauchy principal value integrals

Consider the numerical evaluation of the Cauchy principal value (CPV) integrals

$$J(kf; \lambda) = \fint_{-1}^{1} k(x)\frac{f(x)}{x - \lambda}dx, \quad \lambda \in (-1, 1). \tag{13}$$

In [11] the problem has been investigated, following the "subtracting singularity" approach.

Assuming that $J(k; \lambda)$ exists for $\lambda \in (-1, 1)$, the integral (13) can be written in the form

$$\begin{aligned} J(kf; \lambda) &= \int_{-1}^{1} k(x)g_\lambda(x)dx + f(\lambda)J(k; \lambda) \\ &= \mathcal{I}(kg_\lambda) + f(\lambda)J(k; \lambda), \end{aligned}$$

where

$$g_\lambda(x) = g(x;\lambda) = \begin{cases} \frac{f(x)-f(\lambda)}{x-\lambda} & x \neq \lambda \\ f'(\lambda) & x = \lambda \text{ and } f'(\lambda) \text{ exists} \\ 0 & \text{otherwise} . \end{cases}$$

Therefore, approximating $\mathcal{I}(kg_\lambda)$ by $\mathcal{I}(kW_n g_\lambda)$ we can write [11]

$$J(kf;\lambda) = J_n(kf;\lambda) + E_n(kf;\lambda),$$

where

$$J_n(kf;\lambda) = \mathcal{I}(kW_n g_\lambda) + f(\lambda)J(k;\lambda) .$$

For any $\lambda \in (-1, 1)$ we define a family of functions $\bar{M}_d(z;k) = \{g \in C(I\backslash\lambda), \exists G : G$ is continuous nondecreasing in $[-1;\lambda)$, continuous non increasing in $(\lambda, 1]; kG \in \mathbf{L}_1(I), |g| < G$ in $I\}$.

We assume

$$N_\delta(\lambda) = \{x : \lambda - \delta \leq x \leq \lambda + \delta\} ,$$

where $\delta > 0$ is such that $N_\delta(\lambda) \subset I$.

We denote by $\mathbf{H}_\mu(I)$, $\mu \in (0, 1]$, the set of Hölder continuous functions

$$\begin{aligned} \mathbf{H}_\mu(I) &= \{g \in C(I) : |g(x_1) - g(x_2)| \\ &\leq L|x_1 - x_2|^\mu, \forall x_1, x_2 \in I, L > 0\} \end{aligned}$$

and by $\mathbf{D}T(I)$ the set of Dini type functions

$$\mathbf{D}T(I) = \{g \in \mathbf{C}(I) : \int_0^{l(I)} \omega(g;t)t^{-1}dt < \infty\}$$

where $l(I)$ is the length of I and ω denotes the usual modulus of continuity.

The following convergence results for the quadrature rules $J_n(kf;\lambda)$, under different hypotheses for the function f, are derived in [11].

THEOREM 3. *For any $\lambda \in (-1, 1)$, let $f \in \mathbf{H}_1\big(N_\delta(\lambda) \cap \mathbf{R}(I)\big)$ and $k \in \mathbf{L}_1(I)$. Then, for l.u. $\{\Pi_n\}$, $E_n(kf;\lambda) \to 0$ as $n \to \infty$.*

THEOREM 4. *Let $f \in \mathbf{H}_\mu(I)$, $0 < \mu < 1$, $k \in \mathbf{L}_1(I) \cap \mathbf{C}\big(N_\delta(\lambda)\big)$. Let h and p be the greatest and the smallest integers such that $\tau_h < \lambda$, $\tau_p > \lambda$. We denote by τ^* the node closest to λ*

$$\tau^* = \begin{cases} \tau_h & \text{if } \lambda - \tau_h \leq \tau_p - \lambda \\ \tau_p & \text{if } \lambda - \tau_h > \tau_p - \lambda \end{cases}$$

and we suppose that there exists some positive constant C, such that

$$|\tau^* - \lambda| > C \max\{(\tau_h - \tau_{h-1}), (\tau_{p+1} - \tau_p)\},$$

then, for l.u. $\{\Pi_n\}$,

$$E_n(kf;\lambda) \to 0$$

as $n \to \infty$.

