RENDICONTI DEL SEMINARIO MATEMATICO

Università e Politecnico di Torino

CONTENTS

Volume 61, N. 1 **2003**

Rend. Sem. Mat. Univ. Pol. Torino
Vol. 61, 1 (2003)

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EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS FOR A MODEL OF CONTAMINANT FLOW IN POROUS MEDIUM

Abstract. This paper deals with the existence and uniqueness of the weak periodic solution for a model of transport of a pollutant flow in a porous medium. Our model is described by means of a nonlinear degenerate parabolic problem. To prove the existence of periodic solutions, we use as preliminary steps the Schauder fixed point theorem for the Poincaré map of a nondegenerate initial–boundary value problem associated to ours and the a–priori estimates deduced on these solutions. Our uniqueness result follows from a more general result which shows the continuous dependence of solutions with respect to the data. As another consequence of this general result we prove a comparison principle for periodic solutions.

1. Introduction

In this paper we consider a nonlinear parabolic problem which arises from a model of transport for a pollutant flow in a porous medium (see [3]).

$$(P)\left\{\begin{array}{c} u_t = div(\nabla\varphi(u) - \psi(u)\mathbf{V}(x,t)), \text{ in } Q_T := \Omega\times(0,T) \\ (\nabla\varphi(u) - \psi(u)\mathbf{V}(x,t))\cdot\mathbf{n} = g(x,t), \text{ on } S_T := \partial\Omega\times(0,T) \\ u(x,t+\omega) = u(x,t), \text{ in } Q_T,\ T\geq\omega>0 \end{array}\right\}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $\mathbf{n}$ denotes the outward unit normal vector on the boundary $\partial\Omega$. The increased demand for water in various parts of the word, makes very important the problem of the water quality for the development and use of water resources.

Special attention should be devoted to the pollution of groundwater in acquifers and surface water. The term pollutant shall be used to denote dissolved matter carried with water. We deal with the transport of mass of certain solute that moves with the water in the interstices of an inhomogeneous porous medium. At every point within a porous medium, we have the product $\psi(u)\mathbf{V}(x,t)$ between the liquid velocity $\mathbf{V}(x,t)$ and a nonlinear function $\psi(u)$ of the concentration u of the pollutant. The term $\psi(u)\mathbf{V}(x,t)$, represents the advective flux i.e. the flux carried by the water at the velocity $\mathbf{V}(x,t)$. The fundamental balance equation for the transport of a pollutant concentration in a porous medium, is given by the adventive–dispersion equation $u_t = div(\nabla\varphi(u) - \psi(u)\mathbf{V}(x,t))$.

2. Preliminaries

We study the problem (P) under the assumptions:

$$H_\varphi)\ \varphi \in C([0,\infty)) \cap C^1((0,\infty)), \varphi(0) = 0,\ \varphi'(s) > 0 \text{ for } s \neq 0\ ;$$

$H_\psi)$ $\psi \in C([0,\infty)) \cap C^1((0,\infty))$, $\psi(0) = 0$, ψ locally Lipschitz continuous;
$H_V)$ $\mathbf{V} \in \prod_{i=1}^n C(\overline{Q}_T) \cap C^1(Q_T)$, $\mathbf{V}(x,.)$ is ω–periodic, $div\mathbf{V}(x,t) = 0$ in Q_T and $\mathbf{V}(x,t)\cdot\mathbf{n} < 0$ on S_T;
$H_g)$ $g \in L^\infty(S_T)$, $g > 0$, $g(x,.)$ is ω–periodic and admits an extension on all Q_T such that $g_x \in L^\infty(Q_T)$.

REMARK 1. The assumptions $H_\varphi)$ and $H_\psi)$ include both the case of degenerate equations i.e. $\varphi'(0) = 0$ and $\psi'(0) = \pm\infty$, while the assumptions $H_g)$ allows to apply the result of [5, thm. 6.2].

DEFINITION 1. *A function $u \in C([0,T]; L^2(\Omega)) \cap L^\infty(Q_T)$, is a periodic weak solution to (P), if $u(x,t+\omega) = u(x,t)$, $\varphi(u) \in L^2((0,T); H^1(\Omega))$ and*

$$\int_0^T \int_\Omega (u\zeta_t + \varphi(u)\Delta\zeta + \psi(u)\mathbf{V}(x,t)\cdot\nabla\zeta)dxdt + \tag{1}$$

$$\int_0^T \int_{\partial\Omega} (g(x,t)\zeta - \varphi(u)\frac{\partial\zeta}{\partial\mathbf{n}})dSdt = \int_\Omega (u(x,T)\zeta(x,T) - u(x,0)\zeta(x,0))dx$$

for any ζ such that ζ, ζ_t, $\Delta\zeta \in L^2(Q_T)$ and $\frac{\partial\zeta}{\partial\mathbf{n}} \in L^2(S_T)$.

The existence of the positive weak periodic solutions for the problem (P) shall be obtained as the limit of approximated periodic solutions whose existence is showed by means of the Schauder fixed point theorem, applied to the Poincaré map of a nondegenerate initial–boundary value problem associated to (P). In the light of what has been said, we begin by proving the existence of the positive periodic solutions for the approximated nondegenerate problem

$$(P_\varepsilon)\left\{\begin{array}{c} u_{\varepsilon t} = div(\nabla\varphi_\varepsilon(u_\varepsilon) - \psi_\varepsilon(u_\varepsilon)\mathbf{V}(x,t)),\ \text{in } Q_T \\ (\nabla\varphi_\varepsilon(u_\varepsilon) - \psi_\varepsilon(u_\varepsilon)\mathbf{V}(x,t))\cdot\mathbf{n} = g_\varepsilon(x,t),\ \text{on } S_T \\ u_\varepsilon(x,t+\omega) = u_\varepsilon(x,t),\ \text{in } Q_T \end{array}\right\}$$

where

$$H_{\varphi_\varepsilon})\ \varphi_\varepsilon \in C^1([0,\infty)), \varphi_\varepsilon(0) = 0, \varphi'_\varepsilon(s) \geq \varepsilon, \varphi_\varepsilon(s) = \varphi(s) \text{ if } s \geq \varepsilon/2$$

and $\varphi_\varepsilon \to \varphi$ uniformly on compact sets of R_+ as $\varepsilon \to 0^+$;
$H_{\psi_\varepsilon})$ $\psi_\varepsilon \in C^1([0,\infty))$, $\psi_\varepsilon(0) = 0$, with $\psi_\varepsilon(s) = \psi(s)$ if $s \geq \varepsilon/2$
and $\psi_\varepsilon \to \psi$ uniformly on compact sets of R_+ as $\varepsilon \to 0^+$;
$H_{g_\varepsilon})$ $g_\varepsilon \in C^\infty(\overline{Q}_T)$, $g_\varepsilon > 0$, $g_\varepsilon(x,.)$ is ω–periodic and $g_\varepsilon \to g$

uniformly on compact sets of $\overline{Q}_T$ as $\varepsilon \to 0^+$.

The existence of the positive periodic solutions to (P_ε) derives from the Schauder fixed point theorem for the Poincaré map of the associated initial–boundary value problem

$$(P'_\varepsilon)\left\{\begin{array}{c} u_{\varepsilon t} = div(\nabla\varphi_\varepsilon(u_\varepsilon) - \psi_\varepsilon(u_\varepsilon)\mathbf{V}(x,t)),\ \text{in } Q_T \\ (\nabla\varphi_\varepsilon(u_\varepsilon) - \psi_\varepsilon(u_\varepsilon)\mathbf{V}(x,t))\cdot\mathbf{n} = g_\varepsilon(x,t),\ \text{on } S_T \\ u_\varepsilon(x,0) = u_{0\varepsilon},\ \text{in } \Omega \end{array}\right\}$$

where

$H_{0\varepsilon})$ $u_{0\varepsilon} \in C^2(\overline{\Omega})$, $u_{0\varepsilon} \geq 0$ for all $x \in \overline{\Omega}$ and satisfies the compatibility condition $(\nabla\varphi_\varepsilon(u_{0\varepsilon}(x)) - \psi_\varepsilon(u_{0\varepsilon}(x))\mathbf{V}(x,0))\cdot\mathbf{n} = g_\varepsilon(x,0)$, on $\partial\Omega$.

The uniqueness of the positive weak periodic solutions, follows from a more general result which shows the continuous dependence of the solutions with respect to the data. This result shall be established extending, to our periodic case, the method utilized in [4], [6], [8] for the study of the Cauchy or the Cauchy–Dirichlet problems. As a conclusive fact of this extention, we show a comparison principle for the periodic solutions. According to the knowledges of the author, the topic considered here has not been discussed previously, in the literature. Related papers to ours are [1] where the blow–up in finite time is studied for a problem of reaction–diffusion and [2] where the existence and uniqueness of the solution for a non periodic problem (P) is showed in a unbounded domain Ω. See also [9].

3. Existence and uniqueness for the approximating problem

The classical theory of parabolic equations asserts that the problem (P'_ε) has a unique solution $u_\varepsilon \in C^{2,1}(\overline{Q}_T)$. Moreover, problem (P'_ε) has ε as a lower–solution if we assume that

$$-\psi(\varepsilon)\mathbf{V}(x,t)\cdot\mathbf{n} \leq g_\varepsilon(x,t),\ \text{on } S_T\ .$$

If we suppose that there exists a constant $M > 0$ such that $\psi(M) > 0$ and

$$-\psi(M)\mathbf{V}(x,t)\cdot\mathbf{n} \geq g_\varepsilon(x,t),\ \text{on } S_T\ ,$$

then, M is an upper–solution for (P'_ε).

If $u_{0\varepsilon}$ verifies

$$\varepsilon \leq u_{0\varepsilon}(x) \leq M\ ,\ \text{for all } x \in \overline{\Omega}\ , \tag{1}$$

the comparison principle asserts that

$$\varepsilon \leq u_\varepsilon(x,t) \leq M\ ,\ \text{in } \overline{Q}_T\ . \tag{2}$$

For $\varphi(u_\varepsilon)$ holds this uniform estimate

PROPOSITION 1. *There exists a constant $C > 0$, independent of ε, such that*

$$\int_0^T \|\varphi(u_\varepsilon)\|^2_{H^1(\Omega)}dt \leq C\ . \tag{3}$$

Proof. Multiply the equation in (P'_ε) by $\varphi(u_\varepsilon)$ and integrate by parts using Young's inequality, we have

$$\frac{d}{dt}\int_\Omega \Phi_\varepsilon(u_\varepsilon)dx + \frac{1}{2}\int_\Omega |\nabla\varphi(u_\varepsilon)|^2 dx \leq \tag{4}$$

$$\int_{\partial\Omega} g_\varepsilon(x,t)\varphi(M)dS + \frac{1}{2}\int_\Omega \|\mathbf{V}(x,t)\|^2_{R^n}|\psi(u_\varepsilon)|^2 dx \ ,$$

where $\Phi_\varepsilon(u_\varepsilon) := \int_\varepsilon^{u_\varepsilon}\varphi(s)ds$. Integrating (4) over $(0, T)$, one has

$$\int_\Omega \Phi_\varepsilon(u_\varepsilon(x,T))dx - \int_\Omega \Phi_\varepsilon(u_{0\varepsilon}(x))dx + \frac{1}{2}\int_0^T\int_\Omega |\nabla\varphi(u_\varepsilon)|^2 dxdt \leq C_1$$

and from

$$\frac{d}{du_\varepsilon}|\varphi(u_\varepsilon)|^2 = 2\varphi(u_\varepsilon)\varphi'(u_\varepsilon) \leq 2C_2\varphi(u_\varepsilon)\ ,$$

$(C_2 = \sup\{|\varphi'(\xi)|, \varepsilon < \xi < M\})$, one obtains

$$|\varphi(u_\varepsilon)|^2 \leq 2C_2\Phi_\varepsilon(u_\varepsilon) + |\varphi(\varepsilon)|^2\ .$$

By (1) follows that

$$\int_0^T\int_\Omega |\varphi(u_\varepsilon)|^2 xdt + \frac{1}{2}\int_0^T\int_\Omega |\nabla\varphi(u_\varepsilon)|^2 dxdt \leq C\ .$$

□

Taking into account that $u_{0\varepsilon} \in C^2(\overline{\Omega})$, we can utilize the regularity result given in [5], which establishes that the sequence of solutions u_ε is equicontinuous in $\overline{Q}_T$.

PROPOSITION 2. ([5]). *If $u_{0\varepsilon}$ is continuous on $\overline{\Omega}$, then the sequence $\{u_\varepsilon\}$ of solutions of (P'_ε) is equicontinuous in $\overline{Q}_T$ in the sense that there exists $\omega_0 : R_+ \to R_+$, $\omega_0(0) = 0$ continuous and nondecreasing such that*

$$|u_\varepsilon(x_1,t_1) - u_\varepsilon(x_2,t_2)| \leq \omega_0(|x_1 - x_2| + |t_1 - t_2|^{1/2})\ , \tag{5}$$

for any (x_1, t_1), $(x_2, t_2) \in \overline{Q}_T$.

In order to mobilize the Schauder fixed point theorem, we introduce the closed and convex set

$$K_\varepsilon := \{w \in C(\overline{\Omega}) : \varepsilon \leq w(x) \leq M,\ \forall x \in \overline{\Omega}\}$$

and the Poincaré map associated to the problem (P'_ε), defined as follows

$$\mathcal{F}(u_{0\varepsilon}(.)) = u_\varepsilon(., \omega)$$

where u_ε is the unique solution of (P'_ε).

From the formula (2) and the Proposition 2, we deduce that

i) $\mathcal{F}(K_\varepsilon) \subset K_\varepsilon$

ii) $\mathcal{F}(K_\varepsilon)$ is relatively compact in $C(\overline{\Omega})$.

It remains to prove that

iii) $\mathcal{F}|_{K_\varepsilon}$ is continuous.

PROPOSITION 3. *If $u^n_{0\varepsilon}$, $u_{0\varepsilon} \in K_\varepsilon$ and $u^n_{0\varepsilon} \to u_{0\varepsilon}$ uniformly on $\overline{\Omega}$ as $n \to \infty$, then, if u^n_ε and u_ε are solutions of (P'_ε) with initial data $u^n_{0\varepsilon}$ and $u_{0\varepsilon}$ respectively, we have that $u^n_\varepsilon(.,t)$ converges to $u_\varepsilon(.,t)$ uniformly as $n \to \infty$ for any $t \in [0,T]$.*

Proof. Multiplying the equation in (P'_ε) by $sgn(u^n_\varepsilon - u_\varepsilon)$ and integrating over Q_t, we get

$$\frac{d}{dx}\int_0^t \int_\Omega |u^n_\varepsilon(x,s) - u_\varepsilon(x,s)|dxds = 0$$

i.e.

$$\int_\Omega |u^n_\varepsilon(x,t) - u_\varepsilon(x,t)|dx = \int_\Omega |u^n_{0\varepsilon}(x) - u_{0\varepsilon}(x)|dx.$$

The uniform convergence of $u^n_{0\varepsilon}(x) \to u_{0\varepsilon}(x)$ when $n \to \infty$, implies that u^n_ε strongly converges to u_ε in $L^1(\Omega)$ as n goes to infinity. Consequently, for a subsequence, $u^n_\varepsilon(x,t)$ converges to $u_\varepsilon(x,t)$ a.e. $x \in \Omega$. Since $\varepsilon \le u^n_\varepsilon(x,t) \le M$, the Lebesgue theorem allows to conclude that $u^n_\varepsilon \to u_\varepsilon$ in $L^p(\Omega)$, for any $1 \le p \le \infty$. The uniform convergence of $u^n_\varepsilon(.,t)$ to $u_\varepsilon(.,t)$ when $n \to \infty$ is due to the fact that $u^n_\varepsilon(.,t)$, $u_\varepsilon(.,t) \in C(\overline{\Omega})$ for any $t \in [0,T]$.

□

Now, we can apply the Schauder fixed point theorem and conclude that the Poincaré map has at least one point, which is a periodic solution of (P'_ε). Closing this section, we state our main result

THEOREM 1. *If the assumptions $H_\varphi) - H_g)$ hold, there exist positive weak ω–periodic solutions to the problem (P).*

Proof. When $\varepsilon \to 0^+$, the above estimates and the compactness result yield

(6) $\quad u_\varepsilon \to u$ uniformly on $\overline{Q}_T$, by the Ascoli–Arzelà theorem

and

(7) $\quad u_\varepsilon \to u$ strongly in $L^2(Q_T)$, because of (6) and (2) .

From (4), one has

(8) $$\varphi(u_\varepsilon) \to \varphi(u) \text{ in } L^2((0,T); H^1(\Omega)),$$

while (6) and the Lebesgue theorem imply that

(9) $$\varphi(u_\varepsilon) \to \varphi(u) \text{ in } L^2(Q_T)\ .$$

Finally, assumption $H_{g_\varepsilon})$ gives

$$g_\varepsilon(x,t) \to g(x,t) \text{ uniformly on } \partial\Omega \times [0,T]\,. \tag{10}$$

This easily leads to conclude that u satisfies (1).

□

4. Uniqueness and comparison principle

To obtain the uniqueness of the solutions, we need some preliminary inequalities. Let u_ε and v_ε be any positive ω–periodic solutions of (P_ε) with boundary data g_ε, g_ε^* respectively, such that

$$\varepsilon \le \max\{u_\varepsilon(x,t), v_\varepsilon(x,t)\}$$

then,

$$\int_0^T \int_\Omega [(u_\varepsilon - v_\varepsilon)\zeta_t + (\varphi(u_\varepsilon) - \varphi(v_\varepsilon))\Delta\zeta + (\psi(u_\varepsilon) - \psi(v_\varepsilon))\mathbf{V}(x,t)\cdot\nabla\zeta]dxdt + \tag{11}$$

$$\int_0^T \int_{\partial\Omega} (g_\varepsilon(x,t) - g_\varepsilon^*(x,t))\zeta(x,t)dSdt - \int_0^T \int_{\partial\Omega} (\varphi(u_\varepsilon) - \varphi(v_\varepsilon))\frac{\partial\zeta}{\partial\mathbf{n}})dSdt =$$

$$\int_\Omega (u_\varepsilon(x,T) - v_\varepsilon(x,T))\zeta(x,T)dx - \int_\Omega (u_\varepsilon(x,0) - v_\varepsilon(x,0))\zeta(x,0)dx,$$

for any ζ such that $\zeta, \zeta_t, \Delta\zeta \in L^2(Q_T)$ and $\frac{\partial\zeta}{\partial\mathbf{n}} \in L^2(S_T)$.

Define

$$A_\varepsilon(x,t) := \frac{\varphi(u_\varepsilon) - \varphi(v_\varepsilon)}{u_\varepsilon - v_\varepsilon} = \int_0^1 \varphi'(\theta u_\varepsilon(x,t) + (1-\theta)v_\varepsilon(x,t))d\theta$$

$$B_\varepsilon(x,t) := \frac{\psi(u_\varepsilon) - \psi(v_\varepsilon)}{u_\varepsilon - v_\varepsilon} = \int_0^1 \psi'(\theta u_\varepsilon(x,t) + (1-\theta)v_\varepsilon(x,t))d\theta,$$

so that (11) becomes

$$\int_0^T \int_\Omega (u_\varepsilon - v_\varepsilon)(\zeta_t + A_\varepsilon(x,t)\Delta\zeta + B_\varepsilon(x,t)\mathbf{V}(x,t)\cdot\nabla\zeta)dxdt = \tag{12}$$

$$\int_\Omega (u_\varepsilon(x,T) - v_\varepsilon(x,T))\zeta(x,T)dx - \int_\Omega (u_\varepsilon(x,0) - v_\varepsilon(x,0))\zeta(x,0)dx +$$

$$\int_0^T \int_{\partial\Omega} (g_\varepsilon^*(x,t) - g_\varepsilon(x,t))\zeta(x,t)dSdt + \int_0^T \int_{\partial\Omega} (\varphi(u_\varepsilon) - \varphi(v_\varepsilon))\frac{\partial\zeta}{\partial\mathbf{n}})dSdt.$$

There exist some positive constants α and L depending only on ε and M, such that (see [6])

$$\varepsilon \le A_\varepsilon(x,t) \le \alpha := \sup\{\varphi'(s), \varepsilon \le s \le M\}\,, \quad \forall\,(x,t) \in \overline{Q}_T$$

$$|B_\varepsilon(x,t)| \le L := \sup\{|\psi'(s)|, \varepsilon \le s \le M\}\,, \quad \forall\,(x,t) \in \overline{Q}_T.$$

Let $\zeta_{\varepsilon,m}$ be the solution of the backward linear parabolic problem with smooth coefficients

$$(P_{\zeta_{\varepsilon,m}}) \left\{ \begin{array}{c} \zeta_{\varepsilon,mt} + A_{\varepsilon,m}(x,t)\Delta\zeta_{\varepsilon,m} + B_{\varepsilon,m}(x,t)\mathbf{V}(x,t)\cdot\nabla\zeta_{\varepsilon,m} = f, \text{ in } Q_T \\ \zeta_{\varepsilon,m}(x;T) = \Theta(x), \text{ in } \overline{\Omega} \\ \nabla\zeta_{\varepsilon,m}\cdot\mathbf{n} = 0, \text{ on } S_T \end{array} \right\}$$

with $A_{\varepsilon,m}$, $B_{\varepsilon,m}$, $f \in C^\infty(\overline{Q}_T)$, $A_{\varepsilon,m} \to A_\varepsilon$, $B_{\varepsilon,m} \to B_\varepsilon$ uniformly on $\overline{Q}_T$ as m goes to infinity and $\Theta \in C_0^\infty(\overline{\Omega})$, $0 \le \Theta(x) \le 1$.

