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T. Gramchev*

PERTURBATIVE METHODS IN SCALES OF BANACH SPACES: APPLICATIONS FOR GEVREY REGULARITY OF SOLUTIONS TO SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS

Abstract. We outline perturbative methods in scales of Banach spaces of Gevrey functions for dealing with problems of the uniform Gevrey regularity of solutions to partial differential equations and nonlocal equations related to stationary and evolution problems. The key of our approach is to use suitably chosen Gevrey norms expressed as the limit for $N \to \infty$ of partial sums of the type

$$\sum_{\alpha\in\mathbb{Z}_+^n, |\alpha|\le N} \frac{T^{|\alpha|}}{(\alpha!)^\sigma} \|D_x^\alpha u\|_{H^s(\mathbb{R}^n)}$$

for solutions to semilinear elliptic equations in $\mathbb{R}^n$. We also show (sub)exponential decay in the framework of Gevrey functions from Gelfand-Shilov spaces $S_\nu^\mu(\mathbb{R}^n)$ using sequences of norms depending on two parameters

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^n, |\alpha|+|\beta|\le N} \frac{\varepsilon^{|\beta|}T^{|\alpha|}}{(\alpha!)^\mu(\beta!)^\nu} \|x^\beta D^\alpha u\|_{H^s(\mathbb{R}^n)}.$$

For solutions $u(t,\cdot)$ of evolution equations we employ norms of the type

$$\sum_{\alpha\in\mathbb{Z}_+^n, |\alpha|\le N} \sup_{0<t<T} \left(\frac{t^\theta(\rho(t))^{|\alpha|}}{(\alpha!)^\sigma} \|D_x^\alpha u(t,\cdot)\|_{L^p(\mathbb{R}^n)}\right)$$

for some $\theta \ge 0$, $1 < p < \infty$, $\rho(t) \searrow 0$ as $t \searrow 0$.

The use of such norms allows us to implement a Picard type scheme for seemingly different problems of uniform Gevrey regularity and to reduce the question to the boundedness of an iteration sequence $z_N(T)$ (which is one of the N-th partial sums above) satisfying inequalities of the type

$$z_{N+1}(T) \le \delta_0 + C_0 T z_N(T) + g(T; z_N(T))$$

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with T being a small parameter, and g being at least quadratic in u near $u = 0$.

We propose examples showing that the hypotheses involved in our abstract perturbative approach are optimal for the uniform Gevrey regularity.

1. Introduction

The main aim of the present work is to develop a unified approach for investigating problems related to the uniform G^σ Gevrey regularity of solutions to PDE on the whole space $\mathbb{R}^n$ and the uniform Gevrey regularity with respect to the space variables of solutions to the Cauchy problem for semilinear parabolic systems with polynomial nonlinearities and singular initial data. Our approach works also for demonstrating exponential decay of solutions to elliptic equations provided we know a priori that the decay for $|x| \to \infty$ is of the type $o(|x|^{-\tau})$ for some $0 < \tau \ll 1$.

The present article proposes generalizations of the body of iterative techniques for showing Gevrey regularity of solutions to nonlinear PDEs in Mathematical Physics in papers of H.A. Biagioni* and the author.

We start by recalling some basic facts about the Gevrey spaces. We refer to [50] for more details. Let $\sigma \geq 1$, $\Omega \subset \mathbb{R}^n$ be an open domain. We denote by $G^\sigma(\mathbb{R}^n)$ (the Gevrey class of index σ) the set of all $f \in C^\infty(\Omega)$ such that for every compact subset $K \subset\subset \Omega$ there exists $C = C_{f,K} > 0$ such that

$$\sup_{\alpha \in \mathbb{Z}_+^n} \left(\frac{C^{|\alpha|}}{(\alpha!)^\sigma} \sup_{x \in K} |\partial_x^\alpha f(x)| \right) < +\infty,$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Throughout the present paper we will investigate the regularity of solutions of stationary PDEs in $\mathbb{R}^n$ in the frame of the L^2 based uniformly Gevrey G^σ functions on $\mathbb{R}^n$ for $\sigma \geq 1$. Here $f \in G^\sigma_{un}(\mathbb{R}^n)$ means that for some $T > 0$ and $s \geq 0$

$$\sup_{\alpha \in \mathbb{Z}_+^n} \left(\frac{T^{|\alpha|}}{(\alpha!)^\sigma} \|\partial_x^\alpha f\|_s \right) < +\infty, \tag{1}$$

where $\|f\|_s = \|f\|_{H^s(\mathbb{R}^n)}$ stands for a $H^s(\mathbb{R}^n) = H_2^s(\mathbb{R}^n)$ norm for some $s \geq 0$. In particular, if $\sigma = 1$, we obtain that every $f \in G^1_{un}(\mathbb{R}^n)$ is extended to a holomorphic function in $\{z \in \mathbb{C}^n;\ |Im\, z| < T\}$. Note that given $f \in G^\sigma_{un}(\mathbb{R}^n)$ we can define

$$\rho_\sigma(f) = \sup\{T > 0 : \text{ such that (1) holds}\}. \tag{2}$$

One checks easily by the Sobolev embedding theorem and the Stirling formula that the definition (2) is invariant with respect to the choice of $s \geq 0$. One may call $\rho_\sigma(f)$ the uniform G^σ Gevrey radius of $f \in G^\sigma_{un}(\mathbb{R}^n)$.

*She has passed away on June 18, 2003 after struggling with a grave illness. The present paper is a continuation following the ideas and methods contained in [6], [7] and especially [8] and the author dedicates it to her memory.

We will use scales of Banach spaces of G^σ functions with norms of the following type

$$\sum_{k=0}^{\infty} \frac{T^{|k|}}{(k!)^\sigma} \sum_{j=0}^{n} \|D_j^k u\|_s, \quad D_j = D_{x_j}.$$

For global L^p based Gevrey norms of the type (1) we refer to [8], cf. [27] for local L^p based norms of such type, [26] for $|f|_\infty := \sup_\Omega |f|$ based Gevrey norms for the study of degenerate Kirchhoff type equations, see also [28] for similar scales of Banach spaces of periodic G^σ functions. We stress that the use $\sum_{j=1}^{n} \|D_j^k u\|_s$ instead of $\sum_{|\alpha|=k} \|D_x^\alpha u\|_s$ allows us to generalize with simpler proofs hard analysis type estimates for $G^\sigma_{un}(\mathbb{R}^n)$ functions in [8].

We point out that exponential G^σ norms of the type

$$\|u\|_{\sigma,T;exp} := \sqrt{\int_{\mathbb{R}^n} e^{2T|\xi|^{1/\sigma}} |\hat{u}(\xi)|^2 \, d\xi}$$

have been widely (and still are) used in the study of initial value problems for weakly hyperbolic systems, local solvability of semilinear PDEs with multiple characteristics, semilinear parabolic equations, (cf. [23], [6], [30] for $\sigma = 1$ and [12], [20], [27], [28] when $\sigma > 1$ for applications to some problems of PDEs and Dynamical Systems).

The abstract perturbative methods which will be exposed in the paper aim at dealing with 3 seemingly different problems. We write down three model cases.

1. First, given an elliptic linear constant coefficients partial differential operator P in $\mathbb{R}^n$ and an entire function f we ask whether one can find $s_{cr} > 0$ such that

$$(P1) \qquad \begin{array}{c} Pu + f(u) = 0, \; u \in H^s(\mathbb{R}^n), \, s > s_{cr} \\ \text{implies} \\ u \in \mathcal{O}\{z \in \mathbb{C}^n : |\Im z| < T\} \text{ for some } T > 0 \end{array}$$

while for (some) $s < s_{cr}$ the implication is false.

Recall the celebrated KdV equation

$$(3) \qquad u_t - u_{xxx} - auu_x = 0 \qquad x \in \mathbb{R}, \; t > 0, \; a > 0$$

or more generally the generalized KdV equation

$$(4) \qquad u_t - u_{xxx} + au^p u_x = 0 \qquad x \in \mathbb{R}, \; t > 0 \; a > 0$$

where p is an odd integer (e.g., see [34] and the references therein). We recall that a solution u in the form $u(x,t) = v(x+ct)$, $v \neq 0$, $c \in \mathbb{R}$, is called solitary (traveling) wave solution. It is well known that v satisfies the second order Newton equation (after plugging $v(x+ct)$ in (4) and integrating)

$$(5) \qquad v'' - cv + \frac{a}{p+1} v^{p+1} = 0,$$

and if $c > 0$ we have a family of explicit solutions

$$v_c(x) = \frac{C_{p,a}}{(\cosh((p-1)\sqrt{c}x))^{2/(p+1)}} \qquad x \in \mathbb{R}, \tag{6}$$

for explicit positive constant $C_{p,a}$.

Incidentally, $u_c(t,x) = e^{ict}v_c(x)$, $c > 0$ solves the nonlinear Schrödinger equation

$$iu_t - u_{xx} + a|u|^p u = 0 \qquad x \in \mathbb{R},\ t > 0\ a > 0 \tag{7}$$

and is called also stationary wave solution cf. [11], [34] and the references therein.

Clearly the solitary wave v_c above is uniformly analytic in the strip $|\Im x| \leq T$ for all $0 < T < \pi/((p-1)\sqrt{c})$. One can show that the uniform G^1 radius is given by $\rho_1[v_s] = \pi/(((p-1)\sqrt{c})$.

In the recent paper of H. A. Biagioni and the author [8] an abstract approach for attacking the problem of uniform Gevrey regularity of solutions to semilinear PDEs has been proposed. One of the key ingredients was the introduction of L^p based norms of $G^\sigma_{un}(\mathbb{R}^n)$ functions which contain infinite sums of fractional derivatives in the non-analytic case $\sigma > 1$. Here we restrict our attention to simpler L^2 based norms and generalize the results in [8] with simpler proofs. The hard analysis part is focused on fractional calculus (or generalized Leibnitz rule) for nonlinear maps in the framework of $L^2(\mathbb{R}^n)$ based Banach spaces of uniformly Gevrey functions $G^\sigma_{un}(\mathbb{R}^n)$, $\sigma \geq 1$. In particular, we develop functional analytic approaches in suitable scales of Banach spaces of Gevrey functions in order to investigate the $G^\sigma_{un}(\mathbb{R}^n)$ regularity of solutions to semilinear equations with Gevrey nonlinearity on the whole space $\mathbb{R}^n$:

$$Pu + f(u) = w(x), \qquad x \in \mathbb{R}^n \tag{8}$$

where P is a Gevrey G^σ pseudodifferential operator or a Fourier multiplier of order m, and $f \in G^\theta$ with $1 \leq \theta \leq \sigma$. The crucial hypothesis is some G^σ_{un} estimates of commutators of P with $D^\alpha_j := D^\alpha_{x_j}$

If $n = 1$ we capture large classes of dispersive equations for solitary waves (cf. [4], [21], [34], [42], for more details, see also [1], [2], [10] and the references therein).

Our hypotheses are satisfied for: $P = -\Delta + V(x)$, where the real potential $V(x) \in G^\sigma_{un}(\mathbb{R}^n)$ is real valued, bounded from below and $\lim_{|x|\to\infty} V(x) = +\infty$; P being an arbitrary linear elliptic differential operator with constant coefficients. We allow also the order m of P to be less than one (cf. [9] for the so called fractal Burgers equations, see also [42, Theorem 10, p.51], where G^σ, $\sigma > 1$, classes are used for the Whitham equation with antidissipative terms) and in that case the Gevrey index σ will be given by $\sigma \geq 1/m > 1$. We show $G^\sigma_{un}(\mathbb{R}^n)$ regularity of every solution $u \in H^s(\mathbb{R}^n)$ with $s > s_{cr}$, depending on n, the order of P and the type of nonlinearity. For general analytic nonlinearities, $s_{cr} > n/p$. However, if $f(u)$ is polynomial, s_{cr} might be taken less than $n/2$, and in that case s_{cr} turns out to be related to the critical index of the singularity of the initial data for semilinear parabolic equations, cf. [15], [5] [49] (see also [25] for $H^s(\mathbb{R}^n) := H^s_2(\mathbb{R}^n)$, $0 < s < n/2$ solutions in $\mathbb{R}^n$, $n \geq 3$, to semilinear elliptic equations).

The proof relies on the nonlinear Gevrey calculus and iteration inequalities of the type $z_{N+1}(T) \leq z_0(T) + g(T, z_N(T))$, $N \in \mathbb{Z}_+$, $T > 0$ where $g(T, 0) = 0$ and

$$z_N(T) = \sum_{k=0}^{N} \frac{T^k}{(k!)^\sigma} \sum_{j=1}^{n} \|D_j^k u\|_s. \tag{9}$$

Evidently the boundedness of $\{z_N(T)\}_{N=1}^{\infty}$ for some $T > 0$ implies that $z_{+\infty}(T) = \|u\|_{\sigma,T;s} < +\infty$, i.e., $u \in G^\sigma_{un}(R^n)$. We recover the results of uniform analytic regularity of dispersive solitary waves (cf. J. Bona and Y. Li, [11], [40]), and we obtain $G^\sigma_{un}(\mathbb{R}^n)$ regularity for $u \in H^s(\mathbb{R}^n)$, $s > n/2$ being a solution of equations of the type $-\Delta u + V(x)u = f(u)$, where $f(u)$ is polynomial, $\nabla V(x)$ satisfies (1) and for some $\mu \in \mathbb{C}$ the operator $(-\Delta + V(x) - \mu)^{-1}$ acts continuously from $L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. An example of such $V(x)$ is given by $V(x) = V_\sigma(x) =< x >^\rho \exp(-\frac{1}{|x|^{1/(\sigma-1)}})$ for $\sigma > 1$, and $V(x) =< x >^\rho$ if $\sigma = 1$, for $0 < \rho \leq 1$, where $< x >= \sqrt{1 + x^2}$. In fact, we can capture also cases where $\rho > 1$ (like the harmonic oscillator), for more details we send to Section 3.