THEOREM 5. *Let* $f \in \mathbf{C}^1(I)$, $k \in \mathbf{L}_1(I)$. *Then*

$$E_n(kf;\lambda) \to 0 \text{ uniformly in } \lambda, \text{ as } n \to \infty.$$

However, if $k \in \mathbf{L}_1(I) \cap \mathbf{D}T(-1,1)$, *then* $J(kf;\lambda)$ *exists for all* $\lambda(-1,1)$. *Besides*

$$J_n(kf;\lambda) \to J(kf;\lambda) \text{ as } n \to \infty$$

uniformly for all $\lambda \in (-1,1)$.

Moreover in [14] it has been proved that $J(\omega_{\alpha,\beta}W_n;\lambda) \to J(kf;\lambda)$ uniformly with respect to $\lambda \in (-1,1)$, for $\omega_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha,\beta > -1$, and $f(x) \in \mathbf{H}_\rho(-1,1)$, $0 < \rho \leq 1$.

3.3. The Hadamard finite part integrals

We consider the evaluation of the finite part integrals of the form

$$\bar{J}(\omega_{\alpha,\beta}f) = \neq\!\!\!\!\!\!\int_I \frac{\omega_{\alpha,\beta}(x)f(x)}{x+1}dx, \tag{14}$$

where $\alpha > -1$, $-1 < \beta \leq 0$ and $\neq\!\!\!\!\!\!\int$ denotes the Hadamard finite part (HFP).

It is well known that a sufficient condition so that (14) exists is

$$f \in \mathbf{H}_\mu(I), \quad 0 < \mu \leq 1, \quad \mu + \beta > 0\,.$$

We recall that [25]

$$\bar{J}(\omega_{\alpha,\beta}f) = \int_{-1}^{1} \omega_{\alpha,\beta}(x)\frac{f(x)-f(-1)}{x+1}dx + f(-1)\neq\!\!\!\!\!\!\int_{-1}^{1}\frac{\omega_{\alpha,\beta}(x)}{x+1}dx, \tag{15}$$

where, denoting $c_j = \frac{d^j}{dx^j}\frac{(1-x)^j}{j!}\Big|_{x=-1}$, $j = 0,1,\ldots$, we obtain for the HFP in (15),

$$\neq\!\!\!\!\!\!\int_{-1}^{1}\frac{\omega_{\alpha,\beta}(x)}{x+1}dx = \begin{cases} \log 2 & \text{if } \alpha=\beta=0 \\ c_0\log 2 + \sum_{j=1}^{\infty}\frac{c_j}{j!}2^j & \text{if } \beta=0, \alpha\neq 0 \\ \frac{\alpha+\beta+1}{\beta}2^{\alpha+\beta}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} & \text{if } \alpha>-1,\ -1<\beta<0, \end{cases}$$

where Γ is the gamma function.

Approximating f by $W_n f$ in (14) we obtain the quadrature rule [5]:

$$\bar{J}(\omega_{\alpha,\beta}f) = \bar{J}_n(f) + \bar{E}_n(f), \tag{16}$$

where

$$\bar{J}_n(f) = \sum_{i=0}^{n} \bar{v}_i(\omega_{\alpha,\beta}) f(\tau_i)$$

with $\bar{v}_i(\omega_{\alpha,\beta}) = \bar{J}(\omega_{\alpha,\beta} w_i)$, and

$$\bar{E}_n(f) = \bar{J}(\omega_{\alpha,\beta}(f - W_n f)).$$

A computational procedure for evaluating $\bar{v}_i(\omega_{\alpha,\beta})$ is given in [6].

Denoting by $\mathbf{H}^s_\mu(I)$ the set of the functions $f \in \mathbf{C}^s(I)$ having $f^{(s)} \in \mathbf{H}_\mu(I)$, in [5] the following theorem has been proved.

THEOREM 6. *Let $f \in \mathbf{H}^s_\mu(I)$, $0 \le s \le m-1$, and $\mu + \beta > 0$ if $s = 0$. Then, as $n \to \infty$:*

$$||\bar{E}_n(f)||_\infty = \begin{cases} O(H_n^{s+\mu+\beta}) & \text{if } \beta < 0 \\ O(H_n^{s+\mu}|\log H_n|) & \text{if } \beta = 0\,. \end{cases}$$

Consider now HFP integrals of the form:

$$J^*(\omega_{\alpha,\beta} f; \lambda; p) = \oint\!\!\!\!\!\!= _I \omega_{\alpha,\beta}(x) \frac{f(x)}{(x-\lambda)^{p+1}}, \quad \lambda \in [-1,1], \quad p \ge 1 \tag{17}$$

If $f \in \mathbf{H}^p_\mu(I)$, then $J^*(\omega_{\alpha,\beta} f; \lambda; p)$ exists.