Also for $A_{\varepsilon,m}$ and $B_{\varepsilon,m}$ hold

$$\varepsilon \le A_{\varepsilon,m}(x,t) \le \alpha,\ \forall\,(x,t) \in \overline{Q}_T$$

$$|B_{\varepsilon,m}(x,t)| \le L,\ \forall\,(x,t) \in \overline{Q}_T.$$

The existence, uniqueness and regularity of $\zeta_{\varepsilon,m}(x,t)$ as solution of $(P_{\zeta_{\varepsilon,m}})$ follow from the classical theory of linear parabolic equations with smooth coefficients (see [7]).

For the solution $\zeta_{\varepsilon,m}$ the following estimates hold

LEMMA 1. *Let $\zeta_{\varepsilon,m}(x,t)$ be the solution of $(P_{\zeta_{\varepsilon,m}})$, then*

$$\max_{\overline{Q}_T} |\zeta_{\varepsilon,m}(x,t)| \le k_1 = k_1(\|f\|_{\infty,\overline{Q}_T})\,. \tag{13}$$

$$\int_0^T \int_\Omega |\nabla\zeta_{\varepsilon,m}(x,t)|^2 dxdt \le k \tag{14}$$

$$\int_0^T \int_\Omega |\Delta\zeta_{\varepsilon,m}(x,t)|^2 dxdt \le k \tag{15}$$

where $k := k(\varepsilon, L, \|f\|_{2,Q_T})$. Moreover, if $f \le 0$ one has

$$0 \le \zeta_{\varepsilon,m}(x,t)\,, \quad \forall\,(x,t) \in \overline{Q}_T\,. \tag{16}$$

Proof. Inequalities (13) and (16) are a straightforward consequence of the maximum principle. To prove (14), multiply the equation in $(P_{\zeta_{\varepsilon,m}})$ by $\Delta\zeta_{\varepsilon,m}$ and integrate by parts over $\overline{\Omega} \times [\tau, T]$. This yields

$$-\int_\tau^T \int_\Omega \nabla\zeta_{\varepsilon,m}(x,t)\nabla\zeta_{\varepsilon,mt}(x,t)dxdt + \int_\tau^T \int_\Omega A_{\varepsilon,m}(x,t)|\Delta\zeta_{\varepsilon,m}(x,t)|^2 dxdt +$$

$$\int_\tau^T \int_\Omega B_{\varepsilon,m}(x,t)\Delta\zeta_{\varepsilon,m}(x,t)\mathbf{V}(x,t)\cdot\nabla\zeta_{\varepsilon,m}(x,t)dxdt = \int_\tau^T \int_\Omega f(x,t)\Delta\zeta_{\varepsilon,m}(x,t)dxdt\,.$$

Applying Young's inequality, we get

$$(17)\qquad -\frac{1}{2}\int_\Omega |\nabla\Theta(x)|^2dx + \frac{1}{2}\int_\Omega |\nabla\zeta_{\varepsilon,m}(x,\tau)|^2dx +$$

$$\frac{\varepsilon}{4}\int_\tau^T\int_\Omega |\nabla\zeta_{\varepsilon,m}(x,t)|^2dxdt \le$$

$$\frac{C_3L^2}{\varepsilon}\int_\tau^T\int_\Omega |\nabla\zeta_{\varepsilon,m}(x,\tau)|^2dxdt + \frac{1}{2\varepsilon}\int_\tau^T\int_\Omega |f(x,t)|^2dxdt\ ,$$

($C_3 = \sup\{\|\mathbf{V}(x,t)\|^2_{R^n},\ (x,t)\in\overline{Q}_T\}$) from which

$$\frac{1}{2}\int_\Omega |\nabla\zeta_{\varepsilon,m}(x,\tau)|^2dx \le \frac{1}{2}\int_\Omega |\nabla\Theta(x)|^2dx +$$

$$\frac{1}{2\varepsilon}\int_\tau^T\int_\Omega |f(x,t)|^2dxdt + \frac{C_3L^2}{\varepsilon}\int_\tau^T\int_\Omega |\nabla\zeta_{\varepsilon,m}(x,\tau)|^2dxdt\ .$$

Gronwall's inequality and integration with respect to τ gives (14) and therewith by substitution in (17)), the (15).

□

PROPOSITION 4. *For any $f\in C^\infty(\overline{Q}_T)$ and any $\Theta\in C_0^\infty(\overline{\Omega})$, $0\le\Theta(x)\le 1$ we have*

$$(18)\quad \int_\Omega (u_\varepsilon(x,T)-v_\varepsilon(x,T))\Theta(x)dx - \int_0^T\int_\Omega (u_\varepsilon(x,t)-v_\varepsilon(x,t))f(x,t)dxdt \le$$

$$k_1\left(\int_\Omega |u_\varepsilon(x,0)-v_\varepsilon(x,0)|dx + \int_0^T\int_{\partial\Omega} |g_\varepsilon(x,t)-g^*_\varepsilon(x,t)|dSdt\right)\ .$$

If $f\le 0$, then

$$(19)\quad \int_\Omega (u_\varepsilon(x,T)-v_\varepsilon(x,T))\Theta(x)dx - \int_0^T\int_\Omega (u_\varepsilon(x,t)-v_\varepsilon(x,t))f(x,t)dxdt \le$$

$$k_1\left(\int_\Omega (u_\varepsilon(x,0)-v_\varepsilon(x,0))^+dx + \int_0^T\int_{\partial\Omega} (g_\varepsilon(x,t)-g^*_\varepsilon(x,t))^+dSdt\right)\ .$$

Proof. Substituting $\zeta_{\varepsilon,m}$ in (12) we obtain

$$(20)\qquad \int_0^T\int_\Omega (u_\varepsilon-v_\varepsilon)[f(x,t)+(A_\varepsilon(x,t)-A_{\varepsilon,m}(x,t))\Delta\zeta_{\varepsilon,m}(x,t)+$$

$$(B_\varepsilon(x,t)-B_{\varepsilon,m}(x,t))\mathbf{V}(x,t)\cdot\nabla\zeta_{\varepsilon,m}]dxdt =$$

$$\int_\Omega (u_\varepsilon(x,T)-v_\varepsilon(x,T))\Theta(x)dx - \int_\Omega (u_\varepsilon(x,0)-v_\varepsilon(x,0))\zeta_{\varepsilon,m}(x,0)dx +$$

$$\int_0^T \int_{\partial\Omega} (g_\varepsilon^*(x,t) - g_\varepsilon(x,t))\zeta_{\varepsilon,m}(x,t)dSdt\ .$$

By Lemma 1, one concludes that

$$\int_\Omega (u_\varepsilon(x,T) - v_\varepsilon(x,T))\Theta(x)dx - \tag{21}$$

$$\int_0^T \int_\Omega (u_\varepsilon(x,t) - v_\varepsilon(x,t)) f(x,t)dxdt \le k_1 \Bigg(\int_\omega |u_\varepsilon(x,0) - v_\varepsilon(x,0)|dx +$$

$$\int_0^T \int_{\partial\Omega} |g_\varepsilon(x,t) - g_\varepsilon^*(x,t)|dSdt \Bigg) +$$

$$\left(\max_{\overline{Q}_T} |u_\varepsilon(x,t) - v_\varepsilon(x,t)| \right) \Bigg\{ \max_{\overline{Q}_T} |A_\varepsilon(x,t) - A_{\varepsilon,m}(x,t)|(kT|\Omega|)^{1/2} +$$

$$\max_{\overline{Q}_T} |B_\varepsilon(x,t) - B_{\varepsilon,m}(x,t)|\sqrt{k} \left(\int_0^T \int_\Omega \|\mathbf{V}(x,t)\|_{R^n}^2 dxdt \right)^{1/2} \Bigg\}\ .$$

Passing to the limit in (19) as $m \to \infty$, one obtains the desired result.

$\square$

COROLLARY 1. *Let u and v be any periodic solutions to (P) with boundary data g, g^*, respectively. Then one has*

$$\int_\Omega (u(x,T) - v(x,T))^+ dx \le k_1 \Bigg(\int_0^T \int_{\partial\Omega} (g(x,t) - g^*(x,t))^+ dSdt + \tag{22}$$

$$\int_\Omega (u(x,0) - v(x,0))^+ dx \Bigg)$$

and the continuous dependence result

$$\int_0^T \int_\Omega |u(x,t) - v(x,t)|^2 dxdt \le k_1 \Bigg(\int_\Omega |u(x,0) - v(x,0)|dx + \tag{23}$$

$$\int_0^T \int_{\partial\Omega} |g(x,t) - g^*(x,t)|dSdt \Bigg)\ .$$

Proof. Choosing $f(x,t) \equiv 0$ and $\Theta = \Theta_j \in C_0^\infty(\overline{\Omega})$, with $\Theta_j \to sgn_+(u_\varepsilon(x,T) - v_\varepsilon(x,T))$ in $L^1(\Omega)$ as $j \to \infty$, inequality (17) gives

$$\int_\Omega (u_\varepsilon(x,T) - v_\varepsilon(x,T))^+ dx \le k_1 \Bigg(\int_\Omega (u_\varepsilon(x,0) - v_\varepsilon(x,0))^+ dx + \tag{24}$$

$$\int_0^T \int_{\partial\Omega} (g_\varepsilon(x,t) - g_\varepsilon^*(x,t))^+ dSdt \Bigg)\ .$$

Taking the limit as $\varepsilon \to 0^+$ in (24), we have (22). The (23) is deduced by $\Theta \equiv 0$, $f = f_j \in C^\infty(\overline{Q}_T)$, $f_j \to -(u_\varepsilon - v_\varepsilon)$ in $L^2(Q_T)$ as $j \to \infty$ and letting $\varepsilon \to 0^+$.

$\square$

Our main result of this section is

THEOREM 2. *Problem (P) has a unique positive ω–periodic weak solution.*

Proof. Choosing $T = n\omega$ in (23), the periodicity of u and v gives us

$$n\int_0^\omega \int_\Omega |u(x,t) - v(x,t)|^2 dxdt \le k_1\Bigg(\int_\Omega |u(x,0) - v(x,0)|dx \le k_2$$

for any $n \in N$, hence $u = v$.

□

Finally, we can to show a comparison result for the positive ω–periodic weak solutions

COROLLARY 2. *Let u and v be positive ω–periodic weak solutions to (P) with boundary data g, g^*. If $g \le g^*$, then $u \le v$ in Q_ω.*

Proof. Under the above assumptions, formula (17), with $\Theta \equiv 0$, $f_j \in L^\infty(Q_T)$, $f_j \to sgn_+(u(x,t) - v(x,t))$ in $L^\infty(Q_T)$, says in the limit $\varepsilon \to 0^+$

$$\int_0^T \int_\Omega (u(x,t) - v(x,t))^+ dxdt \le k_1\Bigg(\int_\Omega (u(x,0) - v(x,0))^+ dx +$$

$$\int_0^T \int_{\partial\Omega} (g(x,t) - g^*(x,t))^+ dSdt\Bigg) .$$

Taking $T = n\omega$, one has

$$n\int_0^\omega \int_\Omega (u(x,t) - v(x,t))^+ dxdt \le 2k_1 N|\Omega| ,$$

where $N \ge \max\{\|u\|_{\infty,Q_T}, \|v\|_{\infty,Q_T}\}$, which implies $u(x,t) \le v(x,t)$ in Q_ω.

□

Acknowledgment. The author is very grateful to the referee.

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MSC Classification: 35K65, 35D05, 35B10, 76S05.

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Lavoro pervenuto in redazione il 20.01.2003 e, in forma definitiva, il 26.05.2003.

Rend. Sem. Mat. Univ. Pol. Torino
Vol. 61, 1 (2003)

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ON THE SETS OF SEQUENCES THAT ARE STRONGLY $\alpha-$BOUNDED AND $\alpha-$CONVERGENT TO NAUGHT WITH INDEX p.

Abstract. In this paper we deal with sets of sequences generalizing the well known spaces $w_{\infty}^{p}(\lambda) = \left\{X / C(\lambda)\left(|X|^{p}\right) \in l_{\infty}\right\}$ and $c_{\infty}(\lambda) = (w_{\infty}(\lambda))_{\Delta(\lambda)}$. We consider the set $\left(w_{\alpha}^{p}(\lambda)\right)_{\Delta(\mu)}$ and the cases when the operators $C(\lambda)$ and $\Delta(\mu)$ are replaced by their transposes. These results generalize in a certain sense those given in [4, 10, 11, 13, 14, 16].

1. Notations and preliminary results.

For a given infinite matrix $A = (a_{nm})_{n,m\geq 1}$ we define the operators A_n for any integer $n \geq 1$, by

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm} x_m$$

where $X = (x_m)_{m\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the infinite linear system

$$A_n(X) = b_n \quad n = 1, 2, ... \tag{1}$$

where $B = (b_n)_{n\geq 1}$ is a one-column matrix and X the unknown, see [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Equation (1) can be written in the form $AX = B$, where $AX = (A_n(X))_{n\geq 1}$. In this paper we shall also consider A as an operator from a sequence space into another sequence space.

A Banach space E of complex sequences with the norm $\|\|_E$ is a BK space if each projection $P_n : X \to P_n X = x_n$ is continuous. A BK space E is said to have AK, (see [17]), if for every $B = (b_m)_{m\geq 1} \in E$, $B = \sum_{m=1}^{\infty} b_m e_m$, where $e_m = (0, \ldots, 1, 0, \ldots)$, 1 being in the m-th position, i.e.

$$\left\| \sum_{m=N+1}^{\infty} b_m e_m \right\|_E \to 0 \quad (n \to \infty).$$

s, c_0, c, l_∞ are the sets of all sequences, the set of sequences that converge to zero, that are convergent and that are bounded respectively. cs and l_1 are the sets of convergent and absolutely convergent series respectively. We shall use the set

$$U^{+*} = \left\{(u_n)_{n\geq 1} \in s \;/\quad u_n > 0 \;\forall n\right\}.$$

Using Wilansky's notations [17], we define for any sequence $\alpha = (\alpha_n)_{n\geq 1} \in U^{+*}$ and for any set of sequences E, the set

$$\alpha * E = \left\{(x_n)_{n\geq 1} \in s \;/\quad \left(\frac{x_n}{\alpha_n}\right)_n \in E\right\}.$$

Writing

$$\alpha * E = \begin{cases} s_\alpha \text{ if } E = l_\infty, \\ \overset{\circ}{s}_\alpha \text{ if } E = c_0, \\ s_\alpha^{\bullet} \text{ if } E = c, \end{cases}$$

we have for instance

$$\alpha * c_0 = \overset{\circ}{s}_\alpha = \left\{(x_n)_{n\geq 1} \in s \;/\quad x_n = o(\alpha_n) \quad n \to \infty\right\}.$$

Each of the spaces $\alpha * E$, where $E \in \{l_\infty, c_0, c\}$, is a BK space normed by

$$\|X\|_{s_\alpha} = \sup_{n\geq 1}\left(\frac{|x_n|}{\alpha_n}\right), \tag{2}$$

and $\overset{\circ}{s}_\alpha$ has AK, see [10].

Now let $\alpha = (\alpha_n)_{n\geq 1}$ and $\beta = (\beta_n)_{n\geq 1} \in U^{+*}$. By $S_{\alpha,\beta}$ we denote the set of infinite matrices $A = (a_{nm})_{n,m\geq 1}$ such that

$$(a_{nm}\alpha_m)_{m\geq 1} \in l_1 \text{ for all } n \geq 1 \text{ and } \sum_{m=1}^{\infty} |a_{nm}|\,\alpha_m = O(\beta_n) \;\; (n \to \infty).$$

$S_{\alpha,\beta}$ is a Banach space with the norm

$$\|A\|_{S_{\alpha,\beta}} = \sup_{\nu\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}|\,\frac{\alpha_m}{\beta_n}\right).$$

Let E and F be any subsets of s. When A maps E into F we shall write $A \in (E, F)$, see [2]. So for every $X \in E$, $AX \in F$, ($AX \in F$ will mean that for each $n \geq 1$ the series defined by $y_n = \sum_{m=1}^{\infty} a_{nm}x_m$ is convergent and $(y_n)_{n\geq 1} \in F$). It has been proved in [13] that $A \in \left(s_\alpha, s_\beta\right)$ iff $A \in S_{\alpha,\beta}$. So we can write that $\left(s_\alpha, s_\beta\right) = S_{\alpha,\beta}$.

When $s_\alpha = s_\beta$ we obtain the Banach algebra with identity $S_{\alpha,\beta} = S_\alpha$, (see [1, 4, 5]) normed by $\|A\|_{S_\alpha} = \|A\|_{S_{\alpha,\alpha}}$.

We also have $A \in (s_\alpha, s_\alpha)$ if and only if $A \in S_\alpha$. If $\|I - A\|_{S_\alpha} < 1$, we shall say that $A \in \Gamma_\alpha$. Since S_α is a Banach algebra with identity, we have the useful result: if $A \in \Gamma_\alpha$, A is bijective from s_α into itself.

If $\alpha = (r^n)_{n\geq 1}$, Γ_α, S_α, s_α, $\overset{\circ}{s}_\alpha$ and $s_\alpha^\bullet$ are replaced by Γ_r, S_r, s_r, $\overset{\circ}{s}_r$ and $s_r^\bullet$ respectively (see [1, 4, 5, 6, 7, 8]). When $r = 1$, we obtain $s_1 = l_\infty$, $\overset{\circ}{s}_1 = c_0$ and $s_1^\bullet = c$, and putting $e = (1, 1, ...)$ we have $S_1 = S_e$. It is well known, see [2] that

$$(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1.$$

For any subset E of s, we put

$$AE = \{Y \in s \;/\; \exists X \in E \quad Y = AX\}.$$

If F is a subset of s, we shall denote

$$F(A) = F_A = \{X \in s \;/\; Y = AX \in F\}.$$

We can see that $F(A) = A^{-1}F$.

2. Some properties of the operators $\Delta(\lambda)$, $\Delta^+(\lambda)$ and Σ^+ relative to the sets s_α, $\overset{\circ}{s}_\alpha$ and $s_\alpha^\bullet$.

Here we shall deal with the operators represented by $C(\lambda)$, $C^+(\lambda)$, $\Delta(\lambda)$ and $\Delta^+(\lambda)$.

Let $U = \left\{(u_n)_{n\geq 1} \in s \;/\; u_n \neq 0 \;\forall n\right\}$. We define $C(\lambda) = (c_{nm})_{n,m\geq 1}$ for $\lambda = (\lambda_n)_{n\geq 1} \in U$, by

$$c_{nm} = \begin{cases} \dfrac{1}{\lambda_n} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

So, we put $C^+(\lambda) = C(\lambda)^t$. It can be proved that the matrix $\Delta(\lambda) = \left(c'_{nm}\right)_{n,m\geq 1}$ with

$$c'_{nm} = \begin{cases} \lambda_n & \text{if } m = n, \\ -\lambda_{n-1} & \text{if } m = n-1 \text{ and } n \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

is the inverse of $C(\lambda)$, see [13]. Similarly we put $\Delta^+(\lambda) = \Delta(\lambda)^t$. If $\lambda = e$ we get the well known operator of first difference represented by $\Delta(e) = \Delta$ and it is usually written $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$ and Δ and Σ belong to any given space S_R with $R > 1$. Writing $D_\lambda = (\lambda_n \delta_{nm})_{n,m\geq 1}$, (where $\delta_{nm} = 0$ for $n \neq m$ and $\delta_{nn} = 1$ otherwise), we have $\Delta^+(\lambda) = D_\lambda \Delta^+$. So for any given $\alpha \in U^{+*}$, we see that if $\frac{\alpha_{n-1}}{\alpha_n}\left|\frac{\lambda_n}{\lambda_{n-1}}\right| = O(1)$, then $\Delta^+(\lambda) = D_\lambda \Delta^+ \in \left(s_{\left(\frac{\alpha}{|\lambda|}\right)}, s_\alpha\right)$. Since $Ker\Delta^+(\lambda) \neq 0$, we are led to define the set

$$s_\alpha^*\left(\Delta^+(\lambda)\right) = s_\alpha\left(\Delta^+(\lambda)\right) \bigcap s_{\left(\frac{\alpha}{|\lambda|}\right)} = \left\{X = (x_n)_{n\geq 1} \in s_{\left(\frac{\alpha}{|\lambda|}\right)} \;/\; \Delta^+(\lambda)X \in s_\alpha\right\}.$$

It can be easily seen that

$$s^*_{\left(\frac{\alpha}{|\lambda|}\right)}\left(\Delta^+(e)\right) = s^*_{\left(\frac{\alpha}{|\lambda|}\right)}\left(\Delta^+\right) = s_\alpha^*\left(\Delta^+(\lambda)\right).$$

We obtain similar results with the set $\overset{\circ}{s}{}_\alpha^*\left(\Delta^+(\lambda)\right) = \overset{\circ}{s}_\alpha\left(\Delta^+(\lambda)\right) \bigcap \overset{\circ}{s}_{\left(\frac{\alpha}{|\lambda|}\right)}$.