We point out that our results imply also uniform analytic regularity $G^1_{un}(\mathbb{R})$ of the $H^2(\mathbb{R})$ solitary wave solutions $r(x - ct)$ to the fifth order evolution PDE studied by M. Groves [29] (see Remark 2 for more details).

Next, modifying the iterative approach we obtain also new results for the analytic regularity of stationary type solutions which are bounded but not in $H^s(\mathbb{R}^n)$. As an example we consider Burgers' equation (cf. [32])

$$u_t - \nu u_{xx} + u u_x = 0, \qquad x \in \mathbb{R},\ t > 0 \tag{10}$$

which admits the solitary wave solution $\varphi_c(x + ct)$ given by

$$\varphi_c(x) = \frac{2c}{ae^{-cx} + 1}, \qquad x \in \mathbb{R}. \tag{11}$$

for $a \geq 0$, $c \in \mathbb{R} \setminus 0$. Clearly φ_c extends to a holomorphic function in the strip $|\Im x| < \pi/|c|$ while $\lim_{x \to \operatorname{sign}(c)\infty} \varphi_c(x) = 2c$ and therefore $\varphi_c \notin L^2(\mathbb{R})$. On the other hand

$$\varphi_c'(x) = \frac{2cae^{-cx}}{(ae^{-cx} + 1)^2}, \qquad x \in \mathbb{R}. \tag{12}$$

One can show that $\varphi_c' \in G^1_{un}(\mathbb{R}^n)$. It was shown in [8], Section 5, that if a bounded traveling wave satisfies in addition $v' \in H^1(\mathbb{R})$ then $v' \in G^1_{un}(\mathbb{R}^n)$. We propose generalizations of this result. We emphasize that we capture as particular cases the bore-like solutions to dissipative evolution PDEs (Burgers' equation, the Fisher-Kolmogorov equation and its generalizations cf. [32], [37], [31], see also the survey [55] and the references therein).

We exhibit an explicit recipe for constructing strongly singular solutions to higher order semilinear elliptic equations with polynomial nonlinear terms, provided they have suitable homogeneity properties involving the nonlinear terms (see Section 6). In such a way we generalize the results in [8], Section 7, where strongly singular solutions of $-\Delta u + cu^d = 0$ have been constructed. We give other examples of weak nonsmooth solutions to semilinear elliptic equations with polynomial nonlinearity which are in $H^s(\mathbb{R}^n)$, $0 < s < n/2$ but with $s \leq s_{cr}$ cf. [25] for the particular case of $-\Delta u + cu^{2k+1} = 0$ in $\mathbb{R}^n$, $n \geq 3$. The existence of such classes of singular solutions are examples which suggest that our requirements for initial regularity of the solution are essential in order to deduce uniform Gevrey regularity. This leads to, roughly speaking, a kind of dichotomy for classes of elliptic semilinear PDE's in $\mathbb{R}^n$ with polynomial nonlinear term, namely, that any solution is either extendible to a holomorphic function in a strip $\{z \in \mathbb{C}^n : |Imz| \leq T\}$, for some $T > 0$, or for some specific nonlinear terms the equation admits solutions with singularities (at least locally) in $H^s_p(\mathbb{R}^n)$, $s < s_{cr}$.

2. The second aim is motivated by the problem of the type of decay - polynomial or exponential - of solitary (traveling) waves (e.g., cf. [40] and the references therein), which satisfy frequently nonlocal equations. We mention also the recent work by P. Rabier and C. Stuart [48], where a detailed study of the pointwise decay of solutions to second order quasilinear elliptic equations is carried out (cf also [47]).

The example of the solitary wave (6) shows that we have both uniform analyticity and exponential decay. In fact, by the results in [8], Section 6, one readily obtains that v_c defined in (6) belongs to the Gelfand–Shilov class $S^1(\mathbb{R}^n) = S^1_1(\mathbb{R}^n)$. We recall that given $\mu > 0$, $\nu > 0$ the Gelfand-Shilov class $S^\nu_\mu(\mathbb{R}^n)$ is defined as the set of all $f \in G^\mu(\mathbb{R}^n)$ such that there exist positive constants C_1 and C_2 satisfying

$$|\partial_x^\alpha f(x)| \leq C_1^{|\alpha|+1}(\alpha!)^\nu e^{-C_2|x|^{1/\mu}}, \qquad x \in \mathbb{R}^n, \alpha \in \mathbb{Z}^n_+. \tag{13}$$

We will use a characterization of $S^\nu_\mu(\mathbb{R}^n)$ by scales of Banach spaces with norms

$$|||f|||_{\mu,\nu;\varepsilon,T} \quad = \quad \sum_{j,k\in\mathbb{Z}^n_+} \frac{\varepsilon^{|j|}T^{|k|}}{(j!)^\nu (k!)^\mu}\|x^j D^k_x u\|_s.$$

In particular, $S^\nu_\mu(\mathbb{R}^n)$ contains nonzero functions iff $\mu+\nu \geq 1$ (for more details on these spaces we refer to [24], [46], see also [17], [18] for study of linear PDE in $S_\theta(\mathbb{R}^n) := S^\theta_\theta(\mathbb{R}^n)$).

We require three essential conditions guaranteeing that every solution $u \in H^s(\mathbb{R}^n)$, $s > s_{cr}$ of (8) for which it is known that it decays polynomially for $|x| \to \infty$ necessarily belongs to $S^\mu_\nu(\mathbb{R}^n)$ (i.e., it satisfies (13) or equivalently $|||u|||_{\mu,\nu;\varepsilon,T} < +\infty$ for some $\varepsilon > 0$, $T > 0$). Namely: the operator P is supposed to be invertible; f has no linear term, i.e., f is at least quadratic near the origin; and finally, we require that the $H^s(\mathbb{R}^n)$ based norms of commutators of P^{-1} with operators of the type $x^\beta D^\alpha_x$ satisfy certain analytic–Gevrey estimates for all $\alpha, \beta \in \mathbb{Z}^n_+$. The key is again an iterative approach, but this time one has to derive more subtle estimates involving partial sums for the Gevrey norms $|||f|||_{\mu,\nu;\varepsilon,T}$ of the type

$$z_N(\mu,\nu;\varepsilon,T) \quad = \quad \sum_{|j+k|\leq N} \frac{\varepsilon^{|j|}T^{|k|}}{(j!)^\nu (k!)^\mu}\|x^j D^k_x u\|_s.$$

The (at least) quadratic behaviour is crucial for the aforementioned gain of the rate of decay for $|x| \to 0$ and the technical arguments resemble some ideas involved in the Newton iterative method. If $\mu = \nu = 1$ we get the decay estimates in [8], and as particular cases of our general results we recover the well known facts about polynomial and exponential decay of solitary waves, and obtain estimates for new classes of stationary solutions of semilinear PDEs. We point out that different type of G^1_{un} Gevrey estimates have been used for getting better large time decay estimates of solutions to Navier–Stokes equations in $\mathbb{R}^n$ under the assumption of initial algebraic decay (cf. M. Oliver and E. Titi [44]).

As it concerns the sharpness of the three hypotheses, examples of traveling waves for some nonlocal equations in Physics having polynomial (but not exponential) decay for $|x| \to 0$ produce counterexamples when (at least some of the conditions) fail.

3. The third aim is to outline iterative methods for the study of the Gevrey smoothing effect of semilinear parabolic systems for positive time with singular initial data. More precisely, we consider the Cauchy problem of the type

$$\partial_t u + (-\Delta)^m u + f(u) = 0, \;\; u|_{t=0} = u^0, \quad t > 0, \; x \in \Omega, \tag{14}$$

where $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{T}^n$. We investigate the influence of the elliptic dissipative terms of evolution equations in $\mathbb{R}^n$ and $\mathbb{T}^n$ on the critical L^p, $1 \le p \le \infty$, index of the singularity of the initial data u^0, the analytic regularity with respect to $x \in \Omega$ for positive time and the existence of self-similar solutions. The approach is based again on the choice of suitable L^p based Banach spaces with timedepending Gevrey norms with respect to the space variables x and then fixed point type iteration scheme.

The paper is organized as follows. Section 2 contains several nonlinear calculus estimates for Gevrey norms. Section 3 presents an abstract approach and it is dedicated to the proof of uniform Gevrey regularity of a priori $H^s(\mathbb{R}^n)$ solutions u to semilinear PDEs, while Section 4 deals with solutions u which are bounded on $\mathbb{R}^n$ such that $\nabla u \in H^s(\mathbb{R}^n)$. We prove Gevrey type exponential decay results in the frame of the Gelfand-Shilov spaces $S^\mu_\nu(\mathbb{R}^n)$ in Section 5. Strongly singular solutions to semilinear elliptic equations are constructed in Section 6. The last two sections deal with the analytic-Gevrey regularizing effect in the space variables for solutions to Cauchy problems for semilinear parabolic systems with polynomial nonlinearities and singular initial data.

2. Nonlinear Estimates in Gevrey Spaces

Given $s > n$, $T > 0$ we define

$$G^\sigma(T; H^s) = \{v : \|v\|_{\sigma,T;s} := \sum_{k=0}^{\infty} \sum_{j=1}^{n} \frac{T^k}{(k!)^\sigma} \|D^k_{x_j} v\|_s < +\infty\}, \tag{15}$$

and

$$G^\sigma_\infty(T;H^s) = \{v : |||v|||_{\sigma,T;s} = \|v\|_{L^\infty} + \sum_{k=0}^{+\infty}\sum_{j=0}^{n} \frac{T^k}{(k!)^\sigma}\|D_j^k\nabla v\|_s < +\infty\}. \tag{16}$$

We have

LEMMA 1. *Let $s > n/2$. Then the spaces $G^\sigma(T;H^s)$ and $G^\sigma_\infty(T;H^s)$ are Banach algebras.*

We omit the proof since the statement for $G^\sigma(T;H^s)$ is a particular case of more general nonlinear Gevrey estimates in [27]) while the proof for $G^\sigma_\infty(T;H^s)$ is essentially the same.

We need also a technical assertion which will play a crucial role in deriving some nonlinear Gevrey estimates in the next section.

LEMMA 2. *Given $\rho \in (0,1)$, we have*

$$\| < D >^{-\rho} D_j^k u\|_s \le \varepsilon\|D_j^k u\|_s + (1-\rho)\left(\frac{\rho}{\varepsilon}\right)^{1/(1-\rho)}\|D_j^{k-1}u\|_s \tag{17}$$

for all $k \in \mathbb{N}$, $s \ge 0$, $u \in H^{s+k}(\mathbb{R}^n)$, $j = 1,\dots,n$, $\varepsilon > 0$. Here $< D >$ stands for the constant p.d.o. with symbol $< \xi >= (1+|\xi|^2)^{1/2}$.

Proof. We observe that $< \xi >^{-\rho} |\xi_j|^k \le |\xi_j|^{k-\rho}$ for $j = 1,\dots,n$, $\xi \in \mathbb{R}^n$. Set $g_\varepsilon(t) = \varepsilon t - t^\rho$, $t \ge 0$. Straightforward calculations imply

$$\min_{g\in\mathbb{R}} g(t) = g((\frac{\rho}{\varepsilon})^{1/1-\rho}) = -(1-\rho)\left(\frac{\rho}{\varepsilon}\right)^{1/(1-\rho)}$$

which concludes the proof. □

We show some combinatorial inequalities which turn out to be useful in for deriving nonlinear Gevrey estimates (cf [8]).

LEMMA 3. *Let $\sigma \ge 1$. Then there exists $C > 0$ such that*

$$\frac{\ell!(\sigma\ell_\mu + r)!\prod_{\nu\ne\mu}(\sigma\ell_\nu)!}{\ell_1!\cdots\ell_j!(\sigma\ell + r)!} \le C^j, \tag{18}$$

for all $j \in \mathbb{N}$, $\ell = \ell_1 + \cdots + \ell_j$, $\ell_i \in \mathbb{N}$, $\mu \in \{1,\dots,j\}$ and $0 \le r < \sigma$, with $k! := \Gamma(k+1)$, $\Gamma(z)$ being the Gamma function.

Proof. By the Stirling formula, we can find two constants $C_2 > C_1 > 0$ such that

$$C_1\frac{k^{k+\frac12}}{e^k} \le k! \le C_2\frac{k^{k+\frac12}}{e^k}$$

for all $k \in \mathbb{N}$. Then the left–hand side in (18) can be estimated by:

$$
\begin{aligned}
& \frac{C_2^{j+1}\ell^{\ell+\frac{1}{2}}(\sigma\ell_\mu+r)^{\sigma\ell_\mu+r+\frac{1}{2}}\prod_{\nu\neq\mu}(\sigma\ell_\nu)^{\sigma\ell_\nu+\frac{1}{2}}}{C_1^{j+1}\ell_1^{\ell_1+\frac{1}{2}}\cdots\ell_j^{\ell_j+\frac{1}{2}}(\sigma\ell+r)^{\sigma\ell+r+\frac{1}{2}}} \\
= \; & \left(\frac{C_2}{C_1}\right)^{j+1}\frac{\ell^{\ell}(\sigma\ell_\mu+r)^{\sigma\ell_\mu+r}\prod_{\nu\neq\mu}(\sigma\ell_\nu)^{\sigma\ell_\nu}}{\prod_{\nu=1}^{j}\ell_\nu^{\ell_\nu}(\sigma\ell+r)^{\sigma\ell+r}}\left[\frac{\ell(\sigma\ell_\mu+r)}{\ell_\mu(\sigma\ell+r)}\right]^{\frac{1}{2}}\sigma^{\frac{j-1}{2}} \\
\leq \; & C_3^j\frac{\ell^{\ell}(\sigma\ell_\mu+r)^{\sigma\ell_\mu+r}\sigma^{\sigma(\ell-\ell_\mu)}[\prod_{\nu\neq\mu}\ell_\nu^{\ell_\nu}]^{\sigma}}{\prod_{\nu=1}^{j}\ell_\nu^{\ell_\nu}(\sigma\ell+r)^{\sigma\ell+r}} \\
\leq \; & C_3^j\frac{\ell^{\ell}(\sigma\ell_\mu+r)^{\sigma\ell_\mu}\sigma^{\sigma(\ell-\ell_\mu)}[\prod_{\nu\neq\mu}\ell_\nu^{\ell_\nu}]^{\sigma-1}}{\ell_\mu^{\ell_\mu}(\sigma\ell+r)^{\sigma\ell}} \\
= \; & C_3^j\frac{\ell^{\ell}(\ell_\mu+\frac{r}{\sigma})^{\sigma\ell_\mu}\left[\prod_{\nu\neq\mu}\ell_\nu^{\ell_\nu}\right]^{\sigma-1}}{\ell_\mu^{\ell_\mu}(\ell+\frac{r}{\sigma})^{\sigma\ell}} \\
= \; & C_3^j\frac{\ell^{\ell}(\ell_\mu+\frac{r}{\sigma})^{(\sigma-1)\ell_\mu}(\ell_\mu+\frac{r}{\sigma})^{\ell_\mu}\left[\prod_{\nu\neq\mu}\ell_\nu^{\ell_\nu}\right]^{\sigma-1}}{(\ell+\frac{r}{\sigma})^{\ell}(\ell+\frac{r}{\sigma})^{(\sigma-1)\ell}\ell_\mu^{\ell_\mu}} \\
\leq \; & C_3^j e^{\frac{r}{\sigma}}\left[\frac{(\ell_\mu+\frac{r}{\sigma})^{\ell_\mu}\prod_{\nu\neq\mu}\ell_\nu^{\ell_\nu}}{(\ell+\frac{r}{\sigma})^{\ell_1+\cdots+\ell_j}}\right]^{\sigma-1} \leq C_3^j e^{\frac{r}{\sigma}}, \qquad N\in\mathbb{N}
\end{aligned}
$$

which implies (18) since $0 < r \leq \sigma$.