In [20, 21] quadrature rules for the numerical evaluation of (17), based on some different type of spline approximation, including the optimal nodal splines, are considered and studied.

In [29] the following theorem has been proved.

THEOREM 7. *Assume that in (17) $\lambda \in (-1,1)$, $p \in \mathbf{N}$ and $f \in H^p_\mu$. Let $\{f_n\}$ be a given sequence of functions such that $f_n \in \mathbf{C}^p(I)$ and*

i) - $||D^j r_n||_\infty = o(1)$ as $n \to \infty$ $j = 0, 1, \ldots, p$, *where* $r_n = f - f_n$

ii) - $D^j r_n(-1) = 0 \quad 0 \le j \le p - \beta; \quad D^j r_n(1) = 0 \quad 0 \le j \le p - \alpha$

iii) - $r_n \in \mathbf{H}^p_\sigma(I), \quad \forall n, \quad 0 < \sigma \le \mu, \quad \sigma + \min(\alpha, \beta) > 0.$

Then

$$J^*(\omega_{\alpha,\beta} f_n; \lambda; p) \to J^*(\omega_{\alpha,\beta} f; \lambda; p) \quad \text{as } n \to \infty \tag{18}$$

uniformly for $\forall \lambda \in (-1,1)$.

If we consider a sequence of optimal nodal splines for approximating the function f, in order to obtain the uniform convergence in (18) of integration rules, we must modify the sequence $\{W_n\}$ in the sequence $\{\hat{W}_n f\}$, for which condition *ii)* is satified.

Therefore, in [15], for $0 \leq s, t \leq p$, are defined two sets of B-splines $\bar{B}_i$, $\bar{B}_{N-i}$ on the knot sets

$$\{x_0, \ldots x_0, x_1, \ldots, x_{s+1}\}, \quad \{x_{N-t-1}, \ldots, x_{N-1}, x_N, \ldots, x_N\}$$

respectively, where $N = (m-1)n$ and x_0, x_N are repeated exactly m times.

Considering that $W_n f(\tau_i) = f(\tau_i)$, $i = 0, n$, one defines

$$g_n(x) := \begin{cases} \sum_{i=1}^{s} d_i \bar{B}_i(x) & x \in [x_0, \ldots, x_{s+1}] \\ 0 & x \in (x_{s+1}, \ldots, x_{N-t-1}) \\ \sum_{i=1}^{t} \tilde{d}_i \bar{B}_{N-i}(x) & x \in [x_{N-t-1}, \ldots, x_N] \end{cases}$$

where d_i, $\tilde{d}_i$ are determined by solving two non-singular triangular systems obtained by imposing

$$g^{(j)}(\tau_0) = r_n^{(j)}(\tau_0) \quad j = 1, 2, \ldots, s$$
$$g_n^{(j)}(\tau_n) = r_n^{(s)}(\tau_n) \quad j = 1, 2, \ldots, t$$

For the sequence $\{\hat{W}_n f = W_n f + g_n\}$, it is possible to prove the following:

THEOREM 8. *Let* $\{\hat{W}_n f\}$ *be a sequence of modified optimal nodal splines and set* $\hat{r}_n = f - \hat{W}_n f$, *then*
$\hat{W}_n f(\tau_i) = f(\tau_i) \quad i = 0, \ldots, n$;
$D^j \hat{r}_n(-1) = 0$, $0 \leq j \leq p - \beta$; $D^j \hat{r}_n(1) = 0$, $0 \leq j \leq p - \alpha$,
$\hat{W}_n g = g$ *if* $g \in \mathbb{P}_m$.
Besides supposing $f \in \mathbf{C}^r(I_k)$, $I_k = [\tau_k, \tau_{k+1}]$, $h_k = \tau_{k+1} - \tau_k$, *for any* $x \in I_k$ *there results:*
$|D^\nu \hat{r}_n(x)| \leq \tilde{k}_\nu h_k^{r-\nu} \omega(D^r f; h_k; I_k), \quad \nu = 0, \ldots, r$
$|D^{r+1} \hat{W}_n f(x)| \leq \tilde{k}_{r+1} h_k^{-1} \omega(D^r f; h_k; I_k)$,
$\hat{r}_n \in \mathbf{H}_\mu^r(I)$.