2.1. Properties of the sequence $C(\alpha)\alpha$.

We shall use the following sets

$$\begin{aligned}
\widehat{C_1} &= \left\{\alpha \in U^{+*} / \quad \frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right) = O(1) \quad (n \to \infty)\right\}, \\
\widehat{C} &= \left\{\alpha \in U^{+*} / \quad \left(\frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right)\right)_{n\geq 1} \in c\right\}, \\
\widehat{C_1^+} &= \left\{\alpha \in U^{+*} \bigcap cs \ / \quad \frac{1}{\alpha_n}\left(\sum_{k=n}^{\infty}\alpha_k\right) = O(1) \quad (n \to \infty)\right\}, \\
\Gamma &= \left\{\alpha \in U^{+*} \ / \ \overline{\lim}_{n\to\infty}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1\right\}
\end{aligned}$$

and

$$\Gamma^+ = \left\{\alpha \in U^{+*} \ / \ \overline{\lim}_{n\to\infty}\left(\frac{\alpha_{n+1}}{\alpha_n}\right) < 1\right\}.$$

Note that $\alpha \in \Gamma^+$ if and only if $\frac{1}{\alpha} \in \Gamma$. We shall see in Proposition 1 that if $\alpha \in \widehat{C_1}$, α tends to infinity. On the other hand we see that $\Delta \in \Gamma_\alpha$ implies $\alpha \in \Gamma$. We also have $\alpha \in \Gamma$ if and only if there is an integer $q \geq 1$ such that

$$\gamma_q(\alpha) = \sup_{n\geq q+1}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1.$$

We obtain the following results in which we put $[C(\alpha)\alpha]_n = \dfrac{1}{\alpha_n}\left(\sum\limits_{k=1}^{n}\alpha_k\right)$.

PROPOSITION 1. *Let $\alpha \in U^{+*}$. Then*

i) $\dfrac{\alpha_{n-1}}{\alpha_n} \to 0$ *if and only if* $[C(\alpha)\alpha]_n \to 1$.

ii) $[C(\alpha)\alpha]_n \to l$ *implies that* $\dfrac{\alpha_{n-1}}{\alpha_n} \to 1 - \dfrac{1}{l}$.

iii) If $\alpha \in \widehat{C_1}$ *then there are* $K > 0$ *and* $\gamma > 1$ *such that* $\alpha_n \geq K\gamma^n$ *for all* n.

iv) $\alpha \in \Gamma$ *implies that* $\alpha \in \widehat{C_1}$ *and there exist a real* $b > 0$ *and an integer* q*, such that*

$$[C(\alpha)\alpha]_n \leq \frac{1}{1-\chi} + b\chi^n \quad \textit{for } n \geq q+1 \textit{ and } \chi = \gamma_q(\alpha) \in]0,1[.$$

v) $\alpha \in \Gamma^+$ *implies* $\alpha \in \widehat{C_1^+}$.

Proof. i), ii), iii) and iv) have been proved in [10].

Assertion v). If $\alpha \in \Gamma^{+}$, there are $\chi' \in]0,1[$ and an integer $q' \geq 1$ such that

$$\frac{\alpha_k}{\alpha_{k-1}} \leq \chi' \quad \text{for } k \geq q'.$$

Then we have for every $n \geq q'$

$$\frac{1}{\alpha_n}\left(\sum_{k=n}^{\infty}\alpha_k\right) = \sum_{k=n}^{\infty}\left(\frac{\alpha_k}{\alpha_n}\right) \leq 1 + \sum_{k=n+1}^{\infty}\left[\prod_{i=0}^{k-n-1}\left(\frac{\alpha_{k-i}}{\alpha_{k-i-1}}\right)\right] \leq \sum_{k=n}^{\infty}\chi'^{k-n} = O(1).$$

This gives the conclusion.

□

REMARK 1. Note that as a direct consequence of Proposition 2.1, we have

$$\widehat{C} \subset \Gamma \subset \widehat{C_1}.$$

We also have $\widehat{C} \neq \Gamma$, see [4]. On the other hand we see that $\widehat{C_1} \bigcap \widehat{C_1^{+}} = \Gamma \bigcap \Gamma^{+} = \phi$.

2.2. The spaces $w_{\alpha}^{p}(\lambda)$, $\overset{\circ}{w}{}_{\alpha}^{p}(\lambda)$ and $w_{\alpha}^{\bullet p}(\lambda)$ for $p > 0$.

In this subsection we recall some results on the sets that generalize the sets $w_{\infty}^{p}(\lambda)$, $w_{0}^{p}(\lambda)$ and $w^{p}(\lambda)$ for given real $p > 0$.

For any given real $p > 0$ and every sequence $X = (x_n)_{n\geq 1}$, we put $|X|^p = (|x_n^p|)_n$ and

$$\begin{aligned}
w_{\alpha}^{p}(\lambda) &= \left\{X \in s \ / \ C(\lambda)\left(|X|^p\right) \in s_{\alpha}\right\},\\
\overset{\circ}{w}{}_{\alpha}^{p}(\lambda) &= \left\{X \in s / \ C(\lambda)\left(|X|^p\right) \in \overset{\circ}{s}_{\alpha}\right\},\\
w_{\alpha}^{\bullet p}(\lambda) &= \left\{X \in s \ / \ X - le^t \in \overset{\circ}{w}{}_{\alpha}^{p}(\lambda) \quad \text{for some } l \in C\right\}.
\end{aligned}$$

For instance we see that

$$w_{\alpha}^{p}(\lambda) = \left\{X = (x_n)_n \in s \ / \ \sup_{n\geq 1}\left(\frac{1}{|\lambda_n|\,\alpha_n}\sum_{k=1}^{n}|x_k|^p\right) < \infty\right\}.$$

If there exist A and $B > 0$, such that $A < \alpha_n < B$ for all n, we get the well known spaces $w_{\alpha}^{p}(\lambda) = w_{\infty}^{p}(\lambda)$, $\overset{\circ}{w}{}_{\alpha}^{p}(\lambda) = w_{0}^{p}(\lambda)$ and $w_{\alpha}^{\bullet p}(\lambda) = w^{p}(\lambda)$, see [14, 15]. In the case when $\lambda = (n)_{n\geq 1}$, the previous sets have been introduced in [3] by Maddox and it is written $w_{\infty}^{p}(\lambda) = w_{\infty}^{p}$, $w_{0}^{p}(\lambda) = w_{0}^{p}$ and $w^{p}(\lambda) = w^{p}$. It is proved that each of the sets w_{0}^{p} and w_{∞}^{p} is a $p-$normed FK space for $0 < p < 1$, (that is a complete linear metric space in which each projection P_n is continuous), and a BK space for

$1 \leq p < \infty$ with respect to the norm

$$\|X\| = \begin{cases} \sup_{\nu \geq 1} \left(\frac{1}{2^{\nu}} \left(\sum_{n=2^{\nu}}^{2^{\nu+1}-1} |x_n|^p \right) \right) & \text{if } 0 < p < 1, \\ \sup_{\nu \geq 1} \left(\frac{1}{2^{\nu}} \left(\sum_{n=2^{\nu}}^{2^{\nu+1}-1} |x_n|^p \right) \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty. \end{cases}$$

w_0^p has the property AK, and every sequence $X = (x_n)_{n \geq 1} \in w^p\left((n)_n\right)$ has a unique representation

$$X = le^t + \sum_{n=1}^{\infty} (x_n - l)\, e_n^t, \text{where } l \in C \text{ is such that } X - le^t \in w_0^p,$$

When $p = 1$, we omit the index p and write $w_\alpha^p(\lambda) = w_\alpha(\lambda)$, $\overset{\circ}{w}_\alpha^p(\lambda) = \overset{\circ}{w}_\alpha(\lambda)$ and $w_\alpha^{\bullet p}(\lambda) = w_\alpha(\lambda)$. It has been proved in [14], that if λ is a strictly increasing sequence of reals tending to infinity then $w_0(\lambda)$ and $w_\infty(\lambda)$ are BK spaces and $w_0(\lambda)$ has AK, with respect to the norm

$$\|X\| = \|C(\lambda)(|X|)\|_{l^\infty} = \sup_n \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} |x_k| \right).$$

Recall the next results given in [10].

THEOREM 1. *Let α and λ be any sequences of U^{+*}.*

i) Consider the following properties

a) $\frac{\alpha_{n-1}\lambda_{n-1}}{\alpha_n \lambda_n} \to 0$;
b) $s_\alpha^\bullet(C(\lambda)) = s_{\alpha\lambda}^\bullet$.
c) $\alpha\lambda \in \widehat{C_1}$;
d) $w_\alpha(\lambda) = s_{\alpha\lambda}$;
e) $\overset{\circ}{w}_\alpha(\lambda) = \overset{\circ}{s}_{\alpha\lambda}$;
f) $w_\alpha^\bullet(\lambda) = \overset{\circ}{s}_{\alpha\lambda}$.

We have a)⇒b), c)⇔d) and c)⇒e) and f).

ii) If $\alpha\lambda \in \widehat{C_1}$, $w_\alpha(\lambda)$, $\overset{\circ}{w}_\alpha(\lambda)$ and $w_\alpha^\bullet(\lambda)$ are BK spaces with respect to the norm

$$\|X\|_{s_{\alpha\lambda}} = \sup_{n \geq 1} \left(\frac{|x_n|}{\alpha_n \lambda_n} \right),$$

and $\overset{\circ}{w}_\alpha(\lambda) = w_\alpha^\bullet(\lambda)$ has AK.

2.3. Properties of some new sets of sequences.

In this subsection we shall characterize the sets $E(\Delta(\mu))$, $E(\Delta^+(\mu))$ for $E \in \{s_\alpha, s_\alpha^\circ, s_\alpha^\bullet\}$, and the sets $w_\alpha^p(\lambda)$, $w_\alpha^{+p}(\lambda)$ and $w_\alpha^{\circ +p}(\lambda)$.

In order to state some new results we need the following lemmas. First recall the well known result.

LEMMA 1. *$A \in (c_0, c_0)$ if and only if*

$$\begin{cases} A \in S_1, \\ \lim_n a_{nm} = 0 \quad \text{for each } m \geq 1. \end{cases}$$

The next result has been shown in [11].

LEMMA 2. *If Δ^+ is bijective from s_α into itself, then $\alpha \in cs$.*

We also need to state the following elementary result.

LEMMA 3. *We have*

$$\Sigma^+(\Delta^+ X) = X \quad \forall X \in c_0 \quad \text{and} \quad \Delta^+(\Sigma^+ X) = X \quad \forall X \in cs.$$

Put now

$$\begin{aligned} w_\alpha^{+p}(\lambda) &= \left\{ X \in s \;/\; C^+(\lambda)\left(|X|^p\right) \in s_\alpha \right\}, \\ w_\alpha^{\circ +p}(\lambda) &= \left\{ X \in s \;/\; C^+(\lambda)\left(|X|^p\right) \in s_\alpha^\circ \right\}, \end{aligned}$$

see [11]. Letting $\beta^- = (\beta_{n-1})_{n \geq 1}$, with $\beta_0 = 1$, for any $\beta = (\beta_n)_{n \geq 1} \in U^{+*}$, we can state the following results.

THEOREM 2. *Let $\alpha \in U^{+*}$, $\lambda, \mu \in U$ and $p > 0$. We successively have*

i) a) $s_\alpha(\Delta(\mu)) = s_{\left(\frac{\alpha}{|\mu|}\right)}$ if and only if $\alpha \in \widehat{C_1}$;

b) $s_\alpha^\circ(\Delta(\mu)) = s^\circ_{\left(\frac{\alpha}{|\mu|}\right)}$ if and only if $\alpha \in \widehat{C_1}$;

c) $s_\alpha^\bullet(\Delta(\mu)) = s^\bullet_{\left(\frac{\alpha}{|\mu|}\right)}$ if and only if $\alpha \in \widehat{C}$.

ii) a) $s_\alpha(\Delta^+(\mu)) = s_{\left(\frac{\alpha}{|\mu|}\right)^-}$ if and only if $\dfrac{\alpha}{|\mu|} \in \widehat{C_1}$;

b) $\dfrac{\alpha_{n-1}}{\alpha_n}\dfrac{\mu_n}{\mu_{n-1}} = o(1)$ implies $s_\alpha^\circ(\Delta^+(\mu)) = s^\circ_{\left(\frac{\alpha}{|\mu|}\right)^-}$;

c) $\dfrac{\alpha}{|\mu|} \in \widehat{C_1^+}$ if and only if $s_\alpha^*(\Delta^+(\mu)) = s_{\left(\frac{\alpha}{|\mu|}\right)}$;

d) $\dfrac{\alpha}{|\mu|} \in \widehat{C_1^+}$ if and only if $s_\alpha^{\circ *}(\Delta^+(\mu)) = s^\circ_{\left(\frac{\alpha}{|\mu|}\right)}$.

iii) a) $s_\alpha\left(\Sigma^+\right)=s_\alpha$ *if and only if* $\alpha\in\widehat{C_1^+}$ *and* $s_\alpha^\circ\left(\Sigma^+\right)=s_\alpha^\circ$ *if and only if* $\alpha\in\widehat{C_1^+}$.

b) $\alpha\in\widehat{C_1^+}$ *if and only if* $w_\alpha^{+p}(\lambda)=s_{(\alpha|\lambda|)^{\frac{1}{p}}}$,

c) if $\alpha\in\widehat{C_1^+}$, *then* $w_\alpha^{\circ+p}(\lambda)=s^\circ_{(\alpha|\lambda|)^{\frac{1}{p}}}$,

d) $\alpha|\lambda|\in\widehat{C_1}$ *if and only if* $w_\alpha^p(\lambda)=s_{(\alpha|\lambda|)^{\frac{1}{p}}}$.

e) If $\alpha|\lambda|\in\widehat{C_1}$, *then* $w_\alpha^{\circ p}(\lambda)=s^\circ_{(\alpha|\lambda|)^{\frac{1}{p}}}$.

Proof. Assertion i) has been proved in [10]. Throughout the proof of part ii) we shall put $\beta=\frac{\alpha}{|\mu|}$.

Assertion ii) a). First we have $s_\alpha\left(\Delta^+(\mu)\right)=s_\beta\left(\Delta^+\right)$. Indeed,

$$X\in s_\alpha\left(\Delta^+(\mu)\right)\Leftrightarrow D_\mu\Delta^+X\in s_\alpha\Leftrightarrow\Delta^+X\in s_\beta\Leftrightarrow X\in s_\beta\left(\Delta^+\right).$$

To get a), it is enough to show that $\beta\in\widehat{C_1}$ if and only if $s_\beta\left(\Delta^+\right)=s_{\beta^-}$. We assume that $\beta\in\widehat{C_1}$. From the inequality

$$\frac{\beta_{n-1}}{\beta_n}\le\frac{1}{\beta_n}\left(\sum_{k=1}^n\beta_k\right)=O(1),$$

we deduce that $\dfrac{\beta_{n-1}}{\beta_n}=O(1)$ and $\Delta^+\in\left(s_{\beta^-},s_\beta\right)$. Then for any given $B\in s_\beta$ the solutions of the equation $\Delta^+X=B$ are given by $x_1=-u$ and

$$-x_n=u+\sum_{k=1}^{n-1}b_k,\text{ for } n\ge 2, \tag{3}$$

where u is an arbitrary scalar. So there exists a real $K>0$, such that

$$\frac{|x_n|}{\beta_{n-1}}=\frac{\left|u+\sum_{k=1}^{n-1}b_k\right|}{\beta_{n-1}}\le\frac{|u|+K\left(\sum_{k=1}^{n-1}\beta_k\right)}{\beta_{n-1}}=O(1),$$

since iii) in Proposition 1 implies $\frac{|u|}{\beta_{n-1}}=O(1)$. So $X\in s_\alpha$ and we conclude that Δ^+ is surjective from s_{β^-} into s_β. Then $\beta=\frac{\alpha}{|\mu|}\in\widehat{C_1}$ implies

$$s_\alpha\left(\Delta^+(\mu)\right)=s_{\left(\frac{\alpha}{|\mu|}\right)^-}.$$

Conversely, assume that $s_\alpha\left(\Delta^+(\mu)\right)=s_{\left(\frac{\alpha}{|\mu|}\right)^-}$. If we take $B=\beta$, we get $x_n=x_1-\sum_{k=1}^{n-1}\beta_k$, where x_1 is an arbitrary scalar and

$$\frac{x_n}{\beta_{n-1}}=\frac{x_1}{\beta_{n-1}}-\frac{1}{\beta_{n-1}}\left(\sum_{k=1}^{n-1}\beta_k\right)=O(1).$$

Putting $x_1 = 0$, we conclude that $\beta \in \widehat{C_1}$.

ii) b) First we have $\Delta^+ \in \left(\overset{\circ}{s}_{\beta^-}, \overset{\circ}{s}_{\beta}\right)$ because $\dfrac{\beta_{n-1}}{\beta_n} = O(1)$. Let us show that Δ^+ is surjective from $\overset{\circ}{s}_{\beta^-}$ into $\overset{\circ}{s}_{\beta}$. For this, let $B = (b_n)_{n\geq 1} \in \overset{\circ}{s}_{\beta}$. The solutions $X = (x_n)_{n\geq 1}$ of the equation $\Delta^+ X = B$ are given by (3). We have

$$\frac{x_n}{\beta_{n-1}} = o(1) - \frac{\sum_{k=1}^{n-1} b_k}{\beta_{n-1}},$$

because from Proposition 2.1, the condition $\dfrac{\beta_{n-1}}{\beta_n} = o(1)$ implies $\beta \in \widehat{C_1}$ and $\beta \to \infty$. Since $B \in \overset{\circ}{s}_{\beta}$ there is a sequence $\nu = (\nu_n)_{n\geq 1} \in c_0$, such that $b_n = \beta_n \nu_n$. Then we have for a real $M > 0$

$$\frac{\left|\sum_{k=1}^{n-1} b_k\right|}{\beta_{n-1}} \leq \frac{1}{\beta_{n-1}}\left(\sum_{k=1}^{n-1} \beta_k \nu_k\right) \quad \text{for all } n \geq 2.$$

It remains to show that $\frac{1}{\beta_{n-1}} \sum_{k=1}^{n-1} \beta_k \nu_k = o(1)$. For this consider any given $\varepsilon > 0$. Since $\beta \to \infty$ there is an integer N such that

$$S_n = \frac{1}{\beta_{n-1}}\left|\sum_{k=1}^{N} \beta_k \nu_k\right| \leq \frac{\varepsilon}{2}$$

for $n > N$, and

$$\sup_{k\geq N+1}(|\nu_k|) \leq \frac{\varepsilon}{2\sup_{n\geq 2}\left([C(\beta)\beta]_{n-1}\right)}.$$

Writing $R_n = \dfrac{1}{\beta_{n-1}}\left|\sum_{k=N+1}^{n-1} \beta_k \nu_k\right|$ for $n > N+2$, we deduce that

$$R_n \leq \left(\sup_{N+1\leq k\leq n-1}(|\nu_k|)\right)[C(\beta)\beta]_{n-1} \leq \frac{\varepsilon}{2}.$$

Finally, we obtain

$$\frac{|x_n|}{\beta_{n-1}} = \left|\frac{1}{\beta_{n-1}}\left(\sum_{k=1}^{N} \beta_k \nu_k\right) + \frac{1}{\beta_{n-1}}\left(\sum_{k=N+1}^{n-1} \beta_k \nu_k\right)\right| \leq S_n + R_n \leq \varepsilon \quad \text{for} \quad n \geq N,$$

and $X \in \overset{\circ}{s}_{\beta^-}$. So we have proved ii) b).

Assertion ii) c). Necessity. Assume that $\beta = \dfrac{\alpha}{|\mu|} \in \widehat{C_1^+}$. Since we have $s_\alpha^*\left(\Delta^+(\mu)\right) = s_\beta^*\left(\Delta^+\right) = s_\beta$, it is enough to show that Δ^+ is bijective from s_β to s_β. We can write that $\Delta^+ \in \left(s_\beta, s_\beta\right)$, since

$$\frac{\beta_{n+1}}{\beta_n} \leq \frac{1}{\beta_n}\left(\sum_{k=n}^{\infty} \beta_k\right) = O(1) \quad (n \to \infty). \tag{4}$$

Further, from $s_\beta \subset cs$, we deduce using Lemma 3 that for any given $B \in s_\beta$, $\Delta^+\left(\Sigma^+ B\right) = B$. On the other hand $\Sigma^+ B = \left(\sum_{k=n}^{\infty} b_k\right)_{n \geq 1} \in s_\beta$, since $\beta \in \widehat{C_1^+}$. So Δ^+ is surjective from s_β into s_β. Finally, Δ^+ is injective because the equation

$$\Delta^+ X = O$$

admits the unique solution $X = O$ in s_β, since $Ker\,\Delta^+ = \left\{ue^t / \quad u \in C\right\}$ and $e^t \notin s_\beta$.

Sufficiency. For every $B \in s_\beta$ the equation $\Delta^+ X = B$ admits a unique solution in s_β. Then from Lemma 2, $\beta \in cs$ and since $s_\beta \subset cs$ we deduce from Lemma 3 that $X = \Sigma^+ B \in s_\beta$ is the unique solution of $\Delta^+ X = B$. Taking $B = \beta$, we get $\Sigma^+ \beta \in s_\alpha$ that is $\beta \in \widehat{C_1^+}$.