□

Given $s > n/2$ we associate two N-th partial sums for the norm in (15)

$$
S_N^{\sigma}[v;T,s] = \sum_{k=0}^{N}\frac{T^k}{(k!)^{\sigma}}\sum_{j=1}^{n}\|D_{x_j}^k v\|_s, \tag{19}
$$

$$
\widetilde{S}_N^{\sigma}[v;T,s] = \sum_{k=1}^{N}\frac{T^k}{(k!)^{\sigma}}\sum_{j=1}^{n}\|D_{x_j}^k v\|_s. \tag{20}
$$

Clearly (19) and (20) yield

$$
S_N^{\sigma}[v;T,s] = \|v\|_s + \widetilde{S}_N^{\sigma}[v;T,s]. \tag{21}
$$

LEMMA 4. *Let $f \in G^{\theta}(Q)$ for some $\theta \geq 1$, where $Q \subset \mathbb{R}^p$ is an open neighbourhood of the origin in $\mathbb{R}^p$, $p \in \mathbb{N}$ satisfying $f(0) = 0$, $\nabla f(0) = 0$. Then for $v \in H^{\infty}(\mathbb{R}^n : \mathbb{R}^p)$ there exists a positive constant A_0 depending on $\|v\|_s$, $\rho_\theta(f|_{B_{|v|_\infty}})$, where B_R stands for the ball with radius R, such that*

$$
\widetilde{S}_N^{\sigma}[f(v);T,s] \leq |\nabla f(v)|_\infty \widetilde{S}_N^{\sigma}[v;T,s] + \sum_{j\in\mathbb{Z}_+^p,\,2\leq|j|\leq N}\frac{A_0^j}{(j!)^{\sigma-\theta}}(\widetilde{S}_{N-1}^{\sigma}[v;T,s])^j, \tag{22}
$$

for $T > 0$, $N \in \mathbb{N}$, $N \geq 2$.

Proof. Without loss of generality, in view of the choice of the H^s norm, we will carry out the proof for $p = n = 1$. First, we recall that

$$\begin{aligned} D^k(f(v(x))) &= \sum_{j=1}^{k} \frac{(D^j f)(v(x))}{j!} \sum_{\substack{k_1+\cdots+k_j=k \\ k_1\geq 1,\cdots,k_j\geq 1}} \prod_{\mu=1}^{j} \frac{D^{k_\mu} v(x)}{k_\mu!} \\ &= f'(v(x)) D^k v(x) \\ &+ \sum_{j=2}^{k} \frac{(D^j f)(v(x))}{j!} \sum_{\substack{k_1+\cdots+k_j=k \\ k_1\geq 1,\cdots,k_j\geq 1}} \prod_{\mu=1}^{j} \frac{D^{k_\mu} v(x)}{k_\mu!}. \end{aligned} \tag{23}$$

Thus

$$\begin{aligned} \widetilde{S}_N^\sigma[f(v); T, s] &\leq \omega_s \|f'(v)\|_s \widetilde{S}_N^\sigma[v; T, s] + \sum_{k=1}^{N} \sum_{j=1}^{k} \frac{\|(D^j f)(v)\|_s)}{(j!)^\theta} \frac{\omega_s^j}{(j)!^{\sigma-\theta}} \\ &\times \sum_{\substack{k_1+\cdots+k_j=k \\ k_1\geq 1,\cdots,k_j\geq 1}} M_{k_1,\ldots,k_j}^{\sigma,j} \prod_{\mu=1}^{j} \frac{T^{k_\mu} \|D^{k_\mu} v\|_s}{(k\mu!)^\sigma} \end{aligned} \tag{24}$$

where ω_s is the best constant in the Schauder Lemma for $H^s(\mathbb{R}^n)$, $s > n/2$, and

$$M_{k_1,\ldots,k_j}^{\sigma,j} = \left(\frac{k_1! \cdots k_j! j!}{(k_1 + \cdots + k_j)!} \right)^{\sigma-1}, \qquad j, k_\mu \in \mathbb{N}, k_\mu \geq 1, \mu = 1, \ldots, j. \tag{25}$$

We get, thanks to the fact that $k_\mu \geq 1$ for every $\mu = 1, \ldots, j$, that

$$M_{k_1,\ldots,k_j}^{\sigma,j} \leq 1, \qquad k_\mu \in \mathbb{N}, k_\mu \geq 1, \mu = 1, \ldots, j \tag{26}$$

(see [27]). Combining (26) with nonlinear superposition Gevrey estimates in [27] we obtain that there exists $A_0 = A_0(f, \|v\|_s) > 0$ such that

$$\omega_s^j \frac{\|(D^j f)(v)\|_s}{(j!)^\theta} \leq A_0^j, \qquad j \in \mathbb{N}. \tag{27}$$

We estimate (24) by

$$
\begin{aligned}
\widetilde{S}^{\sigma}_{N}[f(v); T, s] &\le \omega_s \|f'(v)\|_s \widetilde{S}^{\sigma}_{N}[v; T, s] \\
&+ \sum_{k=2}^{N} \sum_{j=2}^{k} \frac{\|(D^j f)(v)\|_s)}{(j!)^{\theta}} \frac{\omega_s^j}{(j)!^{\sigma-\theta}} \\
&\times \sum_{\substack{k_1+\cdots+k_j=k \\ k_1\ge 1,\cdots,k_j\ge 1}} \prod_{\mu=1}^{j} \frac{T^{k_\mu}\|D^{k_\mu} v\|_s}{(k\mu!)^{\sigma}} \\
&\le \omega_s \|f'(v)\|_s \widetilde{S}^{\sigma}_{N}[v; T, s] \\
&+ \sum_{j=2}^{N} \frac{A_0^j}{(j)!^{\sigma-\theta}} (\widetilde{S}^{\sigma}_{N-1}[v; T, s])^j. \qquad (28)
\end{aligned}
$$

The proof is complete.

□

We also propose an abstract lemma which will be useful for estimating Gevrey norms by means of classical iterative Picard type arguments.

LEMMA 5. *Let $a(T)$, $b(T)$, $c(T)$ be continuous nonnegative functions on $[0, +\infty[$ satisfying $a(0) = 0$, $b(0) < 1$, and let $g(z)$ be a nonzero real–valued nonnegative $C^1[0, +\infty)$ function, such that $g'(z)$ is nonnegative increasing function on $(0, +\infty)$ and*

$$g(0) = g'(0) = 0.$$

Then there exists $T_0 > 0$ such that

a) for every $T \in]0, T_0]$ the set $F_T = \{z > 0; z = a(T) + b(T)z + c(T)g(z)\}$ is not empty.

b) Let $\{z_k(T)\}_1^{+\infty}$ be a sequence of continuous functions on $[0, +\infty[$ satisfying

$$z_{k+1}(T) \le a(T) + b(T)z_k(T) + g(z_k(T)), \;\; z_0(T) \le a(T), \qquad (29)$$

for all $k \in \mathbb{Z}_+$. Then necessarily $z_k(T)$ is bounded sequence for all $T \in]0, T_0]$.

The proof is standard and we omit it (see [8], Section 3 for a similar abstract lemma).

3. Uniform Gevrey regularity of $H^s(\mathbb{R}^n)$ solutions

We shall study semilinear equations of the following type

$$Pv(x) = f[v](x) + w(x), \qquad x \in \mathbb{R}^n \qquad (30)$$

where $w \in G^\sigma(T; H^s)$ for some fixed $\sigma \geq 1$, $T_0 > 0$, $s > 0$ to be fixed later, P is a linear operator on $\mathbb{R}^n$ of order $\tilde{m} > 0$, i.e. acting continuously from $H^{s+\tilde{m}}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$ for every $s \in \mathbb{R}$, and $f[v] = f(v, \ldots, D^\gamma v, \ldots)_{|\gamma| \leq m_0}$, $m_0 \in \mathbb{Z}_+$, with $0 \leq m_0 < \tilde{m}$ and

$$f \in G^\theta(\mathbb{C}^L), \qquad f(0) = 0 \tag{31}$$

where $L = \sum_{\gamma \in \mathbb{Z}_+^n} 1$.

We suppose that there exists $m \in]m_0, \tilde{m}]$ such that P admits a left inverse P^{-1} acting continuously

$$P^{-1} : H^s(\mathbb{R}^n) \to H^{s+m}(\mathbb{R}^n), \qquad s \in \mathbb{R}. \tag{32}$$

We note that since $f[v]$ may contain linear terms we have the freedom to replace P by $P + \lambda$, $\lambda \in \mathbb{C}$. By (32) the operator P becomes hypoelliptic (resp., elliptic if $\tilde{m} = m$) globally in $\mathbb{R}^n$ with $\tilde{m} - m$ being called the loss of regularity (derivatives) of P. We define the critical Gevrey index, associated to (30) and (32) as follows

$$\sigma_{crit} = \max\{1, (m - m_0)^{-1}, \theta\}.$$

Our second condition requires Gevrey estimates on the commutators of P with D_j^k, namely, there exist $s > n/2 + m_0$, $C > 0$ such that

$$\|P^{-1}[P, D_p^k]v\|_s \leq (k!)^\sigma \sum_{0 \leq \ell \leq k-1} \frac{C^{k-\ell+1}}{(\ell!)^\sigma} \sum_{j=1}^n \|D_j^\ell v\|_s \tag{33}$$

for all $k \in \mathbb{N}$, $p = 1, \ldots, n$, $v \in H^{k-1}(\mathbb{R}^n)$.

We note that all constant p.d.o. and multipliers satisfy (33). Moreover, if P is analytic p.d.o. (e.g., cf. [13], [50]), then (33) holds as well for the L^2 based Sobolev spaces $H^s(\mathbb{R}^n)$.

If $v \in H^s(\mathbb{R}^n)$, $s > m_0 + \frac{n}{2}$, solves (30), standard regularity results imply that $v \in H^\infty(\mathbb{R}^n) = \bigcap_{r>0} H^r(\mathbb{R}^n)$.

We can start by $v \in H^{s_0}(\mathbb{R}^n)$ with $s_0 \leq m_0 + \frac{n}{2}$ provided f is polynomial. More precisely, we have

LEMMA 6. *Let $f[u]$ satisfy the following condition: there exist $0 < s_0 < m_0 + \frac{n}{2}$ and a continuous nonincreasing function*

$$\kappa(s),\ s \in [s_0, \frac{n}{2} + m_0[, \quad \kappa(s_0) < m - m_0, \quad \lim_{s \to \frac{n}{p} + m_0} \kappa(s) = 0$$

such that

$$f \in C(H^s(\mathbb{R}^n) : H^{s-m_0-\kappa(s)}(\mathbb{R}^n)), \qquad s \in [s_0, \frac{n}{2} + m_0[. \tag{34}$$

Then every $v \in H^{s_0}(\mathbb{R}^n)$ solution of (30) belongs to $H^\infty(\mathbb{R}^n)$.

Proof. Applying P^{-1} to (30) we get $v = P^{-1}(f[v] + w)$. Therefore, (34) and (32) lead to $v \in H^{s_1}$ with $s_1 = s_0 - m_0 - \kappa(s_0) + m > s_0$. Since the gain of regularity $m - m_0 - \kappa(s) > 0$ increases with s, after a finite number of steps we surpass $\frac{n}{2}$ and then we get $v \in H^{\infty}(\mathbb{R}^n)$.

□

REMARK 1. Let $f[u] = (D_x^{m_0}u)^d,\ d \in \mathbb{N},\ d \geq 2$. In this case $\kappa(s) = (d - 1)(\frac{n}{2} - (s - m_0))$, for $s \in [s_0, \frac{n}{2} + m_0[$, with $\kappa(s_0) < m - m_0$ being equivalent to $s_0 > m_0 + \frac{n}{2} - \frac{m-m_0}{d-1}$. This is a consequence of the multiplication rule in $H^s(\mathbb{R}^n)$, $0 < s < \frac{n}{p}$, namely: if $u_j \in H^{s_j}(\mathbb{R}^n),\ s_j \geq 0,\ \frac{n}{p} > s_1 \geq \cdots \geq s_d$, then

$$\prod_{j=1}^{d} u_j \in H^{s_1+\cdots+s_d-(d-1)\frac{n}{2}}(\mathbb{R}^n),$$

provided

$$s_1 + \cdots + s_d - (d-1)\frac{n}{2} > 0.$$

Suppose now that $f[u] = u^{d-1}D_x^{m_0}u$ (linear in $D_x^{m_0}u$), $m_0 \in \mathbb{N}$. In this case, by the rules of multiplication, we choose $\kappa(s)$ as follows: $s_0 > n/2$ (resp., $s_0 > m_0/2$), $\kappa(s) \equiv 0$ for $s \in]s_0, n/2 + m_0[$ provided $n \geq m_0$ (resp., $n < m_0$); $s_0 \in]n/2 - (m - m_0)/(d-1), n/2[$, $\kappa(s) = (d-1)(n/2 - s)$ for $s \in [s_0, n/2[$, $\kappa(s) = 0$ if $s \in [n/2, n/2 + m_0[$ provided $\frac{n}{p} - \frac{m-m_0}{d-1} > 0$ and $ds_0 - (d-2)n/2 - m_0 > 0$.