Therefore all the conditions of theorem 3.3.2 being satisfied, if $\mu + \min(\alpha, \beta) > 0$, then

$$J^*(\omega_{\alpha,\beta} \hat{W}_n f; \lambda; p) \to J(\omega_{\alpha,\beta} f; \lambda; p) \quad \text{as } n \to \infty$$

uniformly for $\forall \lambda \in (-1, 1)$.

3.4. Integration rules for 2-D CPV integrals

In this section we will consider the numerical evaluation of the following two types of CPV integrals:

$$J_1(f; x_0, y_0) = \fint_R \omega_1(x)\omega_2(y) \frac{f(x, y)}{(x - x_0)(y - y_0)} dx dy \tag{19}$$

where $R = [a, b] \times [\tilde{a}, \tilde{b}]$, $x_0 \in (a, b)$, $y_0 \in (\tilde{a}, \tilde{b})$, and we assume $\omega_1(x) \in \mathbf{L}_1[a, b] \cap \mathbf{D}T(N_\delta(x_0))$, $\omega_2(y) \in \mathbf{L}_1[\tilde{a}, \tilde{b}] \cap \mathbf{D}T(N_\delta(y_0))$; and

$$J_2(\phi; P_0) = \oint_D \Phi(P_0, P)dP, \quad P_0 \in D \tag{20}$$

where D denotes a polygonal region and $\Phi(P_0, P)$ is an integrable function on D except at the point P_0 where it has a second order pole.

For numerically evaluating (19), in [9] the following cubatures based on a sequence of nodal splines (8) have been proposed:

$$J_1(W_{n\tilde{n}} f; x_0, y_0) = \sum_{i=0}^{n} \sum_{\tilde{i}=0}^{\tilde{n}} v_i(x_0)\tilde{v}_{\tilde{i}}(y_0) f(\tau_i, \tilde{\tau}_{\tilde{i}}) ,$$

where $v_i(x_0) = \oint_a^b \omega_1(x) \dfrac{w_i(x)}{x - x_0} dx$, and $\tilde{v}_{\tilde{i}}(y_0) = \oint_{\tilde{a}}^{\tilde{b}} \omega_2(y) \dfrac{\tilde{w}_{\tilde{i}}(y)}{y - y_0} dy$.

We denote by $\mathbf{H}^p_{\mu,\mu}(R)$ the set of continuous functions having all partial derivatives of order $j = 0, \ldots, p$, $p \geq 0$ continuous and each derivative of order p satisfying a Hölder condition, i.e.:

$$|f^{(p)}(x_1, y_1) - f^{(p)}(x_2, y_2)| \leq C(|x_1 - x_2|^\mu + |y_1 - y_2|^\mu), \quad 0 < \mu \leq 1$$

for some constant $C > 0$, and we assume

$$E_{n\tilde{n}}(f; x_0, y_0) = J_1(f; x_0, y_0) - J_1(W_{n\tilde{n}} f; x_0, y_0). \tag{21}$$

In [9] the following convergence theorem has been proved.

THEOREM 9. *Let* $f \in \mathbf{H}^p_{\mu,\mu}$, $0 < \mu \leq 1$, $0 \leq p < m - 1$. *For the remainder term in (21), there results:*

$$E_{n\tilde{n}}(f; x_0, y_0) = O\big((\Delta^*)^{p+\mu-\gamma}\big),$$

where $\gamma \in \mathbb{R}$, $0 < \gamma < \mu$, *small as we like and* Δ^* *has been defined in (9).*

In many practical applications it is necessary that rules, uniformly converging for $\forall (x_0, y_0) \in (-1, 1) \times (-1, 1)$, are available, in particular considering the Jacobi weight type functions

$$\omega_1(x) = (1 - x)^{\alpha_1}(1 + x)^{\beta_1}, \quad \omega_2(y) = (1 - y)^{\alpha_2}(1 + y)^{\beta_2}$$

with $\alpha_i, \beta_i > -1$, $i = 1, 2$, $(x, y) \in R = [-1, 1] \times [-1, 1]$.