As above to prove ii) d) it is enough to verify that $\beta = \frac{\alpha}{|\mu|} \in \widehat{C_1^+}$ if and only if $\overset{\circ}{s}{}_\beta^*\left(\Delta^+\right) = s_\beta$. If $\beta \in \widehat{C_1^+}$, Δ^+ is bijective from $\overset{\circ}{s}_\beta$ into itself. Indeed, we have $D_{\frac{1}{\beta}} \Delta^+ D_\beta \in (c_0, c_0)$ from (4) and Lemma 1. Furthermore, since $\beta \in \widehat{C_1^+}$ we have $\overset{\circ}{s}_\beta \subset cs$ and for every $B \in \overset{\circ}{s}_\beta$, $\Delta^+\left(\Sigma^+ B\right) = B$. From Lemma 1, we have $\Sigma^+ \in \left(\overset{\circ}{s}_\beta, \overset{\circ}{s}_\beta\right)$, so the equation $\Delta^+ X = B$ admits in $\overset{\circ}{s}_\beta$ the solution $X_0 = \Sigma^+ B$ and we have proved that Δ^+ is surjective from $\overset{\circ}{s}_\beta$ into itself. Finally, $\beta \in \widehat{C_1^+}$ implies that $e^t \notin \overset{\circ}{s}_\beta$, so $Ker\,\Delta^+ \bigcap \overset{\circ}{s}_\beta = \{0\}$ and we conclude that Δ^+ is bijective from $\overset{\circ}{s}_\beta$ into itself.

iii) a) comes from ii), since $\alpha \in \widehat{C_1^+}$ if and only if Δ^+ is bijective from s_α into itself and is also bijective from $\overset{\circ}{s}_\alpha$ into itself, and

$$\Sigma^+\left(\Delta^+ X\right) = \Delta^+\left(\Sigma^+ X\right) = X \quad \text{for all } X \in s_\alpha.$$

b) Assume that $\alpha \in \widehat{C_1^+}$. Since $C^+(\lambda) = \Sigma^+ D_{\frac{1}{\lambda}}$, we have

$$w_\alpha^{+p}(\lambda) = \left\{X \in s \;/\; \left(\Sigma^+ D_{\frac{1}{\lambda}}\right)\left(|X|^p\right) \in s_\alpha\right\} = \left\{X \;/\; D_{\frac{1}{\lambda}}\left(|X|^p\right) \in s_\alpha\left(\Sigma^+\right)\right\};$$

and since $\alpha \in \widehat{C_1^+}$ implies $s_\alpha\left(\Sigma^+\right) = s_\alpha$, we conclude that

$$w_\alpha^{+p}(\lambda) = \left\{X \in s \;/\; |X|^p \in D_\lambda s_\alpha = s_{\alpha|\lambda|}\right\} = s_{(\alpha|\lambda|)^{\frac{1}{p}}}.$$

Conversely, we have $(\alpha|\lambda|)^{\frac{1}{p}} \in s_{(\alpha|\lambda|)^{\frac{1}{p}}} = w_{\alpha}^{+p}(\lambda)$. So

$$C^{+}(\lambda)\left[(\alpha|\lambda|)^{\frac{1}{p}}\right]^{p} = \left(\sum_{k=n}^{\infty}\frac{\alpha_k|\lambda_k|}{|\lambda_k|}\right)_{n\geq 1} \in s_{\alpha},$$

i.e. $\alpha \in \widehat{C_1^{+}}$ and we have proved i). We get iii) c) reasoning as above.

iii) d) has been proved in [4]. iii) e) Assume that $\alpha|\lambda| \in \widehat{C_1}$. Then

$$w_{\alpha}^{\circ p}(\lambda) = \left\{X \in s \ / \quad |X|^{p} \in \Delta(\lambda)\, s_{\alpha}^{\circ}\right\}.$$

Since $\Delta(\lambda) = \Delta D_{\lambda}$, we get $\Delta(\lambda)\, s_{\alpha}^{\circ} = \Delta s_{\alpha|\lambda|}^{\circ}$. Now, from i) b) we deduce that $\alpha|\lambda| \in \widehat{C_1}$ implies that Δ is bijective from $s_{\alpha|\lambda|}^{\circ}$ into itself and $w_{\alpha}(\lambda) = s^{\circ}_{(\alpha|\lambda|)^{\frac{1}{p}}}$. We get e) reasoning as above.

$\square$

As a direct consequence of Theorem 2 we obtain the following results given in [11].

COROLLARY 1. *Let $r > 0$ be any real. We get*

$$r > 1 \Leftrightarrow s_r(\Delta) = s_r \Leftrightarrow s_r^{\circ}(\Delta) = s_r^{\circ} \Leftrightarrow s_r\left(\Delta^{+}\right) = s_r.$$

We deduce from the previous section the following.

3. Sets of sequences that are strongly $\alpha-$bounded and $\alpha-$convergent to zero with index p and generalizations.

In this section we deal with sets generalizing the well known sets of sequences that are strongly bounded and convergent to zero.

First we recall some results given in [10].

3.1. Sets $c_{\alpha}(\lambda,\mu)$, $c_{\alpha}^{\circ}(\lambda,\mu)$ and $c_{\alpha}^{\bullet}(\lambda,\mu)$.

If $\alpha = (\alpha_n)_n \in U^{+*}$ is a given sequence, we consider now for $\lambda \in U$, $\mu \in s$ the space

$$c_{\alpha}(\lambda,\mu) = (w_{\alpha}(\lambda))_{\Delta(\mu)} = \{X \in s \ / \ \Delta(\mu)\, X \in w_{\alpha}(\lambda)\}$$

It is easy to see that

$$c_{\alpha}(\lambda,\mu) = \{X \in s \ / \ C(\lambda)\,(|\Delta(\mu)\, X|) \in s_{\alpha}\},$$

that is

$$c_{\alpha}(\lambda,\mu) = \left\{X = (x_n)_n \in s \ / \ \sup_{n\geq 2}\left(\frac{1}{|\lambda_n|\,\alpha_n}\sum_{k=2}^{n}|\mu_k x_k - \mu_{k-1}x_{k-1}|\right) < \infty\right\}.$$

See [10, 11, 13]. Similarly we define the following sets

$$\begin{aligned} \overset{\circ}{c}_\alpha (\lambda, \mu) &= \left\{ X \in s \ / \ C(\lambda) \left(|\Delta(\mu) X| \right) \in \overset{\circ}{s}_\alpha \right\}, \\ c^{\bullet}_\alpha (\lambda, \mu) &= \left\{ X \in s \ / \ X - le^t \in \overset{\circ}{c}_\alpha (\lambda, \mu) \quad \text{for some } l \in C \right\} \end{aligned}$$

Recall that if $\lambda = \mu$ it is written that $c_0(\lambda) = (w_0(\lambda))_{\Delta(\lambda)}$,

$$c(\lambda) = \left\{ X \in s \ / \ X - le^t \in c_0(\lambda) \ \text{ for some } l \in C \right\},$$

and $c_\infty(\lambda) = (w_\infty(\lambda))_{\Delta(\lambda)}$, see [16]. It can be easily seen that

$$c_0(\lambda) = \overset{\circ}{c}_e(\lambda, \lambda), \ c_\infty(\lambda) = c_e(\lambda, \lambda) \ \text{ and } c(\lambda) = c^{\bullet}_e(\lambda, \lambda).$$

These sets are called sets of sequences that are strongly bounded, strongly convergent to 0 and strongly convergent. If $\lambda \in U^{+*}$ is a sequence strictly increasing to infinity, $c(\lambda)$ is a Banach space with respect to

$$\|X\|_{c_\infty(\lambda)} = \sup_{n \geq 1} \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} |\lambda_k x_k - \lambda_{k-1} x_{k-1}| \right)$$

with the convention $x_0 = 0$. Each of the spaces $c_0(\lambda)$, $c(\lambda)$ and $c_\infty(\lambda)$ is a BK space, with respect to the previous norm (see [14]). $c_0(\lambda)$ has AK and every $X \in c(\lambda)$ has a unique representation given by

$$X = le^t + \sum_{k=1}^{\infty} (x_k - l) e_k^t, \tag{5}$$

where $X - le^t \in c_0$. The number l is called the strong $c(\lambda)$-limit of the sequence X.

We obtain the next result given in [10]:

THEOREM 3. *Let α, λ and μ be sequences of U^{+*}.*

i) Consider the following properties

a) $\alpha\lambda \in \widehat{C_1}$;

b) $c_\alpha(\lambda, \mu) = s_{\alpha \frac{\lambda}{\mu}}$;

c) $\overset{\circ}{c}_\alpha(\lambda, \mu) = \overset{\circ}{s}_{\alpha \frac{\lambda}{\mu}}$;

d) $c^{\bullet}_\alpha(\lambda, \mu) = \left\{ X \in s \ / \ X - le^t \in \overset{\circ}{s}_{\alpha \frac{\lambda}{\mu}} \quad \text{for some } l \in C \right\}.$

We have a)⇔b) and a)⇒c) and d).

ii) If $\alpha\lambda \in \widehat{C_1}$, *then* $c_\alpha(\lambda, \mu)$, $\overset{\circ}{c_\alpha}(\lambda, \mu)$ *and* $c_\alpha^\bullet(\lambda, \mu)$ *are BK spaces with respect to the norm*

$$\|X\|_{s_{\alpha\frac{\lambda}{\mu}}} = \sup_{n\geq 1}\left(\mu_n \frac{|x_n|}{\alpha_n \lambda_n}\right).$$

$\overset{\circ}{c_\alpha}(\lambda, \mu)$ *has AK and every* $X \in c_\alpha^\bullet(\lambda, \mu)$ *has a unique representation given by (5), where* $X - le \in \overset{\circ}{s}_{\alpha\frac{\lambda}{\mu}}$.

We immediatly deduce the following:

COROLLARY 2. *Assume that* α, λ *and* $\mu \in U^{+*}$.

i) If $\alpha\lambda \in \widehat{C_1}$ *and* $\mu \in l_\infty$, *then*

$$c_\alpha^\bullet(\lambda, \mu) = \overset{\circ}{s}_{\alpha\frac{\lambda}{\mu}}. \tag{6}$$

ii) $\lambda \in \Gamma \Rightarrow \lambda \in \widehat{C_1} \Rightarrow c_0(\lambda) = \overset{\circ}{s}_\lambda$ *and* $c_\infty(\lambda) = s_\lambda$.

3.2. Generalization

In this subsection we consider spaces generalizing the well known spaces of sequences $c_\infty(\lambda)$ and $c_0(\lambda)$ that are strongly bounded and convergent to naught.

For given real $p > 0$, let us put

$$\begin{aligned}
c_\alpha^p(\lambda, \mu) &= \left(w_\alpha^p(\lambda)\right)_{\Delta(\mu)} = \left\{X \;/\; C(\lambda)\left(|\Delta(\mu) X|^p\right) \in s_\alpha\right\}, \\
c_\alpha^{\cdot+p}(\lambda, \mu) &= \left(w_\alpha^p(\lambda)\right)_{\Delta^+(\mu)} = \left\{X \;/\; C(\lambda)\left(\left|\Delta^+(\mu) X\right|^p\right) \in s_\alpha\right\}, \\
c_\alpha^{+\cdot p}(\lambda, \mu) &= \left(w_\alpha^{+p}(\lambda)\right)_{\Delta(\mu)} = \left\{X \;/\; C^+(\lambda)\left(|\Delta(\mu) X|^p\right) \in s_\alpha\right\}, \\
c_\alpha^{+p}(\lambda, \mu) &= \left(w_\alpha^{+p}(\lambda)\right)_{\Delta^+(\mu)} = \left\{X \;/\; C^+(\lambda)\left(\left|\Delta^+(\mu) X\right|^p\right) \in s_\alpha\right\}.
\end{aligned}$$

When s_α is replaced by $\overset{\circ}{s}_\alpha$ in the previous definitions, we shall write $\widetilde{c_\alpha^p}(\lambda, \mu)$, $\widetilde{c_\alpha^{\cdot+p}}(\lambda, \mu)$, $\widetilde{c_\alpha^{+\cdot p}}(\lambda, \mu)$ and $\widetilde{c_\alpha^{+p}}(\lambda, \mu)$, instead of $c_\alpha^p(\lambda, \mu)$, $c_\alpha^{\cdot+p}(\lambda, \mu)$, $c_\alpha^{+\cdot p}(\lambda, \mu)$ and $c_\alpha^{+p}(\lambda, \mu)$. For instance, it can be easily seen that

$$c_\alpha^p(\lambda, \mu) = \left\{X = (x_n)_{n\geq 1} \;/\; \sup_{n\geq 1}\left[\frac{1}{|\lambda_n|\alpha_n}\left(\sum_{k=1}^n |\mu_k x_k - \mu_{k-1}x_{k-1}|^p\right)\right] < \infty\right\},$$

$$c_\alpha^{\cdot+p}(\lambda, \mu) = \left\{X = (x_n)_{n\geq 1} \;/\; \sup_{n\geq 1}\left[\frac{1}{|\lambda_n|\alpha_n}\left(\sum_{k=1}^n |\mu_k x_k - \mu_{k+1}x_{k+1}|^p\right)\right] < \infty\right\},$$

$$c_\alpha^{+\cdot p}(\lambda, \mu) = \left\{X = (x_n)_{n\geq 1} \;/\; \sup_{n\geq 1}\left[\frac{1}{\alpha_n}\sum_{k=n}^\infty \left(\frac{1}{|\lambda_k|}|\mu_k x_k - \mu_{k-1}x_{k-1}|^p\right)\right] < \infty\right\},$$

$$\widetilde{c_\alpha^{+p}}(\lambda, \mu) = \left\{X = (x_n)_{n\geq 1} \;/\; \lim_{n\to\infty}\left[\frac{1}{\alpha_n}\sum_{k=n}^\infty \left(\frac{1}{|\lambda_k|}|\mu_k x_k - \mu_{k+1}x_{k+1}|^p\right)\right] = 0\right\},$$

with the convention $x_0 = 0$. We shall say that $c_\alpha^p(\lambda, \mu)$ and $\widetilde{c_\alpha^p}(\lambda, \mu)$ are the sets of sequences that are strongly $\alpha-$bounded and $\alpha-$convergent to 0 with index p. If

$\lambda = \mu$, $\alpha = e$ and $p = 1$, then $c_\alpha^p(\lambda, \mu) = c_\infty(\lambda)$ and $\widetilde{c_\alpha^p}(\lambda, \mu) = c_0(\lambda)$ are the sets of sequences that are strongly bounded and strongly convergent to zero.

Now we shall put $\zeta_p = \frac{(\alpha|\lambda|)^{\frac{1}{p}}}{|\mu|} = \left(\frac{(\alpha_n|\lambda_n|)^{\frac{1}{p}}}{|\mu_n|}\right)_{n\geq 1}$, $\zeta_p^- = \left(\frac{(\alpha_{n-1}|\lambda_{n-1}|)^{\frac{1}{p}}}{|\mu_{n-1}|}\right)_{n\geq 1}$ with $\frac{(\alpha_0|\lambda_0|)^{\frac{1}{p}}}{|\mu_0|} = 1$ and $\kappa = \left(\left(\frac{\alpha_{n-1}}{\alpha_n}\left|\frac{\lambda_{n-1}}{\lambda_n}\right|\right)^{\frac{1}{p}}\left|\frac{\mu_n}{\mu_{n-1}}\right|\right)_{n\geq 2}$. From the results of Section 2 we obtain

THEOREM 4. *i) If* $\alpha|\lambda| \in \widehat{C_1}$ *and* $(\alpha|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$, *then*

$$c_\alpha^p(\lambda, \mu) = s_{\zeta_p} \quad \text{and} \quad \widetilde{c_\alpha^p}(\lambda, \mu) = s_{\zeta_p}^\circ. \tag{7}$$

ii) Assume that $\alpha|\lambda| \in \widehat{C_1}$.

a) If $\zeta_p = \dfrac{(\alpha|\lambda|)^{\frac{1}{p}}}{\mu} \in \widehat{C_1}$, *then* $c_\alpha^{\cdot + p}(\lambda, \mu) = s_{\zeta_p^-}$;

b) if $\kappa = 0(1)$, *then* $\widetilde{c_\alpha^{+\cdot p}}(\lambda, \mu) = s_{\zeta_p}^\circ$.

iii) Assume that $\alpha|\lambda| \in \widehat{C_1}$.

a) $(\alpha|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$ *implies*

$$c_\alpha^{+\cdot p}(\lambda, \mu) = s_{\zeta_p} \quad \text{and} \quad \widetilde{c_\alpha^{+\cdot p}}(\lambda, \mu) = s_{\zeta_p}^\circ.$$

iv) Assume that $\alpha \in \widehat{C_1^+}$.

a) if $\zeta_p \in \widehat{C_1}$ *then* $c_\alpha^{+p}(\lambda, \mu) = s_{\zeta_p^-}$;

b) if $\kappa = o(1)$ *then* $\widetilde{c_\alpha^{+p}}(\lambda, \mu) = s_{\zeta_p^-}^\circ$.

Proof. Assertion i). First, we have

$$c_\alpha^p(\lambda, \mu) = \left\{X \;/\; \Delta(\mu)X \in w_\alpha^p(\lambda)\right\};$$

and since $\alpha|\lambda| \in \widehat{C_1}$, we get from iii) d) in Theorem 2, $w_\alpha^p(\lambda) = s_{(\alpha|\lambda|)^{\frac{1}{p}}}$. Thus, using the identities $\Delta(\mu)^{-1} = C(\mu) = D_{\frac{1}{\mu}}\Sigma$ we get $c_\alpha^p(\lambda, \mu) = D_{\frac{1}{\mu}}\Sigma s_{(\alpha|\lambda|)^{\frac{1}{p}}}$; and since $(\alpha|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$, we deduce that Δ is bijective from $s_{(\alpha|\lambda|)^{\frac{1}{p}}}$ into itself, i.e. $\Sigma s_{(\alpha|\lambda|)^{\frac{1}{p}}} = s_{(\alpha|\lambda|)^{\frac{1}{p}}}$ and we conclude that $c_\alpha^p(\lambda, \mu) = s_{\zeta_p}$. By a similar reasoning we obtain $\widetilde{c_\alpha^p}(\lambda, \mu) = s_{\zeta_p}^\circ$.

Assertion ii) a). Here we get

$$c_\alpha^{\cdot + p}(\lambda, \mu) = \left\{X \;/\; \Delta^+(\mu)X \in w_\alpha^p(\lambda)\right\};$$

and since $\alpha\,|\lambda| \in \widehat{C_1}$, we have $w_\alpha^p(\lambda) = s_{(\alpha|\lambda|)^{\frac{1}{p}}}$. So

$$c_\alpha^{\cdot+p}(\lambda,\mu) = s_{(\alpha|\lambda|)^{\frac{1}{p}}}\left(\Delta^+(\mu)\right),$$

and from ii) a) in Theorem 2, we get

$$s_{(\alpha|\lambda|)^{\frac{1}{p}}}\left(\Delta^+(\mu)\right) = s_{\zeta_p^-} \text{ if } \zeta_p \in \widehat{C_1}. \tag{8}$$

This gives the conclusion.

Statement ii) b). As above we obtain using ii) b) in Theorem 2

$$\widetilde{c_\alpha^{+p}}(\lambda,\mu) = s^\circ_{(\alpha|\lambda|)^{\frac{1}{p}}}\left(\Delta^+(\mu)\right) = s^\circ_{\zeta_p^-},$$

since $\kappa = o(1)$.

iii) We have

$$c_\alpha^{+\cdot p}(\lambda,\mu) = \left\{X \,/\, \Delta(\mu)\,X \in w_\alpha^{+p}(\lambda)\right\}.$$

If $\alpha \in \widehat{C_1^+}$, then $w_\alpha^{+p}(\lambda) = s_{(\alpha|\lambda|)^{\frac{1}{p}}}$ and

$$c_\alpha^{+\cdot p}(\lambda,\mu) = C(\mu)\,s_{(\alpha|\lambda|)^{\frac{1}{p}}} = D_{\frac{1}{\mu}}\,\Sigma s_{(\alpha|\lambda|)^{\frac{1}{p}}}.$$

From i) a) in Theorem 2, $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$ implies $\Sigma s_{(\alpha|\lambda|)^{\frac{1}{p}}} = s_{(\alpha|\lambda|)^{\frac{1}{p}}}$ and we conclude that $c_\alpha^{+\cdot p}(\lambda,\mu) = s_{\zeta_p}$. We get $\widetilde{c_\alpha^{+\cdot p}}(\lambda,\mu) = s^\circ_{\zeta_p}$ reasoning as above.

iv) a) Since $w_\alpha^{+p}(\lambda) = s_{(\alpha|\lambda|)^{\frac{1}{p}}}$ for $\alpha \in \widehat{C_1^+}$, we deduce that

$$c_\alpha^{+p}(\lambda,\mu) = \left\{X \,/\, \Delta^+(\mu)\,X \in w_\alpha^{+p}(\lambda) = s_{(\alpha|\lambda|)^{\frac{1}{p}}}\right\} = s_{(\alpha|\lambda|)^{\frac{1}{p}}}\left(\Delta^+(\mu)\right);$$

and we conclude using (8). b) can be obtained reasoning as in ii) b).

□

REMARK 2. Note that the previous sets are BK spaces and we can write for instance that if $\alpha\,|\lambda| \in \widehat{C_1}$ and $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$, then $c_\alpha^{\cdot+p}(\lambda,\mu)$ is a BK space with respect to the norm $\|\,\|_{s_{\xi_p}}$ and $\widetilde{c_\alpha^{\cdot+p}}(\lambda,\mu)$ has AK.

COROLLARY 3. *Assume that $\alpha\,|\lambda| \in \Gamma$. Then*

i) $c_\alpha^p(\lambda,\mu) = c_\alpha^{+\cdot p}(\lambda,\mu) = s_{\zeta_p}$;

ii) $\widetilde{c_\alpha^p}(\lambda,\mu) = s^\circ_{\zeta_p}$ *and* $\widetilde{c_\alpha^{+\cdot p}}(\lambda,\mu) = s^\circ_{\zeta_p}$.