We state the main result on the uniform G^σ regularity of solutions to (30).

THEOREM 1. *Let $w \in G^\sigma(T_0; H^s)$, $s > n/2 + m_0$, $T_0 > 0$, $\sigma \geq \sigma_{crit}$. Suppose that $v \in H^\infty(\mathbb{R}^n)$ is a solution of (30). Then there exists $T_0' \in]0, T_0]$ such that*

$$v \in G^\sigma(T_0'; H^s), \qquad T \in]0, T_0']. \tag{35}$$

In particular, if $m - m_0 \geq 1$, which is equivalent to $\sigma_{crit} = 1$, and $\sigma = 1$, v can be extended to a holomorphic function in the strip $\{z \in \mathbb{C}^n : |Im\ z| < T_0'\}$. If $m < 1$ or $\theta > 1$, then $\sigma_{crit} > 1$ and v belongs to $G^\sigma_{un}(\mathbb{R}^n)$.

Proof. First, by standard arguments we reduce to $(m_0 + 1) \times (m_0 + 1)$ system by introducing $v_j =< D >^j v$, $j = 0, \ldots, m_0$ (e.g., see [33], [50]) with the order of the inverse of the transformed matrix valued–operator P^{-1} becoming $m_0 - m$, while σ_{crit} remains invariant. So we deal with a semilinear system of $m_0 + 1$ equations

$$Pv(x) = f(\kappa_0(D)v_0, \ldots, \kappa_{m_0}(D)v_{m_0}) + w(x), \qquad x \in \mathbb{R}^n$$

where κ_j's are zero order constant p.d.o., $f(z)$ being a G^θ function in $\mathbb{C}^{m_0+1} \mapsto \mathbb{C}^{m_0+1}$, $f(0) = 0$. Since $\kappa_j(D)$, $j = 0, \ldots, m_0$, are continuous in $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, and the nonlinear estimates for $f(\kappa_0(D)v_0, \ldots, \kappa_{m_0}(D)v_{m_0})$ are the same as for

$f(v_0, \dots, v_{m_0})$ (only the constants change), we consider $\kappa_j(D) \equiv 1$. Hence, without loss of generality we may assume that we are reduced to

$$Pv(x) = f(v) + w(x), \qquad x \in \mathbb{R}^n \tag{36}$$

Let $v \in H^\infty(\mathbb{R}^n)$ be a solution to (36). Equation (36) is equivalent to

$$P(D_j^k v) = -[P, D_j^k]v + D_j^k(f(v)) + D_j^k w. \tag{37}$$

which yields

$$D_j^k v = -P^{-1}[P, D_j^k]v + P^{-1}D_j^k(f(v)) + P^{-1}D_j^k w. \tag{38}$$

In view of (33), we readily obtain the following estimates with some constant $C_0 > 0$

$$\frac{T^k}{(k!)^\sigma}\|P^{-1}[P, D_j^k]v\|_s \leq C_0 T \sum_{\ell=0}^{k-1} (C_0 T)^{k-\ell-1} \frac{T^\ell}{(\ell!)^\sigma} \sum_{q=1}^{n} \|D_q^\ell v\|_s \tag{39}$$

for all $k \in \mathbb{N}$, $j = 1, \dots, n$. Therefore

$$\begin{aligned}
S_N^{comm}[v; T] &:= \sum_{k=1}^{N}\sum_{j=1}^{n} \frac{T^k}{(k!)^\sigma}\|P^{-1}[P, D_j^k]v\|_s \\
&\leq \sum_{k=1}^{N}\sum_{\ell=0}^{k-1} (C_0 T)^{k-\ell-1} \frac{T^\ell}{(\ell!)^\sigma} \sum_{q=1}^{n} \|D_q^\ell v\|_s \\
&= nC_0 T \sum_{\ell=1}^{N-1} \frac{T^\ell}{(\ell!)^\sigma} \sum_{q=1}^{n} \|D_q^\ell v\|_s \sum_{k=\ell+1}^{N} (C_0 T)^{k-\ell-1} \\
&\leq \frac{nC_0 T}{1 - C_0 T} S_{N-1}^\sigma[v; T, H^s] \\
&\leq \frac{nC_0 T}{1 - C_0 T}\|v\|_s + \frac{nC_0 T}{1 - C_0 T}\widetilde{S}_{N-1}^\sigma[v; T, H^s] \qquad (40)
\end{aligned}$$

for all $N \in \mathbb{N}$ provided $0 < T < C_0^{-1}$.

Now, since the case $\theta = 1$ is easier to deal with, we shall treat the case $\theta > 1$, hence $\sigma_{cr} > 1$.

Next, by Lemma 2, one gets that for $N_s := \|P^{-1}\|_{H^{s-1/\sigma_{crit}} \to H^s}$

$$\begin{aligned}
\|P^{-1}D_j^k(f(v))\|_s &\leq N_s \||D_j|^{k-1/\sigma_{cr}}(f(v))\|_s \\
&\leq \varepsilon\|D_j^k(f(v))\|_s + C(\varepsilon)\|D_j^{k-1}(f(v))\|_s, \qquad \varepsilon > 0 \qquad (41)
\end{aligned}$$

where

$$C(\varepsilon) = (1 - \rho)(\frac{N_s \rho}{\varepsilon})^{1/1-\rho}.$$

Set

$$L_0 = |f'(v)|_\infty.$$

Therefore, if $N \geq 3$, in view of (22) we can write

$$\widetilde{S}^\sigma_N[v; T, s] \leq \widetilde{S}^\sigma_N[w; T, s] + \frac{nC_0T}{1 - C_0T}\|v\|_s + \frac{nC_0T}{1 - C_0T}\widetilde{S}^\sigma_{N-1}[v; T, H^s]$$

$$+\varepsilon L_0\widetilde{S}^\sigma_N[v; T, s] + \varepsilon \sum_{j=2}^{N} \frac{A_0^j}{(j!)^{\sigma-\theta}}(\widetilde{S}^\sigma_{N-1}[v; T, s])^j \tag{42}$$

$$+C(\varepsilon)T\left(\|f(v)\|_s + \varepsilon L_0\widetilde{S}^\sigma_{N-1}[v; T, s] + \varepsilon \sum_{j=2}^{N-1} \frac{A_0^j}{(j!)^{\sigma-\theta}}(\widetilde{S}^\sigma_{N-2}[v; T, s])^j\right)$$

for $0 < T < \min\{C_0^{-1}, T_0\}$. Now we fix $\varepsilon > 0$ to satisfy

$$\varepsilon L_0 < 1 \tag{43}$$

Then by (42) we obtain that

$$\widetilde{S}^\sigma_N[v; T, s] \leq a(T) + b(T)\widetilde{S}^\sigma_{N-1}[v; T, s] + g(\widetilde{S}^\sigma_{N-1}[v; T, s], T) \tag{44}$$

where

$$a(T) = \frac{\|w\|_{\sigma,T,s} - \|w\|_s + nC_0T(1 - C_0T)^{-1}\|v\|_s + C_\varepsilon T\|f(v)\|_s}{1 - \varepsilon L_0} \tag{45}$$

$$b(T) = T\frac{nC_0 + (1 - C_0T)\varepsilon C_\varepsilon L_0}{(1 - C_0T)(1 - \varepsilon T)} \tag{46}$$

$$c(T)g(z) = \frac{\varepsilon(1 + \varepsilon C(\varepsilon)L_0T)}{1 - \varepsilon L_0}\sum_{j=2}^{\infty} \frac{A_0^j}{(j!)^{\sigma-\theta}}z^j \tag{47}$$

for $0 < T < \min\{C_0^{-1}, T_0\}$. Now we are able to apply Lemma 3 for $0 < T < T_0'$, by choosing T_0' small enough , $T_0' < \min\{T_0, C_0^{-1}\}$ so that the sequence $\widetilde{S}^\sigma_N[v; T_0', s]$ is bounded . This implies the convergence since $\widetilde{S}^\sigma_N[v; T_0', s]$ is nondecreasing for $N \to \infty$.

□

REMARK 2. The operator P appearing in the ODEs giving rise to traveling wave solutions for dispersive equations is usually a constant p.d.o. or a Fourier multiplier (cf. [11], [40], [29]), and in that case the commutators in the LHS of (33) are zero. Let now $V(x) \in G^\sigma(\mathbb{R}^n : \mathbb{R})$, $\inf_{x\in\mathbb{R}^n} V(x) > 0$. Then it is well known (e.g., cf. [52]) that the operator $P = -\Delta + V(x)$ admits an inverse satisfying $P^{-1} : H^s(\mathbb{R}^n) \to H^{s+1}(\mathbb{R}^n)$. One checks via straightforward calculations that the Gevrey commutator hypothesis is satisfied if there exists $C > 0$ such that

$$\|P^{-1}(D_x^\beta V D_x^\gamma u)\|_s \leq C^{|\beta|+1}(\beta!)^\sigma\|D_x^\gamma u\|_s, \tag{48}$$

for all $\beta, \gamma \in \mathbb{Z}_+^n$, $\beta \neq 0$.

We point out that, as a corollary of our theorem, we obtain for $\sigma = 1$ and $f[v] \equiv 0$ a seemingly new result, namely that every eigenfunction $\phi_j(x)$ of $-\Delta + V(x)$ is extended to a holomorphic function in $\{z \in \mathbb{C}^n : |\Im z| < T_0\}$ for some $T_0 > 0$. Next, we have a corollary from our abstract result on uniform G^1 regularity for the $H^2(\mathbb{R})$ solitary wave solutions $r(x + ct)$, $c > 0$ in [29] satisfying

$$Pu = r'''' + \mu r'' + cr = f(r, r', r'') = f_0(r, r') + f_1(r, r')r'', \ \ \mu \in \mathbb{R} \tag{49}$$

with f_j being homogeneous polynomial of degree $d - j$, $j = 0, 1$, $d \geq 3$ and $|\mu| < 2\sqrt{-c}$. Actually, by Lemma 6, we find that every solution r to (49) belonging to $H^s(\mathbb{R})$, $s > 3/2$, is extended to a holomorphic function in $\{z \in \mathbb{C} : |Im\, z| < T\}$ for some $T > 0$.

4. Uniform Gevrey regularity of L^∞ stationary solutions

It is well known that the traveling waves to dissipative equations like Burgers, Fisher–Kolmogorov, Kuramoto–Sivashinsky equations have typically two different nonzero limits for $x \to \pm\infty$ (see the example (6)). Now we investigate the $G^\sigma_{un}(\mathbb{R}^n)$ regularity of such type of solutions for semilinear elliptic equations.

We shall generalize Theorem 4.1 in [8] for G^θ nonlinear terms f. We restrict our attention to (30) for $n = 1$, $m_0 = 0$, $P = P(D)$ being a constant coefficients elliptic p.d.o. or Fourier multiplier of order m.

THEOREM 2. *Let $\theta \geq 1$, $\sigma \geq \theta$, $m - m_0 \geq 1$, $f \in G^\theta(\mathbb{C}^L)$, $f(0) = 0$, $w \in G^\sigma_\infty(T_0; H^s)$ for some $T_0 > 0$. Suppose that $v \in L^\infty(\mathbb{R})$ is a weak solution of (30) satisfying $\nabla v \in H^s(\mathbb{R})$. Then there exists T_0', depending on T_0, $P(D)$, f, $\|v\|_\infty$ and $\|\nabla v\|_s$ such that $v \in G^\sigma_\infty(H^s(\mathbb{R}); T_0')$. In particular, if $\sigma = \theta = 1$ then v can be extended to a holomorphic function in $\{z \in \mathbb{C} : |Im\, z| < T_0'\}$.*

Without loss of generality we suppose that $n = 1$, $m_0 = 0$. It is enough to show that $v' = D_x v \in G^\sigma(H^s, T)$ for some $T > 0$.

We need an important auxiliary assertion, whose proof is essentially contained in [27].

LEMMA 7. *Let $g \in G^\theta(\mathbb{R}^p : \mathbb{R})$, $1 \leq \theta \leq \sigma$, $g(0) = 0$. Then there exists a positive continuous nondecreasing function $G(t)$, $t \geq 0$ such that*

$$\|(D^\alpha g)(v)w\|_s \leq |(D^\alpha g)(v)|_\infty \|w\|_s + G(|v|_\infty)^\alpha (\alpha!)^\theta (\|w\|_{s-1} + \|\nabla v\|^s_{s-1}) \tag{50}$$

for all $v \in (L^\infty(\mathbb{R}^n : \mathbb{R}))^p$, $v' \in (H^s(\mathbb{R}^n : \mathbb{R}))^p$, $w \in (H^s(\mathbb{R}))^p$, $\alpha \in \mathbb{Z}^p_+$, provided $s > n/2 + 1$.