In order to obtain uniform convergence for approximating rules numerically evaluating (19), can be useful to write the integral in the form

$$\begin{aligned} J_1(f; x_0, y_0) &= \oint_R \omega_1(x)\omega_2(y) \frac{f(x, y) - f(x_0, y_0)}{(x - x_0)(y - y_0)} dx dy \\ &+ f(x_0 y_0) J(\omega_1; x_0) J(\omega_2; y_0) \end{aligned} \tag{22}$$

where $J(\omega_1; x_0) = \displaystyle⨍_{-1}^{1} \frac{\omega_1(x)}{x - x_0} dx$, $\quad J(\omega_2; y_0) = \displaystyle⨍_{-1}^{1} \frac{\omega_2(y)}{y - y_0} dy$.

We exploit the results in [31] where, considering a sequence of linear operators $F_{n\tilde{n}}$ approximating f, the integration rule for (22):

$$\begin{aligned} J_1(F_{n\tilde{n}}; x_0, y_0) &= ⨍_R \omega_1(x)\omega_2(y) \frac{F_{n\tilde{n}}(x, y) - F_{n\tilde{n}}(x_0, y_0)}{(x - x_0)(y - y_0)} dx dy \\ &+ f(x_0, y_0) J(\omega_1; x_0) J(\omega_2; y_0) \end{aligned}$$

has been constructed. Denoting $r_{n\tilde{n}} = f - F_{n\tilde{n}}$, and $\Delta_{n\tilde{n}}$ the norm of the partition, with $\lim\limits_{\substack{n \to \infty \\ \tilde{n} \to \infty}} \Delta_{n\tilde{n}} = 0$, the following general theorem of uniform convergence has been proved.

THEOREM 10. *Let* $f \in H^0_{\mu\mu}(R)$, *and assume that the approximation* $F_{n\tilde{n}}$ *to* f *is such that*

i) $r_{n\tilde{n}}(x, \pm 1) = 0 \quad \forall x \in [-1, 1], r_{n\tilde{n}}(\pm 1, y) = 0 \quad \forall y \in [-1, 1]$,

ii) $||r_{n\tilde{n}}||_\infty = O(\Delta^\nu_{n\tilde{n}}), \quad 0 < \nu \leq \mu$,

iii) $r_{n\tilde{n}} \in H^0_\sigma(R), \quad 0 < \sigma \leq \mu$.

If $\rho + \gamma - \bar{\varepsilon} > 0$, where $\rho = \min(\sigma, \nu)$, $\gamma = \min(\alpha_1, \alpha_2, \beta_1, \beta_2)$ and $\bar{\varepsilon}$ is a positive real number as small as we like, then, for the remainder term, $E_{n\tilde{n}} = J_1(f; x_0, y_0) - J_1(F_{n\tilde{n}}; x_0, y_0)$, there results:

$$E_{n\tilde{n}}(f; x_0.y_0) \to 0 \quad \text{as} \quad n \to \infty, \tilde{n} \to \infty$$

uniformly for $\forall (x_0, y_0) \in (-1, 1) \times (-1, 1)$.

If we consider $F_{n\tilde{n}} = W_{n\tilde{n}}(f; x, y)$ only the conditions *ii), iii)*, with $\Delta_{n,\tilde{n}} = \Delta^*$, are satisfied, but we can modify $W_{n\tilde{n}}$ in the form

$$\begin{aligned} \bar{W}_{n\tilde{n}}(f; x, y) &= W_{n\tilde{n}}(f; x, y) + [f(-1, y) - W_{n\tilde{n}}(f; -1, y)] B_{1-m}(x) \\ &+ [f(1, y) - W_{n\tilde{n}}(f; 1, y)] B_{(m-1)n-1}(x) \\ &+ [f(x, -1) - W_{n\tilde{n}}(f; x, -1)] \tilde{B}_{1-m}(y) \\ &+ [f(x, 1) - W_{n\tilde{n}}(f; x, 1)] B_{(m-1)\tilde{n}-1}(y) \,. \end{aligned}$$

Assuming $\bar{r}_{n\tilde{n}}(x, y) = f(x, y) - \bar{W}_{n\tilde{n}}(f; x, y)$, all the condition $i) - iii)$ are verified and then

$$J_1(\bar{W}_{n\tilde{n}}; x_0, y_0) \to J_1(f; x_0, y_0) \quad \text{as } n, \tilde{n} \to \infty$$

uniformly for $\forall (x_0, y_0) \in (-1, 1) \times (-1, 1)$.