Proof. Since $\Gamma \subset \widehat{C_1}$, it is enough to show that $\alpha\,|\lambda| \in \Gamma$ if and only if $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \Gamma$ and apply i) and ii) in Theorem (4). So put $q = 1/p > 0$, $\xi = \alpha\,|\lambda|$ and show that $\xi \in \Gamma$ if and only if $\xi^q \in \Gamma$. If $\xi \in \Gamma$ there is an integer N such that $\sup_{n \geq N+1}\left(\frac{\xi_{n-1}}{\xi_n}\right) < 1$, then

$$\left(\frac{\xi_{n-1}}{\xi_n}\right)^q \leq \left[\sup_{n \geq N+1}\left(\frac{\xi_{n-1}}{\xi_n}\right)\right]^q < 1 \text{ for all } n \geq N+1,$$

and $\xi^q \in \Gamma$. Conversely, assume that $\xi^q \in \Gamma$, that is $\limsup_{n\to\infty}\left(\frac{\xi_{n-1}}{\xi_n}\right)^q < 1$. By a similar reasoning we get

$$\frac{\xi_{n-1}}{\xi_n} \leq \left[\sup_{n \geq N+1}\left(\frac{\xi_{n-1}}{\xi_n}\right)^q\right]^{\frac{1}{q}} < 1 \text{ for all } n \geq N+1,$$

and $\xi \in \Gamma$. We conclude applying Theorem (4).

□

In order to assert the next corollary, we need the following elementary lemma.

LEMMA 4. *Let $q > 0$ be any real and $\alpha \in U^{+*}$ a nondecreasing sequence. Then*

i) $\alpha \in \widehat{C_1}$ implies $\alpha^q \in \widehat{C_1}$, for $q \geq 1$,

ii) $\alpha^q \in \widehat{C_1}$ implies $\alpha \in \widehat{C_1}$, for $0 < q < 1$.

Proof. Let $q \geq 1$. Since α is nondecreasing we see immediatly that for any given integer $n \geq 1$: $\alpha_k \sum_{k=1}^{n}\left(\alpha_n^{q-1} - \alpha_k^{q-1}\right) = \sum_{k=1}^{n}\left(\alpha_n^{q-1}\alpha_k - \alpha_k^q\right) \geq 0$, and

$$\frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right) \geq \frac{1}{\alpha_n^q}\left(\sum_{k=1}^{n}\alpha_k^q\right). \tag{9}$$

Since $\alpha \in \widehat{C_1}$ implies $\frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right) = O(1)$, we obtain i) using the inequality (9). Now, writing $\beta = \alpha^q \in \widehat{C_1}$ and applying i), we get $\alpha = \beta^{\frac{1}{q}} \in \widehat{C_1}$ for $0 < q < 1$. This permits us to conclude for ii).

□

COROLLARY 4. *Assume that $\alpha, \lambda \in U^{+*}$ and $\alpha\,|\lambda|$ is a nondecreasing sequence.*

i) If $p > 1$, then $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$ implies

$$c_\alpha^p(\lambda, \mu) = s_{\zeta_p} \quad \text{and} \quad \widetilde{c_\alpha^p}(\lambda, \mu) = s^\circ_{\zeta_p}; \tag{10}$$

ii) if $0 < p \leq 1$, $\alpha\,|\lambda| \in \widehat{C_1}$ if and only if $c_\alpha^p(\lambda, \mu) = s_{\zeta_p}$.

iii) If $\alpha\,|\lambda| \in \Gamma$*, then (10) holds.*

Proof. i). If $p > 1$, $0 < \frac{1}{p} < 1$ and from Lemma 4, $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$ implies $\alpha\,|\lambda| \in \widehat{C_1}$. So we conclude using i) in Theorem 4.

Assertion ii). The necessity comes from i) in Theorem (4). Sufficiency. First, put

$$\widetilde{\alpha} = \left((-1)^n \frac{(\alpha_n\,|\lambda_n|)^{\frac{1}{p}}}{\mu_n} \right)_{n\geq 1}.$$

We have $\widetilde{\alpha} \in c_\alpha^p(\lambda, \mu) = s_{\zeta_p}$ and using the convention $\alpha_0 = 0$, we can write

$$|\Delta(\mu)\,\widetilde{\alpha}| = \left(\left| \mu_n (-1)^n \frac{(\alpha_n\,|\lambda_n|)^{\frac{1}{p}}}{\mu_n} - \mu_{n-1} (-1)^{n-1} \frac{(\alpha_{n-1}\,|\lambda_{n-1}|)^{\frac{1}{p}}}{\mu_{n-1}} \right| \right)_{n\geq 1}.$$

So

$$|\Delta(\mu)\,\widetilde{\alpha}|^p = \left(\left((\alpha_n\,|\lambda_n|)^{\frac{1}{p}} + (\alpha_{n-1}\,|\lambda_{n-1}|)^{\frac{1}{p}} \right)^p \right)_{n\geq 1}.$$

Then the condition $\Sigma\,|\Delta(\mu)\,\widetilde{\alpha}|^p \in s_{\alpha|\lambda|}$ implies that there is a real $M > 0$ such that for every n:

$$\frac{1}{\alpha_n\,|\lambda_n|} \left(\sum_{k=1}^{n} \alpha_k\,|\lambda_k| \right) \leq \frac{1}{\alpha_n\,|\lambda_n|} \left(\sum_{k=1}^{n} \left((\alpha_k\,|\lambda_k|)^{\frac{1}{p}} + (\alpha_{k-1}\,|\lambda_{k-1}|)^{\frac{1}{p}} \right)^p \right) \leq M.$$

We conclude that $\alpha\,|\lambda| \in \widehat{C_1}$. So ii) can be deduced from Theorem (4).

□

In the next result we shall denote by $c_\alpha^p(\lambda)$ the set $c_\alpha^p(\lambda, \lambda)$.

Consider now the following identities.

(11) $$c_\alpha^p(\lambda) = s_{\left(\alpha^{\frac{1}{p}}|\lambda|^{\frac{1}{p}-1}\right)}$$

(12) $$\widetilde{c_\alpha^p}(\lambda) = s^\circ_{\left(\alpha^{\frac{1}{p}}|\lambda|^{\frac{1}{p}-1}\right)}$$

COROLLARY 5. *Assume that* $\alpha\,|\lambda|$ *is nondecreasing.*

i) If $0 < p \leq 1$*, then*

- *a)* $\alpha\,|\lambda| \in \widehat{C_1}$ *if and only if (11) holds.*
- *b)* $\alpha\,|\lambda| \in \widehat{C_1}$ *implies that (12) holds.*

ii) If $p > 1$*, the condition* $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$ *implies that (11) and (12) hold.*

iii) $\alpha\,|\lambda| \in \Gamma$ *implies (11) and (12).*

Proof. i) a) comes from ii) in Corollary 4, where $\lambda = \mu$. The proof of i) b) comes from Lemma 4 and i) in Theorem (4). ii) comes from i) in Corollary 4. iii) comes from Corollary 3.

□

Now we can give an application which can be considered as corollary.

COROLLARY 6. *i)* $c_\infty^p(\lambda) \neq l_\infty$ *in the following cases:*

a) $0 < p < 1$ *and* $|\lambda| \in \widehat{C_1}$*;*

b) $p > 1$ *and* $|\lambda|^{\frac{1}{p}} \in \widehat{C_1}$.

ii) $c_\infty(\lambda) = l_\infty$ *if and only if* $|\lambda| \in \widehat{C_1}$.

iii) Assume that $\alpha \to \infty$.

a) Let $p > 1$. *If* $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$, *then*

$$c_\alpha^p(\lambda) = l_\infty \quad \textit{implies} \quad |\lambda_n| \to \infty \quad \textit{as } n \to \infty.$$

b) If $0 < p < 1$ *and* $\alpha\,|\lambda| \in \widehat{C_1}$, *then*

$$c_\alpha^p(\lambda) = l_\infty \quad \textit{implies} \quad \lambda \in c_0.$$

Proof. Case a). Since $|\lambda| \in \widehat{C_1}$, we have $c_\infty^p(\lambda) = s_{|\lambda|^{\frac{1}{p}-1}}$. So the identity $c_\infty^p(\lambda) = l_\infty$ implies that there are K_1 and $K_2 > 0$ such that

$$K_1 \leq |\lambda_n|^{\frac{1}{p}-1} \leq K_2 \quad \text{for all } n. \tag{13}$$

Since $\dfrac{1}{p} - 1 > 0$ and $|\lambda| \in \widehat{C_1}$, we deduce that $|\lambda_n|^{\frac{1}{p}-1} \to \infty$ as $n \to \infty$, which is contradictory.

Case b). Here we get $|\lambda_n|^{\frac{1}{p}-1} = o(1)$ and (13) cannot be satisfied. ii) comes from the equivalence a)⇔b) in i) of Theorem 3 in which we put $\alpha = e$ and $\lambda = \mu$.

Assertion iii). Condition a) implies $c_\alpha^p(\lambda) = s_{\left(\alpha^{\frac{1}{p}}|\lambda|^{\frac{1}{p}-1}\right)}$. From the identity $c_\alpha^p(\lambda) = l_\infty$, there exist K_1 and $K_2 > 0$ such that

$$K_1 \leq \alpha_n^{\frac{1}{p}}\,|\lambda_n|^{\frac{1}{p}-1} \leq K_2 \quad \text{and} \quad \frac{K_1}{\alpha_n^{\frac{1}{p}}} \leq |\lambda_n|^{\frac{1}{p}-1} \leq \frac{K_2}{\alpha_n^{\frac{1}{p}}} \quad \text{for all } n \geq 1.$$

Since $\frac{1}{p} - 1 < 0$ we conclude that $|\lambda_n| \to \infty$ as $n \to \infty$. b) can be obtained by a similar reasoning.

□

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AMS Subject Classification: 40H05, 46A15.

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Lavoro pervenuto in redazione il 15.05.2003 e, in forma definitiva, il 08.09.2003.

Rend. Sem. Mat. Univ. Pol. Torino
Vol. 61, 1 (2003)

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LOCAL SOLVABILITY FOR SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS OF CONSTANT STRENGTH

Abstract. The main goal of the present paper is to study the local solvability of semilnear partial differential operators of the form

$$F(u) = P(D)u + f(x, Q_1(D)u, ..., Q_M(D)u),$$

where $P(D)$, $Q_1(D)$, ..., $Q_M(D)$ are linear partial differntial operators of constant coefficients and $f(x, v)$ is a C^∞ function with respect to x and an entire function with respect to v.
Under suitable assumptions on the nonlinear function f and on $P, Q_1, ..., Q_M$, we will solve locally near every point $x^0 \in \mathbb{R}^n$ the next equation

$$F(u) = g, \ \ g \in B_{p,k},$$

where $B_{p,k}$ is a wieghted Sobolev space as in Hörmander [13].

1. Introduction

During the last years the attention in the literature has been mainly addressed to the semilinear case:

$$P(x, D)u + f(x, D^\alpha u)_{|\alpha|\leq m-1} = g(x) \tag{1}$$

where the nonlinear function $f(x, v)$, $x \in \mathbb{R}^n$, $v \in \mathbb{C}^M$, is in $C^\infty(\mathbb{R}^n, \mathcal{H}(\mathbb{C}^M))$ with $\mathcal{H}(\mathbb{C}^M)$ the set of the holomorphic functions in $\mathbb{C}^M$ and where the local solvability of the linear term $P(x, D)$ is assumed to be already known.
See Gramchev-Popivanov[10] and Dehman[4] where, exploiting the fact that the nonlinear part of the equation (1) involves derivatives of order $\leq m - 1$, one is reduced to applications of the classical contraction principle and Brower's fixed point Theorem, provided the linear part is invertible in some sense. The general case of $P(x, D)$ satisfying the $(\mathcal{P})$ condition of Nirenberg and Trèves [21] has been settled in Hounie-Santiago[12], by combining the contraction principle with compactness arguments.
Corcerning the case of linear part with multiple characteristics, we mention the recent results of Gramchev-Rodino[11], Garello[6], Garello[5], Garello-Gramchev-Popivanov-Rodino[7], Garello-Rodino[8], Garello-Rodino[9], De Donno-Oliaro[3], Marcolongo[17], Marcolongo-Oliaro[18], Oliaro[22].

The main goal of the present paper is to study the local solvability of semilinear partial differential operators of the form

$$F(u) = P(D)u + f(x, Q_1(D)u, ..., Q_M(D)u) \tag{2}$$

where $P(D)$, $Q_1(D)$, ..., $Q_M(D)$ are linear partial differential operators with constant coefficients and $f(x, v)$ is as before a C^∞ function with respect to x and an entire function with respect to v.

We will introduce suitable assumptions on the nonlinear function f and on $P, Q_1, ..., Q_M$ in order to solve locally near a point $x^0 \in \mathbb{R}^n$ the next equation

$$F(u) = g, \quad g \in B_{p,k}, \tag{3}$$

where $B_{p,k}$ is a weighted Sobolev space as in Hörmander [13], with $1 \leq p \leq \infty$ and k temperate weight function. Hörmander introduced these spaces exactly in connection with the problem of the solvability of linear partial differential operators with constant coefficients, namely one can find a fundamental solution T of $P(D)$ belonging locally to $B_{\infty,\widetilde{P}}$, see [13].

There are suitable assumptions on the temperate weight function k under which the space $B_{p,k}$ forms an algebra, see [20], [19]. In [20] we have also proved, under the same conditions, invariance after composition with analytic functions. These results allow us to look for a solution u, in a related $B_{p,k}$, giving meaning to the nonlinear term of (2).

More precisely, from basic properties of these spaces (see [13], [24] and the next Section 2 for notations and results), we know that if $u \in B_{p,k\widetilde{P}}$ then $P(D)u \in B_{p,k}$ and $Q_i(D)u \in B_{p,k\widetilde{P}/\widetilde{Q_i}}$ for $i = 1, .., M$. Assuming that $P \succ Q_i$ for every i, we get

$$B_{p,k\widetilde{P}/\widetilde{Q_i}} \hookrightarrow B_{p,k}, \tag{4}$$

then, one should require k satisfying the hypotheses which grant $f(x, Q_1(D)u, ..., Q_M(D)u) \in B_{p,k}$. Under these conditions the equation (3) will be well defined in the classical sense, for $g \in B_{p,k}$ and $u \in B_{p,k\widetilde{P}}$.

In this paper we will prove two theorems about the local solvabilty of (3) under different assumptions on the non-linear function f and the linear terms $P, Q_1, ..., Q_M$, completing the results of [20].

In the first theorem, Theorem 11, we assume $\frac{\widetilde{Q_i}(\xi)}{\widetilde{P}(\xi)} \to 0$ when $|\xi| \to \infty$ for $i = 1, ..., M$. From this hypothesis it follows that the inclusion $B_{p,k\widetilde{P}/\widetilde{Q_i}} \hookrightarrow B_{p,k}$ is compact (see [13]).

However here we assume on the nonlinearity $f(x, 0) = 0$, corresponding to the standard setting in literature; this is essentially weaker than the hypotheses in [20] $f(x_0, v) = 0$ for every $v \in \mathbb{C}^M$.

In the proof, we will generalize a well-known property of the standard Sobolev spaces to the case of the weighted spaces $B_{p,k}$, namely, if k_1 and k_2 are temperate weight functions, $k_2(\xi) \geq N > 0$ for all $\xi \in \mathbb{R}^n$, x^0 is a point in $\mathbb{R}^n$ and

$$\frac{k_2(\xi)}{k_1(\xi)} \to 0 \quad \text{for} \quad |\xi| \to \infty \tag{5}$$

then

$$\|u\|_{p,k_2} \leq C(\epsilon)\|u\|_{p,k_1} \quad \forall u \in B_{p,k_1} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)}) \tag{6}$$

where $\overline{B_\epsilon(x^0)} = \overline{\{x \in \mathbb{R}^n / \|x - x^0\| < \epsilon\}}$ and $C(\epsilon) \to 0$ for $\epsilon \to 0$.
Applying the previous property and the Schauder Fixed Point Theorem, we will conclude that the equation (3) is locally solvable at every $x^0 \in \mathbb{R}^n$, and the solution belongs to $B_{p,k\widetilde{P}}$.
In the second result, Theorem 12, the assumptions are weaker than those in the first theorem, but the functional frame is now limited to suitable spaces $H_k := B_{2,k}$.
More precisely, we suppose $g \in H_k$ having the algebra property, and k of the particular form $k = \psi^r$, with $r \in \mathbb{R}$, $\psi \in C^\infty$ satisfying the "slowly varying" estimates, stronger than the temperate condition. These estimates were introduced by Beals [1], cf. Hörmander [14], in connection with the pseudodifferential calculus. Namely, as a particular case of the classes of Beals [1], we may consider the classes of symbols S^ν_ψ, whose definition is obtained by replacing $(1 + |\xi|)$ in the classic $S^\nu_{1,0}$ bounds with the weight $\psi(\xi)$. The corresponding pseudodifferential operators OPS^ν_ψ have a natural action on the weighted Sobolev spaces H_{ψ^r}. For most of the related properties and definitions recalled in the following let us refer to Rodino [23].
After having transformed the equation (3) into an equivalent fixed point problem, we will deduce, using the assumption $Q_i \prec\prec P$ and applying the properties of the pseudodifferential calculus, that there exists $C(\epsilon)$ such that $C(\epsilon) \to_{\epsilon\to 0} 0$ and

$$\|Q_i(D)u\|_{H_k} \leq C(\epsilon)\|P(D)u\|_{H_k}$$

for $i = 1, ..., M$ and every $u \in H_{k\widetilde{P}} \cap \mathcal{E}'(B_\epsilon(x^0))$, x^0 any point in $\mathbb{R}^n$.
The Contraction Principle will allow us to conclude again that the equation (3) is locally sovable, and the solution belongs to $H_{k\widetilde{P}}$, at every point of $\mathbb{R}^n$.
We end this introduction by giving a simple example of an operator with multiple characteristics to which our Theorems 11 and 12 apply. Namely consider

$$D^4_{x_1}u - L^2u + f(u, (D^2_{x_1} - L)u, (D^2_{x_2} + L)u) = g. \tag{7}$$

where L is a constant vector field in $\mathbb{R}^n$. Our results will provide for (7) solutions in different scales of functional spaces $B_{p,k}$, cf. Section 2, exibiting novelty in comparison with the papers mentioned at the beginning. We also point out that even for some semilinear partial differential equations with simple characteristics Theorem 11 and Theorem 12 imply new results for the local solvability in more general functional spaces.

2. Preliminary results

We begin with a short survey on $B_{p,k}$ spaces.

DEFINITION 1. *A positive function k defined in $\mathbb{R}^n$ is called a temperate weight function if there exist positive constants C and N such that*

$$k(\xi + \eta) \leq (1 + C|\xi|)^N k(\eta)\,; \quad \xi,\ \eta \in \mathbb{R}^n. \tag{8}$$

The set of all such functions will be denoted by K.

EXAMPLE 1. The basic example of a function in K, is the function $\widetilde{P}$ defined by

$$\widetilde{P}^2(\xi) = \sum_{|\alpha|\geq 0} |P^{(\alpha)}(\xi)|^2 \tag{9}$$

where P is a polynomial and so that the sum is finite. Here $P^{(\alpha)} = \partial^\alpha P$. It follows immediately from Taylor's formula that

$$\widetilde{P}(\xi+\eta) \leq (1+C|\xi|)^m \widetilde{P}(\eta)$$

where m is the degree of P and C is a constant depending only on m and the dimension n.

The proofs of the following results are omitted for shortness; let us refer for them to [13].

DEFINITION 2. *If $k \in K$ and $1 \leq p \leq +\infty$, we denote by $B_{p,k}$ the set of all distributions $u \in \mathcal{S}'$ such that the Fourier Transform $\widehat{u}$ is a function and*

$$\|u\|_{p,k} = \left((2\pi)^{-n} \int |k(\xi)\widehat{u}(\xi)|^p d\xi\right)^{1/p} < +\infty. \tag{10}$$

When p=$+\infty$ we shall interpret $\|u\|_{p,k}$ as $ess.sup|k(\xi)\widehat{u}(\xi)|$. We shall also write $H_k := B_{2,k}$, endowed with the natural Hilbert structure.

EXAMPLE 2. The usual Sobolev spaces $H_{(s)}$ correspond to the temperate weight function $k_s(\xi) = (1+|\xi|^2)^{s/2}$, with $p = 2$.

$B_{p,k}$ is a Banach space with the norm (10). We have

$$\mathcal{S} \subset B_{p,k} \subset \mathcal{S}' \tag{11}$$

also in topological sense.

THEOREM 1. *If k_1 and k_2 belong to K and there exists $C > 0$ such that*

$$k_2(\xi) \leq Ck_1(\xi),\ \xi \in \mathbb{R}^n,$$

it follows that $B_{p,k_1} \hookrightarrow B_{p,k_2}$.

REMARK 1. Let $k \in K$, in view of estimate (8) with $\eta = 0$ and Theorem 1, one can find $s \in \mathbb{R}$ such that $B_{p,(1+|\xi|^2)^{s/2}} \hookrightarrow B_{p,k}$.

We shall now study how differential operators with constant coefficients act in the spaces $B_{p,k}$. Recall that if $P(\xi)$ is a polynomial in n variables $\xi_1, ..., \xi_n$ with complex coefficients, then a differential operator $P(D)$ is defined by replacing ξ_j by $D_j = -i\partial/\partial_{x_j}$ and the function $\tilde{P}$ is defined by (9).

THEOREM 2. *If $u \in B_{p,k}$ it follows that $P(D)u \in B_{p,k/\tilde{P}}$.*

THEOREM 3. *If $u_1 \in B_{p,k_1} \cap \mathcal{E}'$ and $u_2 \in B_{\infty,k_2}$, it follows that $u_1 * u_2$ belongs to B_{p,k_1k_2}, and we have the estimate*

$$\|u_1 * u_2\|_{p,k_1k_2} \leq \|u_1\|_{p,k_1}\|u_2\|_{\infty,k_2}. \tag{12}$$

THEOREM 4. *If $u \in B_{p,k}$ and $\phi \in \mathcal{S}$, it follows that $u\phi \in B_{p,k}$ and that*

$$\|\phi u\|_{p,k} \leq \|\phi\|_{1,M_k}\|u\|_{p,k}, \tag{13}$$

where $M_k \in K$ is defined by $M_k(\xi) = \sup_\eta \frac{k(\xi+\eta)}{k(\eta)}$.