Proof of Theorem 2. Write $u = v'$. We observe that $(f(v))' = f'(v)u$ and the hypotheses imply that $g(v) := f'(v) \in L^\infty(\mathbb{R})$ and $u \in H^s(\mathbb{R})$. Thus differentiating k

times we obtain that u satisfies

$$PD^k u \quad = \quad D^k(g(v)u) + D^k w'$$

which leads to

$$D^k u \quad = \quad P^{-1}D^k(g(v)u) + P^{-1}D^k w'.$$

Hence, since $m \geq 1$ and $P^{-1}D$ is bounded in $H^s(\mathbb{R})$ we get the following estimates

$$\begin{aligned}
\|D^k u\|_s &\leq C\|D^{k-1}(g(v)u)\|_s + \|D^{k-1}w'\|_s \\
&\leq \sum_{j=0}^{k-1} \left(\begin{array}{c} k-1 \\ j \end{array} \right) \|D^j(g(v))D^{k-1-j}u\|_s + \|D^{k-1}w'\|_s \\
&\leq \sum_{j=0}^{k-1} \left(\begin{array}{c} k-1 \\ j \end{array} \right) \sum_{\ell=0}^{j} \frac{C^{\ell-1}}{\ell!} \\
&\times \sum_{\substack{p_1+\cdots+p_\ell=j \\ p_1\geq 1,\cdots,p_\ell\geq 1}} \prod_{\mu=1}^{\ell} \frac{\|D^{p_\mu-1}u\|_s}{p_\mu!} \|(D^\ell g)(v))D^{k-1-j}u\|_s + \|D^{k-1}w'\|_s. \qquad (51)
\end{aligned}$$

Now, by Lemma 7 and (51) we get, with another positive constant C,

$$\begin{aligned}
\frac{T^k}{(k!)^\sigma}\|D^k u\|_s &\leq CTk^{-\sigma}\|v\|_s \sum_{j=0}^{k-1} \left(\begin{array}{c} k-1 \\ j \end{array} \right)^{-\sigma+1} \sum_{\ell=0}^{j} \frac{C^{\ell-1}}{(\ell!)^{\sigma-\theta}} \\
&\times \sum_{\substack{p_1+\cdots+p_\ell=j \\ p_1\geq 1,\cdots,p_\ell\geq 1}} \prod_{\mu=1}^{\ell} \frac{T^{p_\mu}|D^{p_\mu-1}u|_s}{(p_\mu!)^\sigma} (G(|v|_\infty))^\ell \frac{T^{k-1-j}|D^{k-1-j}u|_s}{((k-1-j)!)^\sigma} \\
&+ \|w'\|_{\sigma,T;s}. \qquad (52)
\end{aligned}$$

Next, we conclude as in [8].

□

REMARK 3. As a corollary from our abstract theorem we obtain apparently new results on the analytic $G^1_{un}(\mathbb{R})$ regularity of traveling waves of the Kuramoto–Sivashinsky equation cf. [41], and the Fisher–Kolmogorov equation and its generalizations (cf. [37], [31]).

5. Decay estimates in Gelfand–Shilov spaces

In the paper [8] new functional spaces of Gevrey functions were introduced which turned out to be suitable for characterizing both the uniform analyticity and the exponential decay for $|x| \to \infty$. Here we will show regularity results in the framework of the Gelfand–Shilov spaces $S^\nu_\mu(\mathbb{R}^n)$ with

$$\mu \geq 1, \nu \geq 1. \qquad (53)$$

Let us fix $s > n/2$, $\mu, \nu \geq 1$. Then for every $\varepsilon > 0$, $T > 0$ we set

$$D^{\nu}_{\mu}(\varepsilon, T) = \{v \in S(\mathbb{R}^n) : |\!|\!| v |\!|\!|_{\varepsilon,T} < +\infty\}$$

where

$$|\!|\!| v |\!|\!|_{\varepsilon,T} = \sum_{j,k\in\mathbb{Z}^n_+}^{\infty} \frac{\varepsilon^{|j|}T^{|k|}}{(j!)^{\nu}(k!)^{\mu}} \|x^j D^k v\|_s.$$

We stress that (53) implies that $S^{\nu}_{\mu}(\mathbb{R}^n)$ becomes a ring with respect to the pointwise multiplication and the spaces $D^{\nu}_{\mu}(\varepsilon, T)$ become Banach algebras.

Using the embedding of $H^s(\mathbb{R}^n)$ in $L^{\infty}(\mathbb{R}^n)$ and standard combinatorial arguments, we get that one can find $c > 0$ such that

$$|D^k v(x)| \leq cT^{-|k|}(k!)^{\mu} e^{-\varepsilon|x|^{1/\nu}} |\!|\!| v |\!|\!|_{\varepsilon,T}, \; x \in \mathbb{R}^n, k \in \mathbb{Z}^n_+, \; v \in D(\varepsilon, T). \tag{54}$$

Clearly $S^{\nu}_{\mu}(\mathbb{R}^n)$ is inductive limit of $D^{\nu}_{\mu}(\varepsilon, T)$ for $T \searrow 0$, $\varepsilon \searrow 0$.

We set

$$E^{\nu}_{\mu;N}[v; \varepsilon, T] = \sum_{\substack{j,k\in\mathbb{Z}^n_+ \\ |j|+|k|\leq N}} \frac{\varepsilon^{|j|}T^{|k|}}{(j!)^{\nu}(k!)^{\mu}} \|x^j D^k v\|_s.$$

We will study the semilinear equation (30), with $w \in D^{\nu}_{\mu}(\varepsilon_0, T_0)$. The linear operator P is supposed to be of order $\tilde{m} = m$, to be elliptic and invertible, i.e. (32) holds. The crucial hypothesis on the nonlinearity $f(u)$ in order to get decay estimates is the lack of linear part in the nonlinear term. For the sake of simplicity we assume that f is entire function and quadratic near 0, i.e.,

$$f(z) = \sum_{j\in\mathbb{Z}^L_+, |j|\geq 2} f_j z^j \tag{55}$$

and for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$|f_j| \leq C_\delta \delta^{|j|}, \qquad j \in \mathbb{Z}^L_+.$$

Next, we introduce the hypotheses on commutators of P^{-1}.

We suppose that there exist $\mathcal{A}_0 > 0$, and $\mathcal{B}_0 > 0$ such that

$$\|P^{-1}, [P, x^{\beta} D^{\alpha}_x]v\|_s \leq (\alpha!)^{\mu}(\beta!)^{\nu} \sum_{\substack{\rho\leq\alpha,\theta\leq\beta \\ \rho+\theta\neq\alpha+\beta}} \frac{\mathcal{A}_0^{|\alpha-\rho|}\mathcal{B}_0^{|\beta-\theta|}}{(\rho!)^{\mu}(\theta!)^{\nu}} \|x^{\theta} D^{\rho} v\|_s \tag{56}$$

for all $\alpha, \beta \in \mathbb{Z}^n_+$.

The next lemma, combined with well known L^p estimates for Fourier multipliers and L^2 estimates for pseudodifferential operators, indicates that our hypotheses on the commutators are true for a large class of pseudodifferential operators.

LEMMA 8. *Let P be defined by an oscillatory integral*

$$
\begin{aligned}
Pv(x) &= \int e^{ix\xi} P(x,\xi)\hat{v}(\xi) \bar{d}\xi \\
&= \iint e^{i(x-y)\xi} P(x,\xi) v(y) dy \bar{d}\xi, \qquad (57)
\end{aligned}
$$

where $P(x,\xi)$ is global analytic symbol of order m, i.e. for some $C > 0$

$$
\sup_{\alpha,\beta\in\mathbb{Z}^n_+} \left(\sup_{(x,\xi)\in\mathbb{R}^{2n}} \left(\frac{<\xi>^{-m+|\beta|}\, C^{|\alpha|+|\beta|}}{\alpha!\beta!} |D^\alpha_x D^\beta_\xi P(x,\xi)| \right) \right) < +\infty. \qquad (58)
$$

Then the following relations hold

$$
[P, x^\beta D^\alpha_x] v(x) = \alpha!\beta! \sum_{\substack{\rho\le\alpha,\theta\le\beta \\ |\rho+\theta|<|\alpha+\beta|}} \frac{(-1)^{|\beta-\theta|}(-i)^{|\alpha-\rho|}}{(\alpha-\rho)!(\beta-\theta)!} P^{(\beta-\theta)}_{(\alpha-\rho)}(x,D)(x^\theta D^\rho_x v) \qquad (59)
$$

for all $\alpha,\beta\in\mathbb{Z}^n_+$, where $P^{(\beta)}_{(\alpha)}(x,\xi) := D^\beta_\xi \partial^\alpha_x P(x,\xi)$.

Proof. We need to estimate the commutator $[P, x^\beta D^\alpha_x]v = P(x^\beta D^\alpha_x v) - x^\beta D^\alpha_x P(v)$. We have

$$
\begin{aligned}
P(x^\beta D^\alpha_x v) &= \iint e^{i(x-y)\xi} P(x,\xi) y^\beta D^\alpha_y v(y) dy \bar{d}\xi; \\
x^\beta D^\alpha_x P(v) &= \iint x^\beta D^\alpha_x (e^{i(x-y)\xi} P(x,\xi)) v(y) dy \bar{d}\xi \\
&= \iint \sum_{\rho\le\alpha} \binom{\alpha}{\rho} x^\beta e^{i(x-y)\xi} \xi^\rho D^{\alpha-\rho}_x P(x,\xi) v(y) dy \bar{d}\xi \\
&= \iint \sum_{\rho\le\alpha} \binom{\alpha}{\rho} x^\beta (-D_y)^\rho (e^{i(x-y)\xi}) D^{\alpha-\rho}_x P(x,\xi) v(y) dy \bar{d}\xi \\
&= \iint \sum_{\rho\le\alpha} \binom{\alpha}{\rho} x^\beta e^{i(x-y)\xi} D^{\alpha-\rho}_x P(x,\xi) D^\rho_y v(y) dy \bar{d}\xi \\
&= \iint \sum_{\rho\le\alpha} \binom{\alpha}{\rho} D^\beta_\xi (e^{ix\xi})(e^{-iy\xi} D^{\alpha-\rho}_x P(x,\xi)) D^\rho_y v(y) dy \bar{d}\xi \\
&= \iint \sum_{\rho\le\alpha} \binom{\alpha}{\rho} e^{ix\xi} (-D_\xi)^\beta (e^{-iy\xi}) D^{\alpha-\beta}_x P(x,\xi)) D^\rho_y v(y) dy \bar{d}\xi \\
&= \iint \sum_{\substack{\rho\le\alpha \\ \theta\le\beta}} \binom{\alpha}{\rho}\binom{\beta}{\theta} e^{i(x-y)\xi} y^\theta (-1)^{|\beta-\theta|} D^{\beta-\theta}_\xi D^{\alpha-\rho}_x P(x,\xi) D^\rho_y v(y) dy \bar{d}\xi \\
&= \sum_{\substack{\rho\le\alpha \\ \theta\le\beta}} \binom{\alpha}{\rho}\binom{\beta}{\theta} \iint e^{i(x-y)\xi} (-1)^{|\beta-\theta|} D^{\beta-\theta}_\xi D^{\alpha-\rho}_x P(x,\xi) y^\theta D^\rho_y v(y) dy \bar{d}\xi
\end{aligned}
$$

which concludes the proof of the lemma.

□

REMARK 4. We point out that if $P(D)$ has nonzero symbol, under the additional assumption of analyticity, namely $P(\xi) \neq 0$, $\xi \in \mathbb{R}^n$ and there exists $C > 0$ such that

$$\frac{|D_\xi^\alpha(P(\xi))|}{|P(\xi)|} \leq C^{|\alpha|}\alpha!(1+|\xi|)^{-|\alpha|}, \quad \alpha \in \mathbb{Z}_+^n,\ \xi \in \mathbb{R}^n, \tag{60}$$

the condition (56) holds. More generally, (56) holds if $P(x,\xi)$ satisfies the following global Gevrey $S_\nu^\mu(\mathbb{R}^n)$ type estimates: there exists $C > 0$ such that

$$\sup_{(x,\xi)\in R^{2n}} (<\xi>^{-m+|\alpha|} (\alpha!)^\mu (\beta!)^{-\nu} |\partial_x^\alpha \partial_\xi^\beta P(x,\xi)|) \leq C^{|\alpha|+|\beta|+1}, \qquad \alpha, \beta \in \mathbb{Z}_+^n.$$

The latter assertion is a consequence from the results on $L^2(\mathbb{R}^n)$ estimates for p.d.o.-s (e.g., cf. [19]).

Let now $P(D)$ be a Fourier multiplier with the symbol $P(\xi) = 1 + i\ \mathrm{sign}(\xi)\xi^2$ (such symbol appears in the Benjamin-Ono equation). Then (56) fails.

Since our aim is to show (sub)exponential type decay in the framework of the Gelfand-Shilov spaces, in view of the preliminary results polynomial decay in [8], we will assume that

$$<x>^N v \in H^\infty(\mathbb{R}^n), \qquad N \in \mathbb{Z}_+. \tag{61}$$

Now we extend the main result on exponential decay in [8].