Now we consider the integral (20) for which we refer to the results in [5,6]. Since the polygon D can be thought as the union of triangles, each one with the singularity at one vertex, by introducing polar coordinates (r, ϑ) with origin at the singularity P_0, the evaluation of (20) can be reduced to the evaluation of

$$J_2^*(f) = \int_{\vartheta_1}^{\vartheta_2} \left(\fint_0^{R(\vartheta)} \frac{f(r,\vartheta)}{r} dr \right) d\vartheta, \tag{23}$$

where

$$\fint_0^{R(\vartheta)} \frac{f(r,\vartheta)}{r} d\vartheta = \int_0^{R(\vartheta)} \frac{f(r,\vartheta) - f(0,\vartheta)}{r} dr + f(0,\vartheta)\log(R(\vartheta));$$

the integration domain is a triangle (Fig. 1)

$$T = \{(r,\vartheta) : 0 \le r \le R(\vartheta), \quad \vartheta_1 \le \vartheta \le \vartheta_2\}$$

with

$$R(\vartheta) = \begin{cases} \frac{d}{\sin\vartheta - \cos\vartheta} & \text{if } s : y = cx + d \\ \frac{d}{\cos\vartheta} & \text{if } s : x = d\,. \end{cases}$$

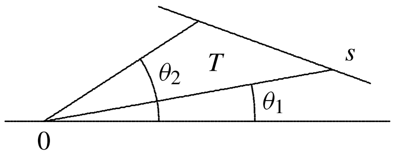

Figure 1. Domain of integration T.

The outer integral in (23) will be approximated by rules of the form considered in section 3.1 with nodes $\Pi_n = \{\tau_i\}_{i=0}^n$ and weights $\{v_i\}_{i=0}^n$; for the inner one we consider rules of the form (16), with $\alpha = \beta = 0$, based on optimal nodal splines of order $\bar{m} \ge 3$, primary knots $\bar{\Pi}_N = \{\bar{\tau}_i = \bar{y}_{(\bar{m}-1)i}\}_{i=0,\dots,N}$ corresponding to the partition

$$\bar{Y}_N = \{-1 = \bar{y}_0 < \bar{y}_1 \cdots < \bar{y}_{(\bar{m}-1)N} = 1\}$$

and we suppose that the norms H_n and $\bar{H}_N$, of Π_n and $\bar{\Pi}_N$, respectively, converges to 0 as n and $N \to \infty$.

We obtain the following rules

$$J_{2,n,N}^*(f) = \frac{\vartheta_2 - \vartheta_1}{2} \sum_{i=0}^{n} v_{in} \left[\sum_{k=0}^{N} \bar{v}_{kN} f(r_{ki}, \xi_i) + f(0,\xi_i) \log\left(\frac{R(\xi_i)}{2}\right) \right] + R_{n,N}(f),$$

where

$$\begin{cases} \xi_i = [(\vartheta_2 - \vartheta_1)/2]\tau_i + (\vartheta_2 + \vartheta_1)/2 & i = 0,\dots,n \\ r_{ki} = [R(\xi_i)/2](\bar{\tau}_{kN}) + [R(\xi_2)/2](\bar{\tau}_{kN} + 1) & i = 0,\dots,N. \end{cases}$$

Let us assume $R = \max_{\vartheta \in [\vartheta_1, \vartheta_2]} |R(\vartheta)|$, $\mathcal{R} = [0, R] \times [\vartheta_1, \vartheta_2]$ and define $m^* = \min(m, \bar{m})$.

We can prove the following theorem:

THEOREM 11. *If* $f \in \mathbf{H}^s_{\mu,\mu}(\mathcal{R})$, $0 < \mu \leq 1$ *and* $0 \leq s \leq m^* - 1$, $\{\Pi_n\}$ *and* $\{\bar{Y}_N\}$ *are sequence of locally uniform partitions, then*

$$||R_{n,N}(f)||_\infty = O(\bar{H}_N^{s+\mu} |\log(\bar{H}_N)| + H_n^{s+\mu-\varepsilon})$$

where ε *is a positive real as small as we like.*

4. A collocation method for weakly singular Volterra equations

Consider the Volterra integral equation of the second kind

$$y(x) = f(x) + \int_0^x k(x,s) y(s) ds \quad x \in I \equiv [0, X] \tag{24}$$

where k is weakly singular kernel, in particular of convolution type of the form $k(x-s)$, where $k \in \mathbf{C}(0, X] \cap \mathbf{L}_1(0, X)$, but $k(t)$ can become unbounded as $t \to 0$.

In [8], for numerically solving (24) a product collocation method, based on optimal nodal splines, has been constructed, for which error analysis and condition number are given.