LEMMA 1. *Let $k \in K$. For every $\phi \in \mathcal{S}$ there exists an equivalent weight function h (i.e. $C^{-1}k(\xi) \leq h(\xi) \leq Ck(\xi)$) such that*

$$\|\phi u\|_{p,h} \leq 2\|\phi\|_{1,1}\|u\|_{p,h}. \tag{14}$$

We stress that the weight h depends on ϕ. We will use Lemma 1 later.

THEOREM 5. *If k_1 and k_2 belong to K and H is a compact set in $\mathbb{R}^n$, the inclusion mapping of $B_{p,k_1} \cap \mathcal{E}'(\mathrm{H})$ into B_{p,k_2} is compact if*

$$\frac{k_2(\xi)}{k_1(\xi)} \to 0 \quad \text{for} \quad |\xi| \to \infty. \tag{15}$$

Conversely, if the mapping is compact for one set H with interior points, it follows that (15) is valid.

The Frèchet space $B^{\mathrm{loc}}_{p,k}$ is defined in the standard way and corresponding properties are valid for it, in particular we have the following variant of Theorem 3.

THEOREM 6. *Let $u_1 \in B_{p,k_1} \cap \mathcal{E}'$ and $u_2 \in B^{loc}_{\infty,k_2}$, it follows that $u_1 * u_2$ belongs to B^{loc}_{p,k_1k_2}.*

Under suitable conditions regarding a temperate weight function k the corresponding space $B_{p,k}$ is an algebra; for the proof of the next theorems, let us refer to [20].

THEOREM 7. *Let $1 < p < +\infty$, $1/p + 1/q = 1$, $u, v \in B_{p,k}$ and $K(\xi,\eta) = \frac{k(\xi)}{k(\xi-\eta)k(\eta)}$ satisfy*

$$\sup_\xi \int |K(\xi,\eta)|^q d\eta \leq C_0 < +\infty. \tag{16}$$

then $uv \in B_{p,k}$ and $\|uv\|_{p,k} \leq C\|u\|_{p,k}\|v\|_{p,k}$.

The previous theorem can be generalized to the invariance of the spaces $B_{p,k}$ under the composition with entire functions.

DEFINITION 3. *We write $f(x, v) \in C^\infty(\mathbb{R}^n_x, \mathcal{H}(\mathbb{C}^M))$, where $\mathcal{H}(\mathbb{C}^M)$ is the set of the entire functions in $\mathbb{C}^M$, if $f(x, v) = \sum_{|\alpha| \geq 0} c_\alpha(x) v^\alpha$ where $c_\alpha(x) \in C^\infty(\mathbb{R}^n)$ and for every compact subset H$\subset \mathbb{R}^n$, there exist $C_\beta(\mathrm{K}), \lambda_\alpha(\mathrm{K}) > 0$ such that $\sup_{x \in \mathrm{K}} |\partial^\beta c_\alpha(x)| < C_\beta \lambda_\alpha$, $\sum_{|\alpha| \geq 0} \lambda_\alpha v^\alpha$ being an entire function of $v \in \mathbb{C}^M$.*

THEOREM 8. *Let $f(x, v) \in C^\infty(\mathbb{R}^n_x, \mathcal{H}(\mathbb{C}^M))$, $u_1, ..., u_M \in B_{p,k}$ with k satisfying the hypotheses of Theorem 7, then $f(x, u_1(x), ..., u_M(x)) \in B^{loc}_{p,k}$.*

Now we recall two particular definitions of comparison between differential polynomials and related theorems of characterization (see Hörmander [13], Trèves [24] for the proofs).

Let Ω be an open subset of $\mathbb{R}^n$, P, Q two differential operators with C^∞ coefficients in Ω.

DEFINITION 4. *We say that P is stronger than Q in Ω if to every relatively compact open subset Ω' of Ω, there exists a constant $C(P, Q, \Omega')$ such that, for all functions $\psi \in C^\infty_0(\Omega')$,*

$$\int |Q(x, D)\psi(x)|^2 dx \leq C(P, Q, \Omega') \int |P(x, D)\psi(x)|^2 dx. \tag{17}$$

DEFINITION 5. *Let x^0 be a point of Ω. We say that P is infinitely stronger than Q at x^0 if to every $\epsilon > 0$ there exists $\eta > 0$ such that, for all functions $\psi \in C^\infty_0(\Omega)$ having their support in the open ball centered at x^0, with radius η,*

$$\int |Q(x, D)\psi(x)|^2 dx \leq \epsilon \int |P(x, D)\psi(x)|^2 dx. \tag{18}$$

In other words, P is infinitely stronger than Q at x^0 if estimate (17) holds for some open neighborhoord Ω' of x^0, and if we may choose the constant $C(P, Q, \Omega')$ so that it converges to zero when Ω' converges to the set $\{x^0\}$. If P is stronger than Q and Q stronger than P in Ω, we say that P and Q are equally strong or equivalent in Ω.
If P and Q have constant coefficients, the validity of Definition 4 does not depend on Ω. In this situation, we simply say that $P(D)$ is stronger than $Q(D)$ and we shall write $P(D) \succ Q(D)$. Similarly, the translation invariance of $P(D)$ and $Q(D)$ implies that if $P(D)$ is infinitely stronger than $Q(D)$ at some point of $\mathbb{R}^n$, this is also true at any other point of $\mathbb{R}^n$. Thus we shall say that $P(D)$ is infinitely stronger than $Q(D)$, and write $P(D) \succ\succ Q(D)$.

THEOREM 9. *Let $P(D)$, $Q(D)$ be two differential polynomials in $\mathbb{R}^n$. The following properties are equivalent:*

(a) $P(D)$ is stronger than $Q(D)$;

(b) the function $\frac{\widetilde{Q}(\xi)}{\widetilde{P}(\xi)}$ is bounded in $\mathbb{R}^n$;

(c) the function $\frac{|Q(\xi)|}{\widetilde{P}(\xi)}$ is bounded in $\mathbb{R}^n$.

REMARK 2. With reference to operator (2), let us observe the following. If we assume that $Q_i \prec P$ for every $i = 1, ..., M$, in view of Theorem 9, we have $\frac{\widetilde{Q}_i(\xi)}{\widetilde{P}(\xi)} \leq C$; therefore for every $k \in K$

$$k(\xi) \leq C \frac{k(\xi)\widetilde{P}(\xi)}{\widetilde{Q}_i(\xi)}. \tag{19}$$

In view of Theorem 1 and (19) we obtain

$$B_{p,\frac{k\widetilde{P}}{\widetilde{Q}_i}} \hookrightarrow B_{p,k}, \quad \forall i = 1, ..., M. \tag{20}$$

We conclude this section by recalling the definition and the main properties of the pseudodifferential operators with symbols in the classes S_ψ, particular case of the classes of Beals [1].

We say that a positive continuous function $\psi(\xi)$ in $\mathbb{R}^n$ is a basic weight function if there are positive constants c, C such that

$$\begin{aligned} &(1) \quad c(1+|\xi|)^c \leq \psi(\xi) \leq C(1+|\xi|), \\ &(2) \quad c \leq \psi(\xi+\theta)\psi(\xi)^{-1} \leq C, \quad \text{if} \quad |\theta|\psi(\xi)^{-1} \leq c. \end{aligned} \tag{21}$$

For $\nu \in \mathbb{R}$ we define S^ν_ψ to be the set of all $a(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which satisfy the estimates

$$|D^\alpha_x D^\beta_\xi a(x,\xi)| \leq c_{\alpha\beta}\psi(\xi)^{\nu-|\beta|}, \quad x \in \mathbb{R}^n, \xi \in \mathbb{R}^n. \tag{22}$$

Let

$$Af(x) = a(x,D)f(x) = (2\pi)^{-n} \int [ix\xi]a(x,\xi)\hat{f}(\xi)d\xi, \quad f \in \mathcal{C}^\infty_0(\mathbb{R}^n), \tag{23}$$

with $a(x,\xi) \in S^\nu_\psi$. The standard rules of the calculus of the pseudo-differential operators hold for operators of the form (23). Let us review shortly the properties which we shall use in the following.

Recall first that for every basic weight function $\psi(\xi)$ we may find a smooth basic weight function $\psi_0(\xi)$, which is equivalent to $\psi(\xi)$ (i.e. $\psi_0(\xi)\psi(\xi)^{-1}$ and $\psi_0(\xi)^{-1}\psi(\xi)$ are bounded in $\mathbb{R}^n$), such that

$$|D^\beta_\xi \psi_0(\xi)| \leq c_\beta \psi_0(\xi)\psi_0(\xi)^{-|\beta|}, \quad \xi \in \mathbb{R}^n. \tag{24}$$

Note that, let ψ be a smooth function satisfying (24) and (21), then ψ is a temperate weight function.

Equivalent basic weight functions define the same class of symbols; therefore we may assume in the following that $\psi(\xi)$ satisfies (24).

Moreover from (24) it follows that $\psi \in S^1_\psi$ and so, for every $r \in \mathbb{R}$, $\psi^r \in S^r_\psi$.

The operator A in (23) maps continuously $\mathcal{C}^\infty_0(\mathbb{R}^n)$ into $\mathcal{C}^\infty(\mathbb{R}^n)$ and it extends to a linear continuous operator from $\mathcal{E}'(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$.

According to the previous notations, write H_{ψ^ν} for the Hilbert space of the distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ which satisfy

$$\|f\|^2_{H_{\psi^\nu}} = \int \psi(\xi)^{2\nu} |\hat{f}(\xi)|^2 d\xi < \infty. \tag{25}$$

If A has symbol in S^ν_ψ, then for every $s \in \mathbb{R}$:

$$A : H_{\psi^{\nu+s}} \to H_{\psi^s} \quad \text{continuously.} \tag{26}$$

A map $A : \mathcal{C}^\infty_0(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is said to be smoothing if it has a continuous extension mapping $\mathcal{E}'(\mathbb{R}^n)$ into $\mathcal{C}^\infty(\mathbb{R}^n)$; for given operators $A_1, A_2 : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ we shall write $A_1 \sim A_2$ if the difference $A_1 - A_2$ is smoothing.
If $a(x, \xi)$ is in $\cap_\nu S^\nu_\psi$, then $a(x, D)$ is smoothing.

THEOREM 10. *Let $a_1(x, \xi)$ be in $S^{\nu_1}_\psi$, let $a_2(x, \xi)$ be in $S^{\nu_2}_\psi$. Then the product $a_1(x, D)a_2(x, D)$ is in $S^{\nu_1+\nu_2}_\psi$ with symbol*

$$a(x, \xi) \sim \sum_\alpha (\alpha!)^{-1} \partial^\alpha_\xi a_1(x, \xi) D^\alpha_x a_2(x, \xi). \tag{27}$$

3. Statement of the main results

We will study the following semilinear partial differential operator (2) where

(1) $P(D)$ is a linear partial differential operator with constant coefficients;

(2) $f(x, v) \in C^\infty(\mathbb{R}^n_x, \mathcal{H}(\mathbb{C}^M))$ and $f(x, 0) = 0$ for every $x \in \mathbb{R}^n$;

(3) $Q_1(D), ..., Q_M(D)$ are linear partial differential operators with constant coefficients such that $Q_i(D) \prec P(D)$ for $i = 1, ..., M$.

We want to solve locally near every point $x^0 \in \mathbb{R}^n$ the equation (3) under the following stronger assumptions on $Q_i(D)$, cf. Introduction.

THEOREM 11. *Let $g \in B_{p,k}$, with k satisfying the assumptions of Theorem 7 and $k(\xi) \geq N > 0$, $\forall \xi \in \mathbb{R}^n$, $1 < p < \infty$. Consider the operator F defined by* (2) *where, for $i = 1, ..., M$,*

$$\frac{\widetilde{Q}_i(\xi)}{\widetilde{P}(\xi)} \to 0, \quad when \ |\xi| \to \infty; \tag{28}$$

then for every $x^0 \in \mathbb{R}^n$ one can find a constant $\epsilon_0 > 0$ and $u^0 \in B_{p,k\tilde{P}}$ such that

$$F(u^0)(x) = g(x), \quad \forall x \in \Omega \tag{29}$$

where $\Omega = \{x \in \mathbb{R}^n / \|x - x^0\| < \epsilon_0\}$.

THEOREM 12. *Let $g \in H_k$, k satisfying the assumptions of Theorem 7 with $p = 2$ be such that $k = \psi^r$ with $r \in \mathbb{R}$, $\psi^r(\xi) \geq N > 0$ for every $\xi \in \mathbb{R}^n$ and assume that* (21) *holds. Consider the operator F defined by* (2) *where, for $i = 1, \dots, M$,*

$$Q_i \prec\prec P; \tag{30}$$

then for every $x^0 \in \mathbb{R}^n$ one can find a constant $\epsilon_0 > 0$ and $u^0 \in H_{k\widetilde{P}}$ such that

$$F(u^0)(x) = g(x), \quad \forall x \in \Omega \tag{31}$$

where $\Omega = \{x \in \mathbb{R}^n / \|x - x^0\| < \epsilon_0\}$.

4. Proof of the results

To prove Theorem 11 we will apply a property of the Sobolev spaces true also for the weighted Sobolev spaces (see [16]).

THEOREM 13. *If k_1 and k_2 belong to K, $k_2 \geq N > 0$, x^0 is a point in $\mathbb{R}^n$ and*

$$\frac{k_2(\xi)}{k_1(\xi)} \to 0 \quad \text{for} \quad |\xi| \to \infty \tag{32}$$

then

$$\|u\|_{p,k_2} \leq C(\epsilon)\|u\|_{p,k_1} \quad \forall u \in B_{p,k_1} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)}) \tag{33}$$

where $\overline{B_\epsilon(x^0)} = \overline{\{x \in \mathbb{R}^n / \|x - x^0\| < \epsilon\}}$ and $C(\epsilon) \to 0$ for $\epsilon \to 0$.

Proof. First of all note that from the Theorem 5 it follows that the injection of $B_{p,k_1} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)})$ into B_{p,k_2} is compact.
To prove the Theorem we have to verify that

$$\sup_{u \in B_{p,k_1} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)})} \frac{\|u\|_{p,k_2}}{\|u\|_{p,k_1}} = C(\epsilon) \quad \text{where} \quad C(\epsilon) \to 0 \tag{34}$$

that is equivalent to prove that

$$\sup_{u \in B_{p,k_1} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)}), \|u\|_{p,k_1} = 1} \|u\|_{p,k_2} = C(\epsilon) \quad \text{where} \quad C(\epsilon) \to 0. \tag{35}$$

We suppose, ab absurdo, that $C(\epsilon)$ does not tend to 0 when ϵ tends to 0, then there exists a sequence $\{\epsilon_\nu\}_{\nu \in \mathbb{N}}$ such that $\epsilon_\nu \to 0$ when $\nu \to \infty$ and $C(\epsilon_\nu) \nrightarrow 0$ when $\nu \to \infty$. We can deduce that there exists a positive constant r and a subsequence $\{\epsilon_{\nu_j}\}_{j \in \mathbb{N}}$ such that $\epsilon_{\nu_j} \to 0$ when $j \to \infty$ and $C(\epsilon_{\nu_j}) > r$ for every j. Then we obtain that

$$\sup_{u \in B_{p,k_1} \cap \mathcal{E}'(\overline{B_{\epsilon_{\nu_j}}(x^0)}), \|u\|_{p,k_1} = 1} \|u\|_{p,k_2} = C(\epsilon_{\nu_j}) > r \quad \text{for every } j. \tag{36}$$

From the definition of supremum there exists a sequence $u_{\nu_j} \in B_{p,k_1}$ with support contained in the ball of fixed center x^0 with radius ϵ_{ν_j} such that

$$\|u_{\nu_j}\|_{p,k_1} = 1 \quad \text{and} \quad \|u_{\nu_j}\|_{p,k_2} \geq r. \tag{37}$$

The sequence $\{u_{\nu_j}\}_{j\in\mathbb{N}}$ is bounded in B_{p,k_1}, then, according to the compactness of the injection of $B_{p,k_1} \cap \mathcal{E}'(\overline{B_{\epsilon_{\nu_j}}(x^0)})$ into B_{p,k_2}, we may assume that there exists a subsequence, still denoted by u_{ν_j}, and a distribution $u \in B_{p,k_2}$ such that

$$\|u_{\nu_j} - u\|_{p,k_2} \to 0 \quad \text{when} \quad j \to \infty. \tag{38}$$

But from the properties of the topology of B_{p,k_2}, see 11, we obtain that

$$\|u_{\nu_j} - u\|_{\mathcal{S}'} \to 0 \quad \text{when} \quad \mathrm{j} \to \infty. \tag{39}$$

Since u necessarily has support contained in $\{x^0\}$, we have $u=\sum_{0\leq|\alpha|\leq m} c_\alpha \delta_{x^0}^{(\alpha)}$. We have two possibilities:

1. $u \equiv 0$, which is absurd, indeed $\|u_{\nu_j}\|_{p,k_2} \to \|u\|_{p,k_2}$ when $j \to \infty$ and $\|u_{\nu_j}\|_{p,k_2} \geq r > 0$ for every $j \in \mathbb{N}$
2. $u \not\equiv 0$, which is absurd, indeed a nontrivial linear combination of derivatives of the distribution δ_{x^0} does not belong to $B_{p,k}$ if $k \geq N > 0$.

□

Proof of Theorem 11. Fix a point $x^0 \in \mathbb{R}^n$, choose

$$\varphi \in C_0^\infty(\mathbb{R}^n), \ \operatorname{supp}\varphi \subset \overline{B_1(0)}, \varphi \equiv 1 \text{ in } \overline{B_{1/2}(0)} \tag{40}$$

and define

$$\varphi_\epsilon(x) = \varphi\left(\frac{x - x^0}{\epsilon}\right). \tag{41}$$

We also introduce the function $\psi \in C_0^\infty(\mathbb{R}^n)$ defined as $\psi(x) = \varphi(x - x^0)$. We observe that from the general theory of the linear partial differential operators with constant coefficients (see [13]) it follows that there exists a fundamental solution $T \in B_{\infty,\tilde{P}}^{loc}$ of the linear part $P(D)$ of the semilinear operator F.

In order to obtain our result we shall replace the weight function k by an equivalent function $h \in K$; obviously we have $B_{p,k} = B_{p,h}$. The function h will be determined later; Lemma 1 will play a crucial role in this connection (see Hörmander [14] Vol. II Theorem 13.3.3 for a similar argument). Let us define

$$B_{p,h;R} = \{u \in B_{p,h} / \|u\|_{p,h} \leq R\}.$$

We can consider the operator

$$\tilde{F}_\epsilon : B_{p,h;R} \to B_{p,h} \tag{42}$$

with $R \geq 2\|g\|_{p,h}$ given by the following espression

$$\tilde{F}_\epsilon[v] = g - \psi f(x, Q_1(D)(\varphi_\epsilon \psi L\psi v), ..., Q_M(D)(\varphi_\epsilon \psi L\psi v)) \tag{43}$$

where $L := T*$, T the fundamental solution of the operator $P(D)$.
Note that, from the hypothesis (28),

$$\frac{h(\xi)}{\frac{h(\xi)\tilde{P}(\xi)}{\tilde{Q}_i(\xi)}} \to 0 \quad \text{when} \quad |\xi| \to \infty \tag{44}$$

and so, for every $i = 1, .., M$, the injection of $B_{p,\frac{h\tilde{P}}{\tilde{Q}_i}} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)})$ into $B_{p,h}$ is compact. In Theorem 11 we have also supposed that $k(\xi) \geq N > 0$, then we obtain that

$$\frac{h(\xi)\tilde{P}(\xi)}{\tilde{Q}_i(\xi)} \geq C > 0 \quad \text{for every } \xi. \tag{45}$$

From (44) and (45) it follows that the temperate weight functions $\frac{h\tilde{P}}{\tilde{Q}_i}$ and h satisfy the hypothesis of the Theorem 13. Note that, if $v \in B_{p,h;R}$ then $\psi v \in B_{p,h} \cap \mathcal{E}'$; therefore $L\psi v \in B^{loc}_{p,h\tilde{P}}$ and $\psi L\psi v \in B_{p,h\tilde{P}}$ so we obtain $\tilde{F}_\epsilon[v] \in B_{p,h}$ and, let x^0 be a fixed point in $\mathbb{R}^n$, for every $i = 1, .., M$, there exists a function C_i of the variable ϵ such that $C_i(\epsilon) \to 0$ when $\epsilon \to \infty$ and

$$\|u\|_{p,\frac{h\tilde{P}}{\tilde{Q}_i}} \leq C_i(\epsilon)\|u\|_{p,h} \quad \forall u \in B_{p,h} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)}). \tag{46}$$

To prove Theorem 11 we will verify that there exists $\epsilon_0 > 0$ such that the corresponding operator defined by

$$\begin{gathered} \tilde{F}_{\epsilon_0} : B_{p,h;R} \to B_{p,h} \\ \tilde{F}_{\epsilon_0}[v] = g - \psi f(x, Q_1(D)(\varphi_{\epsilon_0} \psi L\psi v), ..., Q_M(D)(\varphi_{\epsilon_0} \psi L\psi v)) \end{gathered} \tag{47}$$

verifies the hypotheses of the Schauder Fixed Point Theorem, namely it is continuous, it is defined from $B_{p,h;R}$ to itself and it maps bounded sets into relatively compact sets. First of all we will prove that $\tilde{F}_\epsilon(B_{p,h;R})$ is relatively compact for every $\epsilon > 0$; to this end we will consider a sequence $\{v_j\}_{j\in\mathbb{N}}$ of distributions in $B_{p,h;R}$ and we will verify that there exists a convergent subsequence in $B_{p,k}$ of the sequence $\{\tilde{F}_\epsilon[v_j]\}_{j\in\mathbb{N}}$.
Set $\{u_j\}_{j\in\mathbb{N}} := \{\varphi_\epsilon \psi L\psi v_j\}_{j\in\mathbb{N}}$, then $\|u_j\|_{p,h\tilde{P}} \leq R_1(\epsilon)$. Indeed, by Theorem 3 and Theorem 4,