THEOREM 3. *Let f satisfy (55) and $w \in S_\mu^\nu(\mathbb{R}^n)$ with μ, ν satisfying (53), i.e., $w \in D_\mu^\nu(\varepsilon_0, T_0)$ for some $\varepsilon_0 > 0$, $T_0 > 0$. Suppose that the hypothesis (56) is true. Let now $v \in H^\infty(\mathbb{R}^n)$ satisfy (61) and solve (30) with the RHS w as above. Fix $\varepsilon \in]0, \min\{\varepsilon_0, \mathcal{B}_0^{-1}\}[$. Then we can find $T_0'(\varepsilon) \in]0, \min\{\tilde{T}_0, \mathcal{A}_0^{-1}\}[$ such that $v \in D_\mu^\nu(\varepsilon, T)$ for $T \in]0, T_0'(\varepsilon)[$. In particular, $v \in S_\mu^\nu(\mathbb{R}^n)$.*

Proof. For the sake of simplicity we will carry out the argument in the one dimensional case. We write for $\beta \geq 1$

$$\begin{aligned} P(x^\beta D^\alpha v) &= x^\beta D_x^\alpha w - [P, x^\beta D_x^\alpha]v + x^\beta D_x^\alpha(f(v)) \\ &= x^\beta D_x^\alpha w - [P, x^\beta D_x^\alpha]v \\ &+ D_x(x^\beta D_x^{\alpha-1}(f(v))) - \beta x^{\beta-1} D_x^{\alpha-1}(f(v)). \end{aligned} \tag{62}$$

Thus

$$\begin{aligned} x^\beta D^\alpha v &= P^{-1}x^\beta D_x^\alpha w - P^{-1}[P, x^\beta D_x^\alpha]v \\ &+ P^{-1}D_x(x^\beta D_x^{\alpha-1}(f(v))) - \beta P^{-1}(x^{\beta-1}D_x^{\alpha-1}(f(v))) \end{aligned} \tag{63}$$

which implies, for some constant depending only on the norms of P^{-1} and $P^{-1}D$ in H^s, the following estimates

$$\begin{aligned}
\frac{\varepsilon^\beta T^\alpha}{(\alpha!)^\mu(\beta!)^\nu}\|x^\beta D^\alpha v\|_s &\le C\frac{\varepsilon^\beta T^\alpha}{(\alpha!)^\mu(\beta!)^\nu}\|x^\beta D^\alpha w\|_s \\
&+ \sum_{\substack{\rho\le\alpha,\theta\le\beta\\ \rho+\theta\ne\alpha+\beta}} \frac{\mathcal{A}_0^{\alpha-\rho}\mathcal{B}_0^{\beta-\theta}}{(\rho!)^\mu(\theta!)^\nu}\|x^\theta D^\rho v\|_s \\
&+ C\frac{T}{\alpha^\mu}\frac{\varepsilon^\beta T^{\alpha-1}}{((\alpha-1)!)^\mu(\beta!)^\nu}\|x^\beta D^{\alpha-1}(f(v))\|_s \\
&+ \frac{\varepsilon T}{\beta^{\nu-1}}\frac{\varepsilon^{\beta-1}T^{\alpha-1}}{((\alpha-1)!)^\mu((\beta-1)!)^\nu}\|x^{\beta-1}D^{\alpha-1}(f(v))\|_s
\end{aligned} \tag{64}$$

for all $\alpha,\beta\in\mathbb{Z}_+$, $\beta\ge1$, $\alpha\ge1$.

On the other hand, if $\alpha=0$ we have

$$\begin{aligned}
\frac{\varepsilon^\beta}{(\beta!)^\nu}\|x^\beta v\|_s &\le C\frac{\varepsilon^\beta}{(\beta!)^\nu}\|x^\beta w\|_s \\
&+ \sum_{0\le\theta<\beta}\frac{\mathcal{B}_0^{\beta-\theta}}{(\theta!)^\nu}\|x^\theta v\|_s + C\frac{\varepsilon^\beta}{(\beta!)^\nu}\|x^\beta(f(v))\|_s
\end{aligned} \tag{65}$$

for all $\beta\in\mathbb{Z}_+$, $\beta\ge1$.

Now use the (at least) quadratic order of $f(u)$ at $u=0$, namely, there exist $C_1>0$ depending on s and f, and a positive nondecreasing function $G(t)$, $t\ge0$, such that

$$\frac{\varepsilon^\beta}{(\beta!)^\nu}\|x^\beta f(v)\|_s \le C_1\varepsilon\|xv\|_s G(\|v\|_s)\left(\frac{\varepsilon^{\beta-1}}{((\beta-1)!)^\nu}\|x^{\beta-1}v)\|_s\right) \tag{66}$$

This, combined with (64), allows us to gain an extra $\nu\epsilon>0$ and after summation with respect to $\alpha,\beta,\alpha+\beta\le N+1$, to obtain the following iteration inequalities for some $C_0>0$

$$E^\mu_{\nu;N+1}[v;\varepsilon,T] \le \|v\|_s + C_0\varepsilon\|xv\|_s E^\mu_{\nu;N}[v;\varepsilon,T] + TG(E^\mu_{\nu;N}[v;\varepsilon,T]) \tag{67}$$

for all $N\in\mathbb{Z}_+$. We can apply the iteration lemma taking $0<T\le T_0'(\varepsilon)$ with $0<T_0'(\varepsilon)\ll1$.

□

REMARK 5. We recall that the traveling waves for the Benjamin-Ono equation decay as $O(x^{-2})$ for $|x|\to+\infty$, where $P(\xi)=c+i\ \mathrm{sign}(\xi)\xi^2$ for some $c>0$. Clearly (H_3) holds with $\mu=2$ but it fails for $\mu\ge3$. Next, if a traveling wave solution $\varphi(x)\in H^2(\mathbb{R})$ in [29] decays like $|x|^{-\varepsilon}$ as $x\to\infty$ for some $0<\varepsilon\ll1$, then by Theorem 2 it should decay exponentially and will belong to the Gelfand–Shilov class $S_1^1(\mathbb{R})$. Finally, we recall that $u_\nu(x)=-4\nu x(x^2+\nu^2)^{-1}$, $x\in\mathbb{R}$, solves the stationary

Sivashinsky equation $|D_x|u + \nu\partial_x^2 u = u\partial_x u$, $\nu > 0$ (cf. [51]). Clearly $u_\nu(x)$ extends to a holomorphic function in the strip $|Im\, z| < \nu$ and decays (exactly) like $O(|x|^{-1})$ for $x \to \infty$. Although the full symbol $-|\xi| + \nu\xi^2$ is not invertible in $L^2(\mathbb{R})$, we can invert P in suitable subspaces of odd functions and check that (56) holds iff $\mu = 1$.

6. Strongly singular solutions

First we consider a class of semilinear ODE with polynomial nonlinear terms on the real line

$$P[y](x) := D^m y(x) + \sum_{\ell=2}^{d} \sum_{j \in \{0,1,\ldots,m-1\}^\ell} p_j^\ell D^{j_1} y(x) D^{j_2} y(x) \cdots D^{j_m} y(x) = 0, \tag{68}$$
$$p_j^\ell \in \mathbb{C}$$

where $m, d \in \mathbb{N}$, $m \geq 2$, $d \geq 2$, $Dy(x) = (1/i)y'(x)$. We require the following homogeneity type condition: there exists $\tau > 0$ such that

$$-\tau - m = -\ell\tau - j_1 - \cdots - j_\ell \qquad \text{if } p_j^d \neq 0. \tag{69}$$

Thus, by the homogeneity we obtain after substitution in (68) and straightforward calculations that $y_\pm(x) = c_\pm(\pm x)^{-\tau}$ solves (68) for $\pm x > 0$ provided $c \neq 0$ is zero of the polynomial $P_m^\pm(\lambda)$, where

$$P_{m,\tau}^\pm(\lambda) = \lambda\tau(\tau-1)\cdots(\tau-m+1)(-1)^m + \sum_{\ell=2}^{d} \sum_{j \in \{0,1,\ldots,m-1\}^\ell} \lambda^\ell \tilde{p}_j^\ell(\tau) \tag{70}$$

where

$$\tilde{p}_j^\ell(\tau) = p_j^\ell(-1)^{|j|}\tau(\tau - j_1)\cdots(\tau - j_\ell). \tag{71}$$

If $\tau < 1$ the singularity of the type $|x|^{-\tau}$ near $x = 0$ is in $L^p_{loc}(\mathbb{R})$, $p \geq 1$, provided $p\tau < 1$. In this case we deduce that one can glue together y_+ and y_- into one $y \in L^p_{loc}(\mathbb{R})$ function. However, the products in (68) are in general not in $L^1_{loc}(\mathbb{R})$ near the origin, so we have no real counterexample of singular solutions to (68) on $\mathbb{R}$. We shall construct such solutions following the approach in [8], namely, using homogeneous distributions on the line (for more details on homogeneous distributions see L. Hörmander [33], vol. I). We recall that if $u \in \mathcal{S}'(\mathbb{R})$ is homogeneous distribution of order r, then $u(x) = u_\pm|x|^r$ for $\pm x > 0$, $u_\pm \in \mathbb{C}$, and $\hat{u}(\xi)$ is a homogeneous distribution of order $-1-r$.

Given $\mu > -1$ we set

$$h^\pm_{-1+\mu}(x) := \mathcal{F}^{-1}_{\xi \to x}(H^\mu_\pm(\xi)), \tag{72}$$

where

$$H^\mu_\pm(\xi) = H(\pm\xi)|\xi|^\mu, \tag{73}$$

with $H(t)$ standing for the Heaviside function. Since $\tau > 0$ if $\mu = -1 + \tau > -1$ we get that $|\xi|^{-1+\mu}$ is L^1_{loc} near $\xi = 0$, therefore $H(\pm\xi)|\xi|^{-1+\mu}$ belongs to $\mathcal{S}'(\mathbb{R})$ and $h^{\pm}_{-\tau}$ are homogeneous of degree $-\tau$. Moreover, since $supp(\widehat{h^{+}_{-\tau}}) = [0, +\infty[$ (resp. $supp(\widehat{h^{-}_{-\tau}})(\xi) =]-\infty, 0[$), $h^{+}_{-\tau}$ (resp. $h^{-}_{-\tau}$) satisfies a well known condition, guaranteeing that the product $(h^{+}_{-\tau})^m$ (resp. $(h^{-}_{-\tau})^m$), or equivalently the convolutions

$$\widehat{D^{j_1}h^{+}_{-\tau}} * \cdots * \widehat{D^{j_\ell}h^{+}_{-\tau}}$$

(resp.

$$\widehat{D^{j_1}h^{-}_{-\tau}} * \cdots * \widehat{D^{j_1}h^{-}_{-\tau}})$$

are well defined in $\mathcal{S}'(\mathbb{R})$ for any $m \in \mathbb{N}$ (cf. [43], see also [33]). In view of the equivalence between (68) and (74), the order of homogeneity $-\tau$ of $y_\pm$, and (72), we will look for solutions to

$$\widehat{P[y]}(\xi) = \xi^m \hat{y}(\xi) + \sum_{\ell=2}^{d} \sum_{j\in\{0,1,\ldots,m-1\}^\ell} p^\ell_j \xi^{j_1}\hat{y} * \cdots * \xi^{j_2}\hat{y} \cdots \xi^{j_m}\hat{y} = 0 \tag{74}$$

proportional to $H(\pm\xi)|\xi|^{-1+\tau}$ homogeneous of order $-1+\tau$ with support in $\pm\xi \geq 0$. Following [8], we set

$$h^{\pm,a}_{-\tau}(\xi) = a_\pm H(\pm\xi)|\xi|^{-1+\tau}, \tag{75}$$

with $a_\pm \in \mathbb{C}$ to be determined later on.

Using the homogeneity of $h^{\pm,a}_{-\tau}$, the definition of τ in (69) and the convolutions identities derived in [8], Section 7, we readily obtain that

$$a_\pm \mathcal{F}^{-1}_{\xi\to x}(h^{\pm}_{-\tau}) = c_\pm(-\pm x)^\tau \tag{76}$$

where $c_\pm$ is a complex constant and by substituting $\hat{y}(\xi) = h^{\pm,a}_{-\tau}(\xi)$ in (74)

$$\widehat{P[y]}(\xi) = a_\pm H(\pm\xi)|\xi|^{m+\tau}\tilde{P}^{\pm}_{m,\tau}(a_\pm) = 0 \tag{77}$$

where $\tilde{P}^{\pm}_{m,\tau}(\lambda)$ is a polynomial such that $a_\pm$ is zero of $\tilde{P}^{\pm}_{m,\tau}(\lambda)$ iff $c_\pm$ in (76) is zero of $P^{\pm}_{m,\tau}(\lambda)$ (defined in (70)).

Therefore we have constructed explicit homogeneous solutions to (68)

$$u_{a_\pm}(x) = a_\pm \mathcal{F}_{\xi\to x}(H^{-1+\tau}_{\pm}(\xi)) \tag{78}$$

Consider now a semilinear PDE with polynomial nonlinearities

$$Pu = P_m(D)u + F(u, \ldots, D^\alpha_x u, \ldots)|_{|\alpha|\leq m-1} \tag{79}$$

where $P_m(D)$ is constant linear partial differential operator homogeneous of order m and where F is polynomial of degree $d \geq 2$.

Given $\theta \in S^{n-1}$ we shall define the ODE $P_\theta(D)$ in the following way: let Q be an orthogonal matrix such that $Q^*\theta = (1, 0, \dots, 0)$. Then for $x = y_1$ we have

$$P_\theta(D)u(y_1) = (P_m(D_y)U_\theta + F(U_\theta, \dots, D_y^\alpha U_\theta, \dots)|_{|\alpha| \le m-1}) \tag{80}$$

with $U_\theta(y) = u(< Q, (y_1, 0, \dots, 0) >)$.

Let now $\theta \in S^{n-1}$, $n \ge 2$ and let $L = L_\theta$ be the hyperplane orthogonal to θ. We define, as in [8], $U_\pm^\theta = a_\pm \delta_P \otimes h^\pm_{-\tau,\theta} \in \mathcal{S}'(\mathbb{R})$, $j = 1, \dots, d-1$, by the action on $\phi(x) \in \mathcal{S}(\mathbb{R}^n)$

$$(U^\theta_{\pm;j}, \phi) := \int_{\mathbb{R}^{n-1}_{y'}} \int_{\mathbb{R}_{y_n}} \phi({}^tQy)dy')u_{-\tau,\pm}(y_n)dy_n dy' \tag{81}$$

where Q is an orthogonal matrix transforming θ into $(0, \dots, 0, 1)$. We have proved

PROPOSITION 1. *Suppose that P_θ satisfies the homogeneity property (69) for some $\theta \in S^{n-1}$ and denote by L the hyperplane orthogonal to θ. Then every homogeneous distribution defined by (81) solves (79).*

We propose examples of semilinear elliptic PDEs with singular solutions as above:

1) $Pu = \Delta u + u^d = 0, d \in \mathbb{N}, d \ge 2, \tau = -2/(d-1), \vartheta \in S^{n-1}$;

2) $P = (-\Delta)^m u + D^p_{x_1} u D^q_{x_1} u + u^d = 0$, with $m, d, p, q \in \mathbb{N}$, $d \ge 2$ satisfying $2m = (2m - p - q)(d-1)$. In that case

$$\tau = -\frac{2m}{d-1} = -2m + p + q, \qquad \theta = (1, 0, \dots, 0).$$

7. Analytic Regularization for Semilinear Parabolic Systems

We consider the initial value problem for systems of parabolic equations

$$\partial_t u_j + P_j(D)u_j + \sum_{\ell=1}^{L} \kappa_{j,\ell}(D)(F_{j,\ell}(\vec{u})) = 0, \tag{82}$$

$$\vec{u}|_{t=0} = \vec{u^0}, t > 0, \ x \in \Omega, \ j = 1, \dots, N,$$

where $\vec{u} = (u_1, \dots, u_N)$; $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$. $P_j(D)$ is differential operator of order $m \in 2\mathbb{N}$, $Re(P_j(D))$ is positive elliptic of order m for all $j = 1, \dots, m$.