If we consider a spline $y_n \in \mathbf{S}_{\pi_n}$, written in the form

$$y_n(x) = \sum_{j=0}^{n} \alpha_j w_j(x) \quad \alpha_j \in \mathbb{R}, \quad j = 0, \ldots, n,$$

and we substitute such function in (24), we obtain

$$y_n(x) - \int_0^x k(x,s) y_n(s) ds + r_n(x) = f(x)$$

where $r_n(x)$ is the residual term obtained in approximating y by y_n.

The values α_j are determined by imposing

$$r_n(\tau_j) = 0 \quad j = 0, \ldots, n, \tag{25}$$

i.e. as solution of a linear system of the form

$$\alpha_j[1 - \mu(\tau_j)] - \sum_{\substack{i=0 \\ i \neq j}}^{n} \mu_i(\tau_j)\alpha_i = f(\tau_j) \quad j = 0, \ldots, n,$$

where $\mu_i(\tau_j) = \int_0^{\tau_j} k(\tau_j, s) w_i(s) ds$.

In the quoted paper the explicit form of $\mu_i(\tau_j)$ for different values of i is provided.

Exploiting the properties of the operator W_n, which is a bounded interpolating projection operator, the condition (25) can be rewritten in the form

$$(I - W_n \tilde{K}) y_n = W_n f, \tag{26}$$

where $\tilde{K} y = \int_I \tilde{k}(x,s) y(s) ds$, with

$$\tilde{k}(x,s) = \begin{cases} k(x,s) & 0 \le s \le x \\ 0 & s > x, \end{cases}$$

is a bounded compact operator on $\mathbf{C}(I)$ [1]. Therefore we can deduce that equation (26) has a unique solution and

THEOREM 12. *For all n sufficiently large, say $n \ge N$, the operator $(I - W_n \tilde{K})^{-1}$ from $\mathbf{C}(I)$ to $\mathbf{C}(I)$ exists.*

Moreover it is uniformly bounded, i.e.:

$$\sup_{n \ge N} ||(I - W_n \tilde{K})^{-1}|| \le M < \infty$$

and

$$||y - y_n||_\infty \le ||(I - W_n \tilde{K})^{-1}|| \, ||y - W_n y||_\infty .$$

This leads to $||y - y_n||_\infty$ converging to zero exactly with the same rate of the norm of the nodal spline approximation error.

References

[1] ATKINSON K.E., *The numerical solution of integral equations of the second kind*, Cambridge Mon. on Appl. Comp. Math., Cambridge University Press **33** (1997), 59–672.

[2] BRUNNER H., *The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes*, Math. Comp. **45** (1985), 417–437.

[3] CALIÓ F. AND MARCHETTI E., *On an algorithm for the solution of generalized Prandtl equations*, Numerical Algorithms **28** (2001), 3–10.

[4] CALIÓ F. AND MARCHETTI E., *An algorithm based on Q. I. modified splines for singular integral models*, Computer Math. Appl. **41** (2001), 1579–1588.

[5] DAGNINO C. AND DEMICHELIS V., *Nodal spline integration rules of Cauchy principal value integrals*, Intern. J. Computer Math. **79** (2002), 233–246.

[6] DAGNINO C. AND DEMICHELIS V., *Computational aspects of numerical integration based on optimal nodal spline*, Intern. J. Computer Math. **80** (2003), 243–255.

[7] DAGNINO C., DEMICHELIS V. AND SANTI E., *Numerical integration of singular integrands using quasi-interpolatory splines*, Computing **50** (1993), 149–163.

[8] DAGNINO C., DEMICHELIS V. AND SANTI E., *A nodal spline collocation method for weakly singular Volterra integral equations*, Studia Univ. Babes-Bolyai Math. **48** 3 (2003), 71–82.

[9] DAGNINO C., PEROTTO S. AND SANTI E., *Convergence of rules based on nodal splines for the numerical evaluation of certain Cauchy principal value integrals*, J. Comp. Appl. Math. **89** (1998), 225–235.

[10] DAGNINO C. AND RABINOWITZ P., *Product integration of singular integrands using quasi-interpolatory splines*, Computer Math. Applic. **33** (1997), 59–672.

[11] DAGNINO C. AND SANTI E., *Numerical evaluation of Cauchy principal integrals by means of nodal spline approximation*, Revue d'Anal. Num. Theor. Approx. **27** (1998), 59–69.