(48)
$$\begin{aligned} \|u_j\|_{p,h\tilde{P}} &= \|\varphi_\epsilon \psi(T * \psi v_j)\|_{p,h\tilde{P}} = \|\varphi_\epsilon \psi(\tilde{\psi} T * \psi v_j)\|_{p,h\tilde{P}} \leq C_\epsilon \|\tilde{\psi} T * \psi v_j\|_{p,h\tilde{P}} \\ &\leq C_{1,\epsilon}\|\tilde{\psi} T\|_{\infty,\tilde{P}}\|\psi v_j\|_{p,h} \leq C_{2,\epsilon}\|\tilde{\psi} T\|_{\infty,\tilde{P}}\|v_j\|_{p,h} \leq C_{3,\epsilon} R \end{aligned}$$

where $\widetilde{\psi} \in C_0^\infty$ and $\widetilde{\psi} \equiv 1$ in K= $\{x \in \mathbb{R}^n/(x + \mathrm{supp}\psi v_j) \cap \mathrm{supp}\varphi_\epsilon \psi \neq \emptyset\}$.
Set

$$\{E_i v_j\}_{i=1,...,M}=\{(E_1 v_j, ..., E_M v_j)\}:=\{Q(D)u_j\}=\{(Q_1(D)u_j, ..., Q_M(D)u_j)\};$$

by Theorem 2, we also obtain $\|E_i v_j\|_{p,\frac{h\widetilde{P}}{\widetilde{Q_i}}} \leq R_2(\epsilon)$ for every $i = 1, ..., M$ and $j \in \mathbb{N}$.
The sequences $\{E_i v_j\}_{j\in\mathbb{N}}$ are bounded in $B_{p,\frac{h\widetilde{P}}{\widetilde{Q_i}}}$ and belong to $\mathcal{E}'(\overline{B_\epsilon(x^0)})$, indeed $\{u_j\} \in \mathcal{E}'(\overline{B_\epsilon(x^0)})$ and the differential operators $Q_i(D)$ do not increase the support.
Then, according to the compactness of the injection of $B_{p,\frac{h\widetilde{P}}{\widetilde{Q_i}}} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)})$ into $B_{p,h}$, we may suppose that there exists a subsequence $\{v_{j_\nu}\}_{\nu\in\mathbb{N}}$ of $\{v_j\}_{j\in\mathbb{N}}$ and a distribution z_1 such that

$$\|E_1 v_{j_\nu} - z_1\|_{p,h} \to 0 \quad \text{when} \quad \nu \to \infty.$$

The sequence $\{E_2 v_{j_\nu}\}_{\nu\in\mathbb{N}}$ is a subsequence of $\{E_2 v_j\}_{j\in\mathbb{N}}$ and so is still bounded in $B_{p,\frac{h\widetilde{P}}{\widetilde{Q_2}}}$, then there exists $\{v_{j_{\nu_l}}\}_{l\in\mathbb{N}}$ and a distribution z_2 such that

$$\|E_2 v_{j_{\nu_l}} - z_2\|_{p,h} \to 0 \quad \text{when} \quad l \to \infty.$$

Iterating this process for every $i = 1, .., M$ we will find a subsequence $\{v_{j_m}\}_{m\in\mathbb{N}}$ of $\{v_j\}_{j\in\mathbb{N}}$ and M distributions $z_1, .., z_M$ such that

$$\|E_i v_{j_m} - z_i\|_{p,h} \to 0 \quad \text{when} \quad m \to \infty.$$

From the continuity of the injection of $B_{p,\frac{h\widetilde{P}}{\widetilde{Q_i}}} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)})$ into $B_{p,h}$, for every i, we may assume that the sequences $\{E_i v_{j_m}\}_{m\in\mathbb{N}}$ are bounded in $B_{p,h}$ and so $\|E_i v_{j_m}\|_{p,h} \leq R^*(\epsilon)$ for every $1 \leq i \leq M$ and $m \in \mathbb{N}$.
To complete the proof we have to verify that $\{\psi f(x, E_1 v_{j_m}, .., E_M v_{j_m})\}_{m\in\mathbb{N}}$ is convergent in $B_{p,h}$.
We will prove that the operator

$$\overline{F} : B_{p,h;R^*} \times B_{p,h;R^*} \times ... \times B_{p,h;R^*} \to B_{p,h} \tag{49}$$

defined as

$$\overline{F}[w^1, .., w^M] = \psi f(x, w^1, .., w^M) \tag{50}$$

is sequentially continuous, then

$$\begin{aligned}&\|\widetilde{F}_\epsilon[v_{j_n}] + \overline{F}[z_1, .., z_M] - g\|_{p,h} = \\ &= \|g - \overline{F}[E_1 v_{j_n}, .., E_M v_{j_n}] + \overline{F}[z_1, .., z_M] - g\|_{p,h}\end{aligned}$$

will tend to zero if $\|\overline{F}[E_1 v_{j_n}, .., E_M v_{j_n}] - \overline{F}[z_1, .., z_M]\|_{p,h}$ tends to zero.
Let $\{w_n\}_{n\in\mathbb{N}} := \{(w_n^1, ..., w_n^M)\}_{n\in\mathbb{N}} \in B_{p,h;R^*}\times\ ...\ \times B_{p,h;R^*}$ and $w := (w^1, .., w^M) \in$

$B_{p,h;R^*}\times \ldots \times B_{p,h;R^*}$ be such that $(w_n^1, \ldots w_n^M) \to (w^1, \ldots, w^M)$ for $n \to \infty$, we must show that

$$\overline{F}[w_n^1, \ldots, w_n^M] \to \overline{F}[w^1, \ldots, w^M]. \tag{51}$$

Reminding the Cavalieri Lagrange Formula, we have to estimate

$$\begin{aligned} &\overline{F}[w_n^1, \ldots w_n^M] - \overline{F}[w^1, \ldots w^M]\|_{p,h} = \\ &\|\psi f(x, w_n^1, .., w_n^M) - \psi f(x, w^1, .., w^M)\|_{p,h} = \\ &\|\psi \sum_{i=1}^{M} (w_n^i - w^i) \int_0^1 \partial_{v_i} f(x, w^1 + t(w_n^1 - w^1), .., w_n^M + t(w_n^M - w^M))dt\|_{p,h}. \end{aligned} \tag{52}$$

Set

$$G_i(x, y, z) := \int_0^1 \partial_{v_i} f(x, y^1 + t(z^1 - y^1), .., y^M + t(z^M - y^M))dt \tag{53}$$

and note that $G_i(x, y, z) \in C^\infty(\mathbb{R}_x^n, \mathcal{H}(\mathbb{C}^{2M}))$ for every $i = 1, .., M$.
Then $G_i(x, y, z) = \sum_{|\alpha|\geq 0} a_{i,\alpha}(x)(y, z)^\alpha$ where $a_{i,\alpha}(x) \in C^\infty(\mathbb{R}^n)$ and for every compact subset $\mathrm{H}\subset\subset \mathbb{R}^n$, there exist $C_\beta(\mathrm{H})$, $\lambda_\alpha(\mathrm{H}) > 0$ such that $\sup_{x\in \mathrm{H}}|\partial^\beta a_{i,\alpha}(x)| < C_\beta \lambda_\alpha$, being $\sum_{|\alpha|\geq 0} \lambda_\alpha(y, z)^\alpha$ an entire function.
Applying Theorem 7, we may further estimate (52) by

$$C \sum_{i=1}^{M} \|(w_n^i - w^i)\|_{p,h} \|\psi G_i(x, w_n, w)\|_{p,h}. \tag{54}$$

Set $\mathrm{H} := \operatorname{supp} \psi$, from the Fourier Transform properties we can deduce that there exist $s \in \mathbb{N}$ and $l \in \mathbb{N}$ such that

$$\|\psi a_{i,\alpha}\|_{p,h} \leq A(\mathrm{H}) \|\psi a_{i,\alpha}\|^l_{C^{(s)}}, \ \forall \alpha \in \mathbb{N}. \tag{55}$$

Indeed, let us consider $B_{p,(1+|\xi|^2)^{s'/2}}$ such that $s' \in \mathbb{N}$ and $B_{p,(1+|\xi|^2)^{s'/2}} \hookrightarrow B_{p,h}$; set $1/p + 1/q = 1$, if $p \geq 2$ then $1 \leq q \leq 2$ and the Sobolev space $W^{q,s'} = \{u \in \mathcal{S}'/\,\mathcal{F}^{-1}((1 + |\xi|^2)^{s'/2}\widehat{u}) \in L^q\}$ is included with continuity in $B_{p,(1+|\xi|^2)^{s'/2}}$. Then in this case

$$\begin{aligned} \|\psi a_{i,\alpha}\|_{p,h} &\leq \|\psi a_{i,\alpha}\|_{W^{q,s'}} = \sum_{|\beta|\leq s'} \|\partial^\beta(\psi a_{i,\alpha})\|_{L^q} \\ &\leq \sum_{|\beta|\leq s'} \left(\int_{\mathrm{K}} (\sup(|\partial^\beta(\psi a_{i,\alpha})|)^q)dx\right)^{1/q} \\ &= (\mathrm{meas}\mathrm{H})^{1/q} \sum_{|\beta|\leq s'} \|\partial^\beta(\psi a_{i,\alpha})\|_{C^0} = A\|\psi a_{i,\alpha}\|_{C^{(s')}} \end{aligned}$$

where $A := (\mathrm{measH})^{1/\mathrm{q}}$.
If $1 < p < 2$ we apply the Hölder inequality with $r = 2/p$:

$$\begin{aligned}
\|\psi a_{i,\alpha}\|_{p,h}^{p} &= \int |h(\xi)\widehat{\psi a_{i,\alpha}}(\xi)|^{p} d\xi \\
&= \int ((1+|\xi|^2)^{s''}|h(\xi)\widehat{\psi a_{i,\alpha}}(\xi)|)^{p}\left(\frac{1}{(1+|\xi|^2)^{s''}}\right)^{p} d\xi \\
&\le \left(\int ((1+|\xi|^2)^{s''}|h(\xi)\widehat{\psi a_{i,\alpha}}(\xi)|)^{2} d\xi\right)^{1/r}\left(\int \frac{1}{(1+|\xi|^2)^{s''pr'}}\right)^{1/r'} \\
&\le D\left(\int ((1+|\xi|^2)^{s^*}|\widehat{\psi a_{i,\alpha}}(\xi)|)^{2} d\xi\right)^{1/r} = D\|\psi a_{i,\alpha}\|_{H_{(s^*)}}^{p},
\end{aligned}$$

where r' is such that $1/r+1/r' = 1$, s'' is such that $s''pr' > n/2$ and $s^* = s'+s'' \in \mathbb{N}$.
But we have

$$\|\psi a_{i,\alpha}\|_{H_{(s^*)}} \le A'\|\psi a_{i,\alpha}\|_{C^{(s^*)}},$$

where $A' := \sqrt{\mathrm{measH}}$, so in this case we have get (55) with $l = p$.
Therefore applying the algebra property of $B_{p,h}$, the estimate (55) and the algebra property of $C^{(s)}$ we obtain

$$\begin{aligned}
\|\psi G_i(x, w_n, w)\|_{p,h} &\le A\sum_{\alpha\ge 0}\|\psi a_{i,\alpha}\|_{C^{(s)}}^{l}\|(y,z)^{\alpha}\|_{p,h} \\
&\le A'\|\psi\|_{C^{(s)}}^{l}\sum_{\alpha\ge 0}\|a_{i,\alpha}|_{\mathrm{K}}\|_{C^{(s)}}^{l}\left(C^{|\alpha|}(\|y\|_{p,h},\|z\|_{p,h})^{\alpha}\right) \\
&\le A''(\mathrm{K})\sum_{\alpha\ge 0}\max_{|\beta|\le s}\{C_{\beta}^{l}\}\lambda_{\alpha}(CR^*)^{|\alpha|} \\
&= A'''(\mathrm{K})\sum_{\alpha\ge 0}\lambda_{\alpha}(CR^*)^{|\alpha|} < D.
\end{aligned} \tag{56}$$

Replacing (56) in (54) we have

$$\|\overline{F}[w_n^1,...,w_n^M] - \overline{F}[w^1,...,w^M]\|_{p,h} \le D\sum_{i=1}^{M}\|w_n^i - w^i\|_{p,h}$$

that tends to zero when $\|w_n^i - w^i\|_{p,h} \to 0$ for every $i = 1, ..M$.
As second step of the proof, now we will prove that there exists $\epsilon_0 > 0$ such that the corresponding operator $\widetilde{F}_{\epsilon_0}$ is defined from $B_{p,h;R}$ to itself, namely let $v \in B_{p,h}$ such that $\|v\|_{p,h} \le R$ then $\|\widetilde{F}_{\epsilon_0}[v]\|_{p,h} \le R$.
We have to estimate

$$\|\widetilde{F}_{\epsilon}[v]\|_{p,h} \le \|\psi f(x, Q_1\varphi_\epsilon\psi L\psi v, .., Q_M\varphi_\epsilon\psi L\psi v)\|_{p,h} + \|g\|_{p,h}. \tag{57}$$

We now determine our choice of the equivalent function h (see the beginning of the proof). Namely we take, as a new equivalent weight, h such that $h\tilde{P}$ is equivalent to

$k\tilde{P}$, with k the original weight, and satisfies the estimate (14) in Lemma 1 with $\phi = \varphi_\epsilon$. We obtain

$$\|\varphi_\epsilon \psi L \psi v\|_{p,h\widetilde{P}} \le 2\|\varphi_\epsilon\|_{1,1}\|\psi L \psi v\|_{p,h\widetilde{P}} \le 2A\|\varphi_\epsilon\|_{1,1}\|v\|_{p,h\widetilde{P}}. \tag{58}$$

We introduce the function, Φ_ϵ defined as $\Phi_\epsilon(x) := \varphi(x/\epsilon)$ so from the Fourier Transform properties, we have that $\|\varphi_\epsilon\|_{1,1} = \|\Phi_\epsilon\|_{1,1}$, with φ,φ_ϵ as in (40) and (41). Noting that

$$\widehat{\Phi_\epsilon}(\xi) = \epsilon^n \widehat{\varphi}(\epsilon\xi) \tag{59}$$

we obtain

$$\begin{aligned} \|\varphi_\epsilon\|_{1,1} = \|\Phi_\epsilon\|_{1,1} = \|\widehat{\Phi}_\epsilon\|_{\mathrm{L}^1} &= \int |\epsilon^n \widehat{\varphi}(\epsilon\xi)| d\xi \\ &= \int |\widehat{\varphi}(z)| dz, \end{aligned} \tag{60}$$

so we have proved that $\|\varphi_\epsilon\|_{1,1}$ does not depend on ϵ.
Replacing (60) in (58), we get $\|\varphi_\epsilon \psi L \psi v\|_{p,h\widetilde{P}} \le A'\|v\|_{p,h} = R_3$ and so $\|Q_i \varphi_\epsilon \psi L \psi v\|_{p,\frac{h\widetilde{P}}{Q_i}} \le R_4$.
Applying Theorem 13 we obtain

$$\|Q_i \varphi_\epsilon \psi L \psi v\|_{p,h} \le C_i(\epsilon)\|Q_i \varphi_\epsilon \psi L \psi v\|_{p,\frac{h\widetilde{P}}{Q_i}} \le C_i(\epsilon) R_4$$

with $C_i(\epsilon) \to 0$ when $\epsilon \to 0$, for every $i = 1, .., M$.
Set $Q_i \varphi_\epsilon \psi L \psi v := E_{i,\epsilon} v$ and $C(\epsilon) := \max_{i=1,...,M}\{C_i(\epsilon)\}$, then $C(\epsilon) \to 0$ when $\epsilon \to 0$ and $\|E_{i,\epsilon} v\|_{p,h} \le C(\epsilon) R_4$. But $f(x, v) \in C^\infty(\mathbb{R}^n_x, \mathcal{H}(\mathbb{C}^M))$ with $f(x, 0) = 0$, then, by the same arguments used to obtain (56)

$$\begin{aligned} &\|\psi f(x, Q_1\varphi_\epsilon \psi L \psi v, .., Q_M \varphi_\epsilon \psi L \psi v)\|_{p,h} = \|\psi f(x, E_{1,\epsilon} v, .., E_{M,\epsilon} v)\|_{p,h} \\ &= \|\psi \sum_{|\alpha|>0} c_\alpha(x)(E_{1,\epsilon} v, .., E_{M,\epsilon} v)^\alpha\|_{p,h} \\ &\le \sum_{|\alpha|>0} C_1 \|\psi c_\alpha\|_{p,h}(\|(E_{1,\epsilon} v\|_{p,h}, .., \|E_{M,\epsilon} v\|_{p,h})^\alpha C^{|\alpha|} \\ &\le \sum_{|\alpha|>0} C_2 \lambda_\alpha (R_4 C(\epsilon), .., R_4 C(\epsilon))^\alpha C^{|\alpha|} = \sum_{|\alpha|>0} C_2 \lambda_\alpha (\widetilde{C}(\epsilon))^{|\alpha|} \\ &= \sum_{l>0} C_2 \mu_l (\widetilde{C}(\epsilon))^l \end{aligned} \tag{61}$$

where $\widetilde{C}(\epsilon) \to 0$ when $\epsilon \to 0$ and $\sum_{l>0} \mu_l v^l < +\infty$ for every $v \in \mathbb{C}$.
If we have chosen ϵ sufficiently small, then $\widetilde{C}(\epsilon) < 1$ and we may further estimate

$$\begin{aligned} \sum_{l>0} C_2 \mu_l (\widetilde{C}(\epsilon))^l &= C_2 \widetilde{C}(\epsilon) \sum_{l\ge 0} \mu_{l+1} (\widetilde{C}(\epsilon))^l \\ &\le C_2 \widetilde{C}(\epsilon) \sum_{l\ge 0} \mu_{l+1} = C_4 \widetilde{C}(\epsilon). \end{aligned} \tag{62}$$

Replace (61) and (62) in (57) and choose ϵ_0 such that $\widetilde{C}(\epsilon_0) < 1$ and $\widetilde{C}(\epsilon_0) \leq \frac{R-\|g\|_{p,h}}{C_4}$, then $\|\widetilde{F}_{\epsilon_0}[v]\|_{p,h} \leq R$ for every $v \in B_{p,h;R}$.
At the end it is easy to verify that the operator $\widetilde{F}_{\epsilon_0}$ defined from $B_{p,h;R}$ to itself is continuouos. We leave to the reader to check sequential continuity, by using the preceding arguments.
According to the Schauder Fixed Point Theorem we can assume that there exists a fixed point $v^0 \in B_{p,h;R}$ of the operator $\widetilde{F}_{\epsilon_0}$:

$$v^0 = g - \psi f(x, Q(D)(\varphi_{\epsilon_0}\psi L\psi v^0)).$$

Multiply this espession for the function ψ, by convolution properties we then deduce $PL(\psi v^0) = \psi v^0$, therefore

$$PL(\psi v^0) = \psi g - \psi^2 f(x, Q(D)(\varphi_{\epsilon_0}\psi L\psi v^0));$$

we obtain that the distribution $u^0 := L\psi v^0 \in B^{loc}_{p,h\widetilde{P}}$ satisfies the theorem, namely shrinking ϵ_0 we have

$$Pu^0 = g - f(x, Q(D)u^0)$$

in $\Omega = \{x \in \mathbb{R}^n / \|x - x^0\| < \epsilon_0\}$.

□

Proof of Theorem 12. Without loss of generality we suppose $\psi \in C^\infty(\mathbb{R}^n)$ and, for every multi–index β, there exists c_β such that $|\partial^\beta\psi(\xi)| \leq c_\beta\psi^{1-|\beta|}(\xi)$, cf. (24).