The nonlinear terms $F_{j,\ell} \in C^1(\mathbb{C}^N : \mathbb{C})$, $j = 1, \dots, N$, are homogeneous of order $s_\ell > 1$. We write $ord_z F(z) = s$ for F (positively) homogeneous of order s.

In the case we study the analytic regularity of the solutions for positive time we will assume that $F_{j,\ell}$, $j = 1, \dots, N$, are homogeneous polynomials of degree $s_\ell \ge 2$, namely

$$F_{j,\ell}(z) = \sum_{\beta \in \mathbb{Z}^N_+, |\beta| = s_\ell} F^\beta_{j,\ell} z^\beta, \qquad F^\beta_{j,\ell} \in \mathbb{C}, \ z \in \mathbb{C}^N, \tag{83}$$

for $j = 1, \dots, N$, $\ell = 1, \dots, L$.

The operators $\kappa_{j,\ell}$ satisfy

$$\kappa_{j,\ell}(D) \in \Psi_h^{d_\ell}(\Omega), 0 \le d_\ell < m \tag{84}$$

for $j = 1, \dots, N$, $\ell = 1, \dots, L$. Here $\Psi_h^\nu(\mathbb{R}^n))$ (resp. $\Psi_h^\nu(\mathbb{T}^n))$ stands for the space of all smooth homogeneous p.d.o. on $\mathbb{R}^n$ (resp. restricted on $\mathbb{T}^n$) of order $\nu \ge 0$. We suppose that

$$\text{either } d_\ell > 0 \text{ or } \kappa_{j,\ell}(\xi) \equiv const,\ \ \ell = 1, \dots L, j = 1, \dots, N \quad \text{if } \Omega = \mathbb{T}^n. \tag{85}$$

The initial data $\vec{u^0} \in \mathcal{S}'(\Omega)$ will be prescribed later on. Such systems contain as particular cases semilinear parabolic equations, the Navier-Stokes equations for an incompressible fluid, Burgers type equations, the Cahn-Hilliard equation, the Kuramoto-Sivashinsky type equations and so on.

For given $q \in [1, +\infty]$, $\gamma \ge 0$, $\theta \in \mathbb{R}$, $\mu \ge 1$ and $T \in]0, +\infty]$ we define the analytic-Gevrey type Banach space $A^\gamma_{\theta,q}(T; \mu)$ as the set of all $\vec{u} \in C(]0, T[: (L^q(\Omega))^N)$ such that the norm

$$\|\vec{u}\|_{A^\gamma_{\theta,q}(T;\mu)} \quad = \quad \sum_{\alpha \in \mathbb{Z}^n_+} \frac{\gamma^{|\alpha|}}{\alpha!} \sup_{0<t\le T} (t^{\frac{|\alpha|}{\mu}+\theta} \|\partial^\alpha \vec{u}(t)\|_{L^q}) \tag{86}$$

is finite. The Sobolev embedding theorems and the Cauchy formula for the radius of convergence of power series imply, for $\gamma > 0$, that, if $\vec{u} \in A^\gamma_{\theta,q}(T; \mu)$ then $\vec{u}(t, \cdot) \in \mathcal{O}(\Gamma_{\gamma t^{\frac{1}{\mu}}})$, $t \in]0, T]$ where $\Gamma_\rho := \{x \in \mathbb{C}^n : |Im(x)| < \rho\}$, $\rho > 0$ and $\mathcal{O}(\Gamma)$ stands for the space of all holomorphic functions in Γ, Γ being an open set in $\mathbb{C}^n$, while for $\gamma = 0$, with the convention $0^0 = 1$, we obtain that $A^0_{\theta,q}(T; \mu)$ coincides with the usual Kato-Fujita weighted type space $C_\theta(L^q; T)$ and μ is irrelevant. Given $u \in C(]0, T[: L^1_{loc}(\Omega))$ and $t \in]0, T[$, we define

$$\rho_{[u]}(t) = \sup\{\rho > 0 : u(t, \cdot) \in \mathcal{O}(\Gamma_\rho)\}$$

with $\rho_{[u]}(t) := 0$ if it cannot be extended to a function in $\mathcal{O}(\Gamma_\rho)$ for any $\rho > 0$. Clearly for each $u \in A^\gamma_{\theta,q}(T; \mu)$ we have $\rho_{[u]}(t) \ge \gamma t^{\frac{1}{\mu}}$, $t \in]0, T]$. We define $A^{\gamma,\mathcal{S}'}_{\theta,q}(T; \mu) := C([0, T[: (\mathcal{S}'(\Omega))^N) \bigcap A^\gamma_{\theta,q}(T; \mu)$, $C^{\mathcal{S}'}_\theta(L^q; T) := A^{0,\mathcal{S}'}_{\theta,q}(T; \mu)$. One motivation for the introduction of $A^{\gamma,\mathcal{S}'}_{\theta,q}(T; \mu)$ is that $(L^p(\Omega))^N \ni f \to (-\Delta)^{\frac{k}{2}} E^\Omega_P[f] \in A^{\gamma,\mathcal{S}'}_{\frac{1}{m}(k+\frac{1}{p}-\frac{1}{q}),q}(T; m)$, for all $1 \le p \le q \le +\infty$, $\gamma \ge 0$, $k \ge 0$ where

$$E^\Omega_P[f](t) := e^{-tP(D)} f = \mathcal{F}^{-1}_{\xi\to x}(e^{-tP(\xi)} \hat{f}(\xi)),$$

$f \in (\mathcal{S}'(\Omega))^N$.

We denote by $B^{\rho,\infty}_q(\mathbb{R}^n)$ (resp. $\dot{B}^{\rho,\infty}_q(\mathbb{R}^n)$) the Besov (resp. homogeneous Besov) spaces, cf. [54]. Typically for perturbative methods dealing with (82), given $\gamma \ge 0$,

$\theta \geq 0, q \geq 1$, we want to find the space of all $f \in (\mathcal{S}'(\Omega))^N$ such that

$$E_P[f] \in A^{\gamma}_{\frac{\theta}{m},q}(T;m) \tag{87}$$

for some (all) $T \in]0,+\infty[$ (respectively

$$E_P[f] \in A^{\gamma}_{\frac{\theta}{m},q}(+\infty;m) \tag{88}$$

if P is homogeneous). These spaces depend on Ω, θ, q and m but not on $\gamma \geq 0$ and P and we denote them by $\mathcal{B}_q^{-\theta,\infty}(\Omega) = \mathcal{B}_q^{-\theta,\infty}(\Omega;m)$ (resp. $\dot{\mathcal{B}}_q^{-\theta,\infty}(\Omega) = \dot{\mathcal{B}}_q^{-\theta,\infty}(\Omega;m)$). It is well known, for instance, that $\dot{\mathcal{B}}_q^{-\theta,\infty}(\mathbb{R}^n) = \dot{B}_q^{-\theta,\infty}(\mathbb{R}^n)$ if $P = -\Delta,\ N=1,\ \theta > 0$, cf. [54]. However $\dot{\mathcal{B}}_q^{0,\infty}(\mathbb{R}^n) \neq \dot{B}_q^{0,\infty}(\mathbb{R}^n)$. One shows that $\mathcal{B}_{q_1}^{-\theta_1,\infty}(\Omega) \hookrightarrow \mathcal{B}_{q_2}^{-\theta_2,\infty}(\Omega)$, (resp. $\dot{\mathcal{B}}_{q_1}^{-\theta_1,\infty}(\Omega) \hookrightarrow \dot{\mathcal{B}}_{q_2}^{-\theta_2,\infty}(\Omega)$) if $\theta_2 \geq$ (resp. $\theta_2 =$) $\theta_1 + \frac{n}{q_1} - \frac{n}{q_2}$, $1 \leq q_1 \leq q_2$; $\dot{\mathcal{B}}_q^{-\theta,\infty}(\mathbb{T}^n) \subset \mathcal{S}_0'(\mathbb{T}^n) := \{f \in \mathcal{S}'(\mathbb{T}^n) \text{ s.t. } \int_{\mathbb{T}^n} f = 0\}$ for all $\theta \geq 0$, $1 \leq q \leq \infty$ and if $b(D) \in \Psi_h^r(\mathbb{T}^n)$, $r \geq 0$, then

$$b(D) : L^p(\mathbb{T}^n) \longrightarrow \dot{\mathcal{B}}_q^{-\frac{n}{p}+\frac{n}{q}-r,\infty}(\mathbb{T}^n), \qquad q \geq p > 1$$

and

$$b(D) : \mathcal{M}(\mathbb{T}^n) \longrightarrow \dot{\mathcal{B}}_q^{-n+\frac{n}{q}-r,\infty}(\mathbb{T}^n), \qquad q \geq 1,\ q > 1 \text{ if } r = 0.$$

Set $H^{\rho}_{\mathcal{S}'}(\mathbb{R}^n)$, $\rho \in \mathbb{R}$, to be the space of all Schwartz distributions homogeneous of order ρ.

We put

$$s = \max\{s_1,\ldots,s_\ell\}, \qquad p_{cr} = n \max_{\ell=1,\ldots,L} \frac{s_\ell - 1}{m - d_\ell}$$

and define $\mathcal{C}^{m,s}_{p_{cr}}(n)$ as the set of all (q,θ) s.t. $q \geq \max\{1,p_{cr}\}$, $\theta \geq 0$, $s\theta < m$, $\theta + \frac{n}{q} \leq \frac{n}{p_{cr}}$ with $q > p_{cr}$ if $\theta = 0$; $\partial\mathcal{C}^{m,s}_{p_{cr}}(n) := \{(q,\theta(q)) \in \mathcal{C}^{m,s}_{p_{cr}}(n)\}$, $\theta(q) := \frac{n}{p_{cr}} - \frac{n}{q}\}$; $\dot{\mathcal{C}}^{m,s}_{p_{cr}}(n) = \mathcal{C}^{m,s}_{p_{cr}}(n) \backslash \partial\mathcal{C}^{m,s}_{p_{cr}}(n)$; $q_{max} = \sup\{\tau > q : s(\theta(q) + \frac{n}{q} - \frac{n}{\tau}) < m\}$. Throughout the section we will tacitly assume that $F_{j,\ell}$'s are polynomials as in (83) when we state analytic regularity results for (82) in the framework of the Gevrey spaces $A^{\gamma}_{\frac{\theta}{m},q}(T;m)$, $\gamma > 0$.

THEOREM 4. *There exists an absolute constant $a > 0$ such that:*

i) if $(q,\theta) \in \dot{\mathcal{C}}^{m,s}_{p_{cr}}(n)$ and $\vec{u}^0 \in \mathcal{B}_q^{-\theta,\infty}(\Omega)$ then $\exists T^* > 0$ s.t. (82) admits a solution

$$\vec{u} \in \bigcap_{\gamma \geq 0} A^{\gamma,\mathcal{S}'}_{\frac{\theta}{m},q}(T^* \exp(-a\gamma^{\frac{m-1}{m}});m). \tag{89}$$

The solution is unique in $C^{\mathcal{S}'}_{\frac{\theta}{m}}(L^q;T^*)$ provided $q \geq s$;

ii) if $(q,\theta(q)) \in \partial\mathcal{C}^{m,s}_{p_{cr}}(n)$ *then* $\exists C' > 0$ *s.t. if* $\vec{u^0} \in \mathcal{B}^{-\theta(q),\infty}_q(\Omega)$ *satisfies*

$$\lim_{T\searrow 0} \|E^{\Omega}_P[\vec{u^0}]\|_{C_{\frac{\theta(q)}{m}}(L^q;T)} = c'_0 \le C' \tag{90}$$

then $\exists T' > 0$ *s.t. (82) admits a solution* $\vec{u} \in C^{\mathcal{S}'}_{\frac{\theta(q)}{m}}(L^q;T')$ *satisfying* $\vec{u} \in A^{\gamma}_{\frac{\theta(q)}{m},q}(T'_\gamma;m)$ *for some* $T'_\gamma \in]0,T']$, $\gamma \in [0,(\frac{1}{a}\ln\frac{C'}{c_0})^{\frac{m}{m-1}}]$. *The solution is unique in* $C^{\mathcal{S}'}_{\frac{\theta(q)}{m}}(L^q;T')$ *provided* $q \ge s$.

For the next two theorems we require that the operator $P(D)$ is homogeneous and

$$\frac{m-d_1}{s_1-1} = \ldots = \frac{m-d_\ell}{s_\ell-1} = \frac{n}{p_{cr}}.$$

THEOREM 5. *Let* $(q,\theta(q)) \in \partial\mathcal{C}^{m,s}_{p_{cr}}(n)$. *We claim that there exists* $C'' > 0$ *s.t. if* $\vec{u^0} \in \dot{\mathcal{B}}^{-\theta(q),\infty}_q(\Omega)$ *satisfies*

$$\|E^{\Omega}_P[\vec{u^0}]\|_{C_{\frac{\theta(q)}{m}}(L^q;+\infty)} = c''_0 \le C''_q \tag{91}$$

then (82) admits a global solution

$$\vec{u} \in \bigcap_{0\le\gamma\le\overline{\gamma}} A^{\gamma}_{\frac{\theta(q)}{m},q}(+\infty;m), \qquad \overline{\gamma} := (\frac{1}{a}\ln\frac{C''}{c''_0})^{\frac{m}{m-1}}.$$

Furthermore, the solution is unique if $q \ge s$.