[12] DAHMEN W., GOODMAN T.N.T. AND MICCHELLI C.A., *Compactly supported fundamental functions for spline interpolation*, Numer. Math. **52** (1988), 641–664.

[13] DEMICHELIS V., *Uniform convergence for Cauchy principal value integrals of modified quasi-interpolatory splines*, Intern. J. Computer Math. **53** (1994), 189–196.

[14] DEMICHELIS V., *Convergence of derivatives of optimal nodal splines*, J. Approx. Th. **88** (1997), 370–383.

[15] DEMICHELIS V. AND RABINOWITZ P., *Finite part integrals and modified splines*, to appear in BIT .

[16] DE VILLIERS J.M., *A convergence result in nodal spline interpolation*, J. Approx. Theory **74** (1993), 266–279.

[17] DE VILLIERS J.M. AND ROHWER C.H., *Optimal local spline interpolants*, J. Comput. Appl. Math. **18** (1987), 107–119.

[18] DE VILLIERS J.M. AND ROHWER C.H., *A nodal spline generalization of the Lagrange interpolant*, in: "Progress in Approximation Theory" (Eds. Nevai P. and Pinkus A.), Academic Press, Boston 1991, 201–211.

[19] DE VILLIERS J.M. AND ROHWER C.H., *Sharp bounds for the Lebesgue constant in quadratic nodal spline interpolation*, International Series of Num. Math. **115** (1994), 1–13.

[20] DIETHELM K., *Error bounds for spline-based quadrature methods for strongly singular integrals*, J. Comput. Appl. Math. **89** (1998), 257–261.

[21] DIETHELM K., *Error bounds for spline-based quadrature methods for strongly singular integrals*, J. Comp. Appl. Math. **142** (2002), 449–450.

[22] GORI C. AND SANTI E., *Spline method for the numerical solution of Volterra integral equations of the second kind*, in: "Integral and Integro-differential equation" (Eds. Agarwal R. and Regan D.O.), Ser. Math. Anal. Appl. **2**, Gordon and Breach, Amsterdam 2000, 91–99.

[23] GORI C., SANTI E. AND CIMORONI M.G., *Projector-splines in the numerical solution of integro-differential equations*, Computers Math. Applic. **35** (1998), 107–116.

[24] LYCHE T. AND SCHUMAKER L.L., *Local spline approximation methods*, J. Approx. Theory **15** (1975), 294–325.

[25] MONEGATO G., *The numerical evaluation of a 2-D Cauchy principal value integral arising in boundary integral equation methods*, Math. Comp. **62** (1994), 765–777.

[26] RABINOWITZ P., *Product integration of singular integrals using optimal nodal splines*, Rend. Sem. Mat. Univ. Pol. Torino **51** (1993), 1–9.

[27] RABINOWITZ P., *Application of approximating splines for the solution of Cauchy singular integral equations*, Appl. Num. Math. **15** (1994), 285–297.

[28] RABINOWITZ P., *Optimal quasi-interpolatory splines for numerical integration*, Annals Numer. Math. **2** (1995), 145–157.

[29] RABINOWITZ P., *Uniform convergence results for finite-part integrals*, in: "Workshop on Analysis celebrating the 60^{th} birthday of Peter Vértesi and in memory of Ottò Kis and Apard Elbert", Budapest 2001.

[30] RABINOWITZ P. AND SANTI E., *On the uniform convergence of Cauchy principal values of quasi-interpolating splines*, BIT **35** (1995), 277–290.

[31] SANTI E., *Uniform convergence results for certain two-dimensional Cauchy principal value integrals*, Portugaliae Math. **57** (2000), 191–201.

[32] SANTI E. AND CIMORONI M.G., *On the convergence of projector-splines for the numerical evaluation of certain two-dimensional CPV integrals*, J. Comp. Math. **20** (2002), 113–120.

AMS Subject Classification: 65D30, 65D07.

Catterina DAGNINO, Vittoria DEMICHELIS
Dipartimento di Matematica
Università di Torino
via Carlo Alberto 8
10123 Torino, ITALIA
e-mail: `catterina.dagnino@unito.it`
`vittoria.demichelis@unito.it`

Elisabetta SANTI
Dipartimento di Energetica
Università dell' Aquila
Monteluco di Roio
67040 L'Aquila, ITALIA
e-mail: `esanti@dsiaq1.ing.univaq.it`