THEOREM 6. *Let* $\vec{u}^0 \in (H^{-\frac{n}{p_{cr}}}_{\mathcal{S}'}(\mathbb{R}^n))^N \bigcap \dot{\mathcal{B}}^{-\theta(q),\infty}_q(\mathbb{R}^n)$ *for some* $q \in]\max\{p_{cr},s\},q_{max}[$, *and let* $\|E^{\mathbb{R}^n}_P[\vec{u^0}]\|_{C_{\frac{\theta(q)}{m}}(L^q;+\infty)} = c''_0 \le C''$. *Then the unique solution in the previous theorem satisfies*

$$\vec{u}(t,x) = t^{-\frac{n}{mp_{cr}}}\vec{g}(\frac{x}{\sqrt[m]{t}}),\ t>0 \qquad \vec{g}(z) \in L^q(\mathbb{R}^n)\bigcap(L^\infty(\mathbb{R}^n)\bigcap\mathcal{O}(\Gamma_\gamma))^N, \tag{92}$$

$$\vec{w} := E^{\mathbb{R}^n}_P[\vec{u^0}](1) - \vec{g} \in (L^{\overline{q}}(\mathbb{R}^n)\bigcap\mathcal{O}(\Gamma_\gamma))^N, \qquad \overline{q} = \max\{p_{cr},\frac{q^*}{s}\} \tag{93}$$

for all $\gamma \in [0,(\frac{1}{a}\ln\frac{C''}{c''_0})^{\frac{m}{m-1}}]$. *Assume now that* $F_{j,\ell}$*'s are polynomials and*

$$s_\ell \le p_{cr} \le 2s_\ell,\ \ s < p_{cr},\ \ 2d_\ell \ge m, \qquad \ell = 1,\ldots,L. \tag{94}$$

Then there exists $\varepsilon > 0$ *s.t. for all*

$$\vec{u^0} \in (H^{-\frac{n}{p_{cr}}}_{\mathcal{S}'}(\mathbb{R}^n)\bigcap\dot{B}^{0,\infty}_{p_{cr}}(\mathbb{R}^n))^N$$

with

$$\|\vec{u^0}\|_{\dot{B}^{0,\infty}_{p_{cr}}} = \varepsilon_0 < \varepsilon$$

the IVP (82) has a unique solution

$$\vec{u} \in BC_w([0,+\infty[: (\dot{B}^{0,\infty}_{p_{cr}}(\mathbb{R}^n))^N)$$

satisfying (92) and $\vec{w} \in L^{p_{cr}}(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$. *Furthermore,* $\vec{g} \in (\mathcal{O}(\Gamma_\gamma))^N$ *for* $\gamma \in [0,\overline{\gamma}]$. *Here the subscript* w *in* BC_w *means that we have continuity in the weak topology* $\sigma(\dot{B}^{0,\infty}_{p_{cr}}(\mathbb{R}^n), (\dot{B}^{0,\infty}_{p'_{cr}}(\mathbb{R}^n)))$ *and* $p'_{cr} = \frac{p_{cr}}{p_{cr}-1}$.

8. Sketch of the proofs of the Gevrey regularity for parabolic systems

The main idea is to reduce (82) to the system of integral equations

$$u_j(t) = E^{\Omega}_{P_j}[u^0_j](t) + \sum_{\ell=1}^{L} K^{\Omega}_{j,\ell}[\vec{u}](t), \qquad j = 1,\dots,N \tag{95}$$

where

$$K^{\Omega}_{j,\ell}[\vec{u}](t) = \int_0^t E^{\Omega}_{j,\ell}(t-\tau) * F_{j,\ell}(\vec{u}(\tau))d\tau,$$

$E^{\Omega}_{j,\ell}(t) = \kappa_{j,\ell}(D)E^{\Omega}_{P_j}$. We assume that P_j is homogeneous.

We write a Picard type iterative scheme

$$u^{k+1}_j(t) = E^{\Omega}_{P_j}[u^0_j](t) + \sum_{\ell=1}^{L} K^{\Omega}_{j,\ell}[\vec{u^k}](t), \qquad j = 1,\dots,N \tag{96}$$

for $k = 0,1,\dots$ with $\vec{u^0} := 0$.

We need two crucial estimates, namely for some absolute constant $a > 0$

$$\max_{j,\ell} \|E^{\Omega}_{j,\ell}\|_{A^{\gamma}_{\frac{d_\ell}{m}+\frac{n}{m}(1-\frac{1}{r}),r}(+\infty;m)} \le C_1 \exp(a\gamma^{\frac{m-1}{m}}), \qquad \forall\gamma \ge 0 \tag{97}$$

$$\|K^{\Omega}_{j,\ell}[\vec{u}]\|_{A^{\gamma}_{\theta,q}(T;m)} \le C_2\|E^{\Omega}_{j,\ell}\|_{A^{\gamma}_{\frac{d_\ell}{m}+\frac{n(s_\ell-1)}{mq},\frac{q}{q-s_\ell+1}}(T;m)}(\|\vec{u}\|_{A^{\gamma}_{\theta,q}(T;m)})^{s_\ell}T^{\rho_\ell} \tag{98}$$

where $r \in [1,+\infty]$ (resp. $r \in]1,+\infty]$) if $d_\ell > 0$ or $d_\ell = 0$ and $\kappa_{j,\ell}(\xi) \equiv const$ (resp. $d_\ell = 0$, $\kappa_{j,\ell}(\xi) \not\equiv const$, $\Omega = \mathbb{R}^n$),

$$(q,\theta) \in \partial\mathcal{C}^{m,s}_{p_{cr}}(n), \qquad \rho_\ell = \frac{m - d_\ell - (\theta + \frac{n}{q})(s_\ell - 1)}{m},$$

$C_1 = C_1(r) > 0$, $C_2 = C_2(\{F_{j,\ell}\}, r) > 0$. We note that in the case $\Omega = \mathbb{R}^n$ we have

$$\partial^\alpha_x E^{\mathbb{R}^n}_{j,\ell}(t,x) = \int e^{ix\xi - tP_j(\xi)}\kappa_{j,\ell}(\xi)\xi^\alpha \,\bar{d}\xi = t^{-\frac{d_\ell+|\alpha|+n}{m}}\varphi_{j,\ell}(\frac{x}{t^{\frac{1}{m}}}), \ \bar{d}\xi = (2\pi)^{-n}d\xi, \tag{99}$$

with $\mathcal{F}\varphi_{j,\ell}(\xi) = e^{-P_j(\xi)}\kappa_{j,\ell}(\xi)\xi^\alpha$. If $r \ge 2$ we estimate $\|\varphi_{j,\ell}\|_{L^r}$ by means of the Fourier transformation, the Young theorem and the Stirling formula. For the case $1 \le$

$r < 2$ we deduce the same result using integration by parts, the properties of the Fourier transform of homogeneous functions and the Stirling formula again. The case $\Omega = \mathbb{T}^n$ and $r \in [2, +\infty]$ is evident while (85), (97) for $\Omega = \mathbb{R}^n$, $r = 1$ and the representation

$$(100) \qquad \partial_x^\alpha E_{j,\ell}^{\mathbb{T}^n}(t,x) = \sum_{\xi \in \mathbb{Z}^n} \partial_x^\alpha E_{j,\ell}^{\mathbb{R}^n}(t, x + 2\pi\xi), \qquad x \in \mathbb{T}^n \sim [-\pi, \pi]^n$$

yield the $L^1(\mathbb{T}^n)$ estimate (97) (see [39] for similar arguments). The Riesz-Thorin theorem concludes the proof of (97) for $\Omega = \mathbb{T}^n$. The key argument in showing (98) is a series of nonlinear superposition estimates in the framework of $A^\gamma_{\theta,q}(T; m)$. We note that $m \geq 1$ is essential for the validity of such estimates. Next, for given $R > 0$ we define

$$B_q^\gamma(R : T) = \{\vec{u} \in A^\gamma_{\frac{\theta}{m},q}(T; m) : \|\vec{u}\|_{A^\gamma_{\frac{\theta}{m},q}(T;m)} \leq R\}.$$

At the end we are reduced to find $R > 0$ and $T > 0$ such that

$$(101) \qquad \|E_P^\Omega[\vec{u^0}]\|_{A^\gamma_{\frac{\theta}{m},q}(T;m)} + C_1 \exp(a\gamma^{\frac{m-1}{m}}) \sum_{\ell=1}^{L} T^{\rho_\ell} R^{s_\ell} \quad \leq \quad R,$$

$$(102) \qquad C_2 \exp(a\gamma^{\frac{m-1}{m}}) \sum_{\ell=1}^{L} T^{\rho_\ell} R^{s_\ell - 1} \quad < \quad 1.$$

The estimates (102) allows us to show the convergence of the scheme above which leads to the existence–uniqueness statements for local and global solutions.

The self-similar solutions in the first part of Theorem 6 are obtained by the uniqueness and the homogeneity, while (92) and (93) are deduced by a suitable generalization of arguments used in [49] and [5].

Concerning the last part of Theorem 6, we follow the idea in [16], namely setting $g = v + w$, $v = E_P[u^0](1)$ (we consider the scalar case $g = \vec{g}$, $L = 1$) we obtain for w, an equation modeled by

(103)

$$w = \mathcal{H}_P^\kappa[(v+w)^s], \quad \mathcal{H}_P^\kappa[f] = \int_0^1 \int_{\mathbb{R}^n} \kappa(D) E_P^{\mathbb{R}^n}(1-\tau, y) \tau^{-\frac{ns}{mp_{cr}}} f(\frac{y}{\sqrt[m]{t}}) dy d\tau$$

where $\kappa(D) \in \Psi_h^d(\mathbb{R}^n)$. The condition (94) allows us to generalize Lemma 6, p. 187 in [16], namely we show that $\mathcal{H}$ acts continuously from $L^{\frac{p_{cr}}{s}}(\mathbb{R}^n)$ to $L^{p_{cr}}(\mathbb{R}^n)$ using the Littlewood-Paley analysis and the characterization of the L^p spaces.

We point out, that if $\vec{u^0} \in (H_p^r(\Omega))^N$ and $p > 1$ we show that $\lim_{T\to 0} \|E_P[\vec{u^0}]\|_{A^\gamma_{\frac{\theta}{m},q}(T;m)} = 0$ for all $\theta = r^- + \frac{n}{p} - \frac{n}{q}$, $q \geq \max\{p, \frac{pn}{n-rp}\}$, $\gamma \geq 0$. Thus we recover and/or generalize the known local and global results for the semilinear heat equations when $r \geq r_{cr}(p)$ (see [38], [3], [5], [21] and [49] and the references therein). In particular, we extend the result of THEOREM 2.1 in [39] on the complex Ginzburg-Landau equation in $\mathbb{T}^n$, since THEOREM 2 ii) allows initial data $u^0 \in H_2^{r_{cr}(2)}(\mathbb{T}^n) = H_2^{\frac{n}{2}-\frac{1}{\sigma}}(\Omega)$, provided $\sigma > \max\{\frac{1}{n}, \frac{4}{n+\sqrt{n^2+16n}}\}$. Furthermore, our local results on the analytic regularity yield $\rho_{[u]}(t) = O(t^{\frac{1}{2}})$, $t \searrow 0$ which improves the corresponding results for the Navier-Stokes equation for an incompressible fluid in $\Omega = \mathbb{T}^n$, $n = 2, 3$ while for the Ginzburg-Landau equation we get $\rho_{[u]}(t) = O(t^{\frac{1}{2}})$, $t \searrow 0$, the same rate as in [53], where the initial data are $L^\infty(\mathbb{R}^n)$. If $m = 4$ Theorem 4 and Theorem 5 yield new results for the Cahn-Hilliard equation $\partial_t u + \Delta^2 u + \Delta(u^s) = 0$. Here $p_{cr} = \frac{n(s-1)}{2}$, and $r_{cr}(p_{cr}) \in I_p$ iff $s > \frac{4+n}{n+2}$ which is always fulfilled since $s \geq 2$. Hence if $u^0 = \beta|D|^{r_{cr}(p)}\omega$, $\omega \in L^p(\mathbb{R}^n)$ if $p > 1$, $\omega \in \mathcal{M}(\mathbb{R}^n)$, $p \in [\max\{1, p_{cr}\}, p_{max}[$, $\beta \in \mathbb{R}$, (82) admits unique global solution $u(t,x)$ which belongs to $\mathcal{O}(\Gamma_{\gamma t^{\frac{1}{4}}})$ for all $t > 0$ provided $\|\omega\|_{L^p} < c\exp(-a\gamma^{\frac{1}{4}})$. We could consider fractional derivatives of measures as initial data iff $p_{cr} \leq 1$ which is equivalent to $s \in]\frac{4+n}{2+n}, \frac{n+2}{n}]$.

Our estimates on the analytic regularity globally in $t > 0$ seem to be completely new. We have examples for $\Omega = \mathbb{R}^n$ showing that our estimates on $\rho_{[u]}(t)$ are sharp at least within certain classes of solutions. If $\Omega = \mathbb{T}^n$ we could give in some cases better estimates of $\rho_{[u]}(t)$ as $t \to +\infty$.

Comparing Theorem 6 with the results in [16] for self-similar solutions, we point out that we allow initial data

$$\vec{u^0} \in (H_{S'}^{-1}(\mathbb{R}^n))^N$$

such that $\vec{u^0}|_{S^{n-1}} \notin (L^\infty(S^{n-1}))^N$. We construct also self-similar solutions for the Cahn-Hilliard equation of the form $u(t,x) = t^{-\frac{1}{2(s-1)}} g(\frac{x}{\sqrt[4]{t}})$. As it concerns the last part of Theorem 6, it is an extension of Theorem 2, p. 181 in [16].

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Todor GRAMCHEV
Dipartimento di Matematica
Università di Cagliari
via Ospedale 72
09124 Cagliari, ITALIA
e-mail: todor@unica.it