Fix a point $x^0 \in \mathbb{R}^n$, choose $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\varphi \subset \overline{B_1(x^0)}$, $\varphi \equiv 1$ in $\overline{B_{1/2}(x^0)}$ and $\varphi_\epsilon(x) = \varphi\left(\frac{x-x_0}{\epsilon}\right)$, and consider the operator

$$\widetilde{F}_\epsilon : H_{k;R} \to H_k \tag{63}$$

where $H_{k;R} = \{u \in H_k / \|u\|_{H_k} \leq R\}$, given by the following expression

$$\widetilde{F}_\epsilon(v) = g - \varphi f(x, Q_1(D)(\varphi_\epsilon L\varphi v), ..., Q_M(D)(\varphi_\epsilon L\varphi v)) \tag{64}$$

with $L := T*$, T the fundamental solution of the operator $P(D)$.
We will prove that, fixed $R \geq 2\|g\|_{H_k}$, there exists $\epsilon_0 > 0$ such that the corresponding operator $\widetilde{F_{\epsilon_0}}$ is defined from $H_{k;R}$ to itself and is a contraction, then we will apply the Contraction Priciple to the operator $\widetilde{F_{\epsilon_0}}$. ¸Let $v \in H_{k;R}$, set $u := \varphi_\epsilon L\varphi v$ and $w = (w_1, ..., w_M) = Q(D)u$ with $Q(D)u = (Q_1(D)u, ..., Q_M(D)u)$; we already noted in the proof of Theorem 11 that from Cavalieri Lagrange Theorem we get

$$\|\widetilde{F}_\epsilon(v^1) - \widetilde{F}_\epsilon(v^2)\|_{H_k} \leq C\sum_{j=1}^M \|(Q_j u^1 - Q_j u^2)\|_{H_k} \|(\varphi G_j(x, Qu^1, Qu^2)\|_{H_k} \tag{65}$$

where $G_j(x, v^1, v^2)$ is defined in (53) and belongs to $C^\infty(\mathbb{R}^n_x, \mathcal{H}(\mathbb{C}^{2M}))$ for $j = 1, ...M$. ¸With respect to the proof of Theorem 11 in which we estimated the second

term of the right-hand part of (65) obtaining $\|(\varphi_\epsilon \varphi G_j(x, Qu^1, Qu^2)\|_{H_k} \leq \epsilon D_3$, in the present case we will fix attention on the first term of the right-hand part of (65) and we expect to get from it the small constant ϵ.
Namely we will estimate the term

$$\|Q_j(D)u^1 - Q_j(D)u^2\|_{H_k} = \|Q_j(D)(u^1 - u^2)\|_{H_k}$$

for $j = 1, ..., M$ and for every $u^1, u^2 \in H_{k\widetilde{P}}$ such that $\operatorname{supp} u^1$, $\operatorname{supp} u^2$ are contained in $B_\epsilon(x^0)$.
From the Fourier trasform properties we get

$$\begin{aligned} \|Q_j(D)u\|_{H_k} &= \|k(\xi)\widehat{Q_j(D)}u(\xi)\|_{L^2} = \|Q_j(\xi)\widehat{k(D)u}(\xi)\|_{L^2} \\ &= \|Q_j(D)k(D)u\|_{L^2} \end{aligned} \tag{66}$$

where $k(D)$ has to be interpreted as pseudodifferential operator corresponding to the symbol $k(\xi)$.
Now let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\quad \chi \equiv 1$ on $\operatorname{supp} u$ and $\operatorname{supp} \chi \subset B_{2\epsilon}(x^0)$, then

$$\begin{aligned} \|Q_j(D)u\|_{H_k} &= \|Q_j(D)k(D)\chi u\|_{L^2} \\ &\leq \|Q_j(D)\chi k(D)u\|_{L^2} + \|Q_j(D)Au\|_{H_k} \end{aligned} \tag{67}$$

where we have denoted with A the operator $[k(D), \chi] = k(D)\chi - \chi k(D)$.
Remind the assumption $Q_j \prec\prec P$ for $j = 1, .., M$, i.e. there exists $C(\epsilon) \to 0$ for $\epsilon \to 0$ such that

$$\|Q_j(D)v\|_{L^2} \leq C(\epsilon)\|P(D)v\|_{L^2} \quad \forall v \in C_0^\infty(B_\epsilon(x^0))$$

and note that the support of $\chi k(D)u$ is contained in $B_{2\epsilon}(x^0)$, then it follows

(68)
$$\begin{aligned} \|Q_j(D)u\|_{H_k} &\leq C(2\epsilon)\|P(D)\chi k(D)u\|_{L^2} + \|Q_j(D)Au\|_{L^2} \\ &= C(2\epsilon)(\|P(D)k(D)\chi u\|_{L^2} + \|P(D)Au\|_{L^2}) + \|Q_j(D)Au\|_{L^2}. \end{aligned}$$

With the same considerations used to obtain (66), this can be further estimated by

$$C(2\epsilon)\|P(D)u\|_{H_k} + C(2\epsilon)\|P(D)Au\|_{L^2} + \|Q_j(D)Au\|_{L^2}. \tag{69}$$

Now we have to estimate the terms $\|P(D)Au\|_{L^2}$, $\|Q_j(D)Au\|_{L^2}$.
From the assumption $k = \psi^r$, it follows $k \in S^r_\psi$ and $\chi \in S^0_\psi$, see (22), then, denoting with $a(x, \xi)$ the symbol of the pseudodifferential operator $A = a(x, D)$, from Theorem 10 about the symbolic calculus we get

$$a(x, \xi) = k(\xi)\chi(x) + b(x, \xi) - \chi(x)k(\xi) + c(x, \xi)$$

where $b, c \in S^{r-1}_\psi$, and so $a(x, \xi) \in S^{r-1}_\psi$. We can write

$$\|P(D)Au\|_{L^2} \leq \|AP(D)u\|_{L^2} + \|[P, A]u\|_{L^2},$$

but, from (26) we get

$$\|AP(D)u\|_{L^2} \leq C\|P(D)u\|_{H_{\psi^{r-1}}} \leq C_1\|u\|_{H_{\psi^{r-1}\widetilde{P}}}$$

where we have also applied the properties of the $B_{p,k}$ spaces.
Reminding the condition (1) in (21), we have

$$\psi^{-1}(\xi) \leq c(1+|\xi|)^{-\rho} \quad \text{with} \quad \rho > 0$$

and from the assumption $k = \psi^r$ we obtain

$$\psi^{r-1}(\xi) \leq ck(\xi)(1+|\xi|)^{-\rho}$$

and so

$$\frac{\widetilde{P}(\xi)\psi^{r-1}(\xi)}{\widetilde{P}(\xi)k(\xi)} \to 0 \quad \text{when} \quad |\xi| \to \infty;$$

from Theorem 13 it follows that

$$\|u\|_{H_{\psi^{r-1}\widetilde{P}}} \leq D(\epsilon)\|u\|_{H_{k\widetilde{P}}} \quad \forall u \in H_{k\widetilde{P}} \cap \mathcal{E}'(B_\epsilon(x^0)) \tag{70}$$

with $D(\epsilon) \to 0$ for $\epsilon \to 0$, then

$$\|AP(D)u\|_{L^2} \leq C_1 D(\epsilon)\|u\|_{H_{k\widetilde{P}}}.$$

From Theorem 10 it also follows

$$\|[P, A]u\|_{L^2} \leq C_2 \sum_\alpha \|(D_x^\alpha a)(x, D)(D_\xi^\alpha P)(D)\|_{L^2}$$

where the sum is finite because P is a polynomial in the variable ξ. Now we note that

$$(D_x^\alpha a)(x,\xi) \in S_\psi^{r-1} \quad \forall \alpha \in \mathbb{N}^n$$

therefore

$$\begin{aligned} \|(D_x^\alpha a)(x, D)(D_\xi^\alpha P)(D)u\|_{L^2} &\leq C_3\|(D_\xi^\alpha P)(D)u\|_{H_{\psi^{r-1}}} \\ &\leq C_4\|u\|_{H_{\psi^{r-1}\widetilde{D_\xi^\alpha P}}} \quad \forall \alpha \in \mathbb{N}^n. \end{aligned} \tag{71}$$

But $\widetilde{D_\xi^\alpha P}(\xi) \leq \widetilde{P}(\xi)$ for every $\xi \in \mathbb{R}^n$, then

$$\|u\|_{H_{\psi^{r-1}\widetilde{D_\xi^\alpha P}}} \leq C_5\|u\|_{H_{\psi^{r-1}\widetilde{P}}}.$$

From the same considerations used to obtain (70) it follows

$$\|[P, A]u\|_{L^2} \leq C_5 D(\epsilon)\|u\|_{H_{k\widetilde{P}}}$$

and so

$$\|P(D)Au\|_{L^2} \leq C_1(\epsilon)\|u\|_{H_{k\widetilde{P}}}$$

with $C_1(\epsilon) \to 0$ when $\epsilon \to 0$. In analogous way we estimate

$$\|Q_j(D)Au\|_{L^2} \leq \|AQ_j(D)u\|_{L^2} + \|[Q_j, A]u\|_{L^2} \leq C_2(\epsilon)\|u\|_{H_{k\widetilde{Q}_j}};$$

but, from the assumption $Q \prec\prec P$ it follows $\frac{\widetilde{Q}_j}{\widetilde{P}} < const.$, therefore, applying the properties of the $B_{p,k}$ spaces, we get

$$\|Q_j(D)Au\|_{L^2} \leq C_3(\epsilon)\|u\|_{H_{k\widetilde{P}}}. \tag{72}$$

In conclusion we have obtained that there exists $g_1(\epsilon)$ such that $g_1(\epsilon) \to 0$ when $\epsilon \to 0$ and

$$\|Q_j(D)u\|_{H_k} \leq g_1(\epsilon)\|u\|_{H_{k\widetilde{P}}} \tag{73}$$

for $j = 1, ..., M$ and every $u \in H_{k\widetilde{P}} \cap \mathcal{E}'(B_\epsilon(x^0))$. Replacing (73) in (65) we have

$$\begin{aligned} \|\widetilde{F}_\epsilon(v^1) - \widetilde{F}_\epsilon(v^2)\|_{H_k} &\leq g(\epsilon)\sum_{j=1}^{M}\|u^1 - u^2\|_{H_{k\widetilde{P}}}\|G_j(x, Qu^1, Qu^2)\|_{H_k} \\ &\leq g_1(\epsilon)\sum_{j=1}^{M}\|\varphi_\epsilon L\varphi v^1 - \varphi_\epsilon L\varphi v^2\|_{H_{k\widetilde{P}}}\|G_j(x, Qu^1, ..., Qu^2)\|_{H_k}. \end{aligned} \tag{74}$$

In the proof of Theorem 11 we have already proved, see (58) and (60), that

$$\|\varphi_\epsilon L\varphi v\|_{H_{k\widetilde{P}}} \leq A\|v\|_{H_k} \tag{75}$$

with A independent of ϵ, then

$$\|\widetilde{F}_\epsilon(v^1) - \widetilde{F}_\epsilon(v^2)\|_{H_k} \leq g_2(\epsilon)\|v^1 - v^2\|_{H_k}\sum_{j=1}\|G_j(x, Qu^1, Qu^2)\|_{H_k}$$

with $g_2(\epsilon) \to 0$ for $\epsilon \to 0$. But $v^1, v^2 \in H_{k,R}$, then we have that if $\|v\|_{H_k} \leq R$ then $\|Q_i u\|_{H_k} \leq R_1$ for $i = 1, ..., M$ and

$$\|G_j(x, Qu^1, Qu^2)\|_{H_k} \leq D \quad \text{for} \quad j = 1, ..., M;$$

therefore

$$\|\widetilde{F}_\epsilon(v^1) - \widetilde{F}_\epsilon(v^2)\|_{H_k} \leq g_3(\epsilon)\|v^1 - v^2\|_{H_k}$$

with $g_3(\epsilon) \to 0$ for $\epsilon \to 0$. We can choose ϵ_1 such that for $\epsilon \leq \epsilon_1$ we have $g_3(\epsilon) < 1$ and $\widetilde{F}_\epsilon$ is then a contraction. Now we will prove that there exists $\epsilon_2 > 0$ such that for $\epsilon \leq \epsilon_2$ the corresponding operator $\widetilde{F}_\epsilon$ is defined from $H_{k,R}$ to itself, i.e. let v such that $\|v\|_{H_k} \leq R$ for $\epsilon \leq \epsilon_2$, then $\|\widetilde{F}_\epsilon v\|_{H_k} \leq R$. We have to estimate

$$\begin{aligned} \|\widetilde{F}_\epsilon\|_{H_k} &\leq \|\varphi f(x, Q_1(\varphi_\epsilon L\varphi v), ..., Q_M(\varphi_\epsilon L\varphi v))\|_{H_k} + \|g\|_{H_k} \\ &\leq \left\|\varphi\sum_{|\alpha|>0} c_\alpha(x)(Q_1(\varphi_\epsilon L\varphi v), ..., Q_M(\varphi_\epsilon L\varphi v))^\alpha\right\|_{H_k} + \|g\|_{H_k}. \end{aligned} \tag{76}$$

Applying (61) and (62) and with the same considerations used to obtain (73) we get

$$\|\varphi f(x, Q_1(\varphi_\epsilon L\varphi v), ..., Q_M(\varphi_\epsilon L\varphi v))\|_{H_k} \leq \widetilde{D}R_1 g_1(\epsilon)$$

where we have already chosen ϵ such that $R_1 g_1(\epsilon) < 1$. Consider the constant ϵ_2 such that $R_1 g_1(\epsilon) \leq \frac{R}{2D}$ for $\epsilon \leq \epsilon_2$, then $\|\widetilde{F}_\epsilon v\|_{H_k} \leq R$ for every $v \in H_{k;R}$. Choosing $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$ we have that the corresponding operator $\widetilde{F_{\epsilon_0}}$ is defined from $H_{k;R}$ to itself and is a contraction. According to the Contraction Principle we can assume that there exists a fixed point $v^0 \in H_{k;R}$ of the operator $\widetilde{F_{\epsilon_0}}$ and we may conclude as in the proof of Theorem 11.

□

Acknowledgement. The author is thankful to Professor Luigi Rodino for usuful discussions on the subject of this paper and to the referee for his kind and clear suggestions and recommendations to improve and complete this paper.

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AMS Subject Classification: 35E99.

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Lavoro pervenuto in redazione il 20.06.2002 e, in forma definitiva, il 05.05.2003.

Rend. Sem. Mat. Univ. Pol. Torino
Vol. 61, 1 (2003)

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A NONLINEAR OSCILLATOR MODELING THE TRANSVERSE VIBRATIONS OF A ROD

Abstract. This paper has been originated by the motion analysis of a flexible rod whose extreme (pivoted) sections cannot undergo any displacement, but only rotate. First an historical outline of the rod tranverse vibration analysis is provided. The dynamical elastica of such a system can be easy expressed through a linear fourth order PDE solved by a Fourier series: the rod resulting movement is then known by armonical superposition, but the overall vibration rod's period is analytically unexpressed.

The authors propose an alternative analytic treatment designing a discrete model, made of a single degree of freedom oscillator, ruled by a nonlinear second order ODE, here solved through the elliptic integrals of first and third kind. This greater complexity is balanced by having found a closed formula for computing the overall vibration period by means of a suitable punctual oscillator.

1. Introduction

Physical systems are characterized by their state variable(s) depending both on time and space.

In Mechanics these models are obtained whenever the first mechanical approach of rigid bodies or mass points (*discrete* models controlled by ordinary differential equations, ODE) are replaced by the more realistic *continuous* elastic bodies, ruled by partial differential equations (PDE).

Nevertheless we will follow the reverse path: this paper will start introducing first a continuous deformable body formed by a doubly hinged flexible rod whose extreme (pivoted) sections cannot undergo any displacement, but only rotate. The dynamical elastica of such a system can be easy found (initial/boundary value problem) through a Fourier series, allowing to know the shape and the period of each (i $^{\text{th}}$) of the infinite armonics. We will know the rod resulting movement - due to the equation's linearity - by armonical superposition, *but the overall period of the rod's transverse vibration would be analytically unexpressed*. Therefore an alternative analytic treatment is proposed here, designing a discrete model of the continuous phenomenology. Whilst this latter involves a linear fourth order elementary integrable PDE, the former will imply

*Research supported by MURST grant: *Metodi matematici in economia*

a nonlinear second order ODE, exactly solved first time by us through the elliptic integrals. This greater complexity is balanced by having obtained a closed formula for the overall oscillation period via algebraic expressions including higher transcendental functions. The new problem which we arrive at, is more tractable as far as it concerns the period, which we succeed to close in a formula. The use of the special functions of Mathematical Physics for solving some nonlinear differential equations of dynamics is at the moment a stable research interest of the authors' activity [6, 7]. Notice that the elliptic integrals of the third kind, frequently met in planar Mathematical Physics, are seldom seen in problems of particle motion.

The authors are indebted with their friend professor Aldo Scimone who drew the first three pictures of this paper: they take here the opportunity for thanking him warmly. Finally we also thank the Referee for her (or his) fruitful advice.

2. Historical account of the rod transverse vibration analysis

The motion of the flexible or elastic bodies was studied before 1750 on the basis of special hypotheses (of load and/or geometry) by B. Taylor (1685–1731), Jakob Bernoulli (1654–1705), Daniel Bernoulli (1700–1782), L. Euler (1707-1783) without knowing neither the vibrating string PDE, nor that governing the *transverse* vibrations of a rod. This prevented any mathematical treatment of initial value problems in elasticity before that date. Thus, when J. D'Alembert (1717-1783) established his PDE for the vibrating string,* this had a sensational effect, because he applied the Newton's second law to a continuous body, and the Taylor's truncated formula for the restoring force.

Euler's *Decouverte d'un nouveau principe de Mécanique*† sets down the newtonian principle in the (now familiar) triple scalar form: $m\ddot{x}_i = F_{xi}$, $i = 1, 2, 3$ for the particles. And there it is stated explicitly that, passing to a continuous body, m and F have to be replaced by the differential elements dm and dF.

Let us go now to the transverse oscillations of a flexible rod: even if most of Euler's immediate effort was directed to the rigid bodies and the perfect fluids, nevertheless (from his notebooks) he is believed to have known the PDE ruling the small transverse vibrations $y(x, t)$ of an elastic rod:

$$\frac{1}{c^4}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} = 0 \tag{1}$$

which was to be published by himself several years later. All this is based upon a well known ‡ Truesdell's authoritative study; but the authors of this article did not succeed to see any edition of the notebooks. After a careful screening of Euler's collected works,

*J. Le Rond D'Alembert: *Recherches sur la courbe que forme une corde tendue mise en vibration*, Histoire de l'Academie de Berlin (III), 1747, published in 1749.

†The paper was presented -and presumably read- on sept. 3rd, 1750, at Berlin Academy and is nowadays in vol 5th series 2, of Euler's Collected Works.

‡C. A. Truesdell: *The Rational Mechanics of flexible or elastic bodies, 1638-1788*, (embodied in Euler's Collected Works, introduction to the vol. 10 and 11 of series 2), Zürich 1960.

they detected the equation's first official appearance in the paper *De motu vibratorio laminarum elasticarum ubi plures novae vibrationum species hactenus non pertractatae evolvuntur*[§] . Euler, equating the effects of acceleration and forces, writes (1) under the form:

$$-\left(\frac{d^2 y}{dt^2}\right) + c^4 \left(\frac{d^4 y}{dx^4}\right) = 0,$$

finding the fundamental solution involving circular and exponential functions through six constants.

3. Free vibrations of a doubly hinged rod

Not many continuous systems are known whose vibrational behaviour may be analyzed exactly. This collection includes a string in transverse vibration, a rod in longitudinal or torsional vibration, a beam in transverse ("flexional") vibration, and -finally- certain simply shaped membranes or plates. Such systems, differently linked, have uniform cross section and material, and they vibrate according to linear PDEs in one (or two) spatial dimensions and in the time.

Let us take an uniform rod of length l, volumic mass ρ, cross section area S and $y(x, t)$ be the dynamic deflection of each $x-$point of it, in the time. By considering the motion of a thin slice of the beam, and neglecting the shearing force (which gives a very small contribution to y) as compared to the bending moment, the Euler equation (1) can be written:

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial x^4} = 0, \quad a^2 = \frac{EJ}{S\rho}. \tag{2}$$

E is the Young modulus, and J the second moment of S about the neutral axis through the centroid. The independent variables are the time t and the coordinate x along the rod axis, while the state dependent variable is the transverse displacement:

$$y = y(t, x) : [0, T] \times [0, l] \to \mathbb{R}.$$

It is common practice, [10] pages 137-138, to search the stationary vibrations, namely those not moving along the rod: in such a way the nodes -where the displacement is zero- are fixed. The same happens for those points (antinodes) having the highest displacements. This implies we are looking for a *separable* solution:

$$y(x, t) = X(x)T(t),$$

leading the equation (2) to split into two different ODEs. A fundamental system of

[§]The paper presented in 1772, but published in 17^{th} volume of Novi Commentarii Academiae Scientiarum Petropolitanae of 1773, can now be read in Euler Collected Works, series 2, vol. 11, first part, pages 112/141.

solutions of (2), accordingly, will be:

$$
\begin{aligned}
y(x,t) &= \Big(C_1\cos(hx)+C_2\sin(hx)+C_3\cosh(hx)+C_4\sinh(hx)\Big)\times \\
&\times \Big(A\cos(ft)+B\sin(ft)\Big), \qquad (3)
\end{aligned}
$$

where h and f are two constants homogeneous to the inverse of a length and of a time, respectively. The boundary conditions of a doubly hinged cantilever for $t \geq 0$, are usually found as a consequence of how the rod is linked. In our case the displacement and bending moment shall be zero at both the pivoted points:

$$
\begin{cases}
x=0, & y=0,\\
x=0, & \dfrac{\partial^2 y}{\partial x^2}=0,\\
x=l, & y=0,\\
x=l, & \dfrac{\partial^2 y}{\partial x^2}=0.
\end{cases} \qquad (4)
$$

Applying (4) to (3), one obtains $C_1 = C_3 = C_4 = 0$, and then y will be given by the series:

$$y(x,t)=\sum_{i=1}^{\infty} y_i=\sum_{i=1}^{\infty}\sin(p_i x)\left[A_i\cos(\omega_i t)+B_i\sin(\omega_i t)\right], \qquad (5)$$

with:

$$\omega_i = a\frac{\pi^2 i^2}{l^2} \quad \text{and} \quad a=\sqrt{\frac{EJ}{S\rho}},$$

where $p_i = i\,\pi\,l^{-1}$, $i = 1, 2, 3, \ldots$ and x marks whichever point along the rod. The constants A_i and B_i shall be determined by the initial conditions:

$$
\begin{cases}
y(x,0)=y_0,\\
\dfrac{\partial y}{\partial t}(x,0)=v_0.
\end{cases}
$$

And then:

$$y_0=\sum_{i=1}^{\infty}A_i\sin(p_i x); \qquad v_0=\sum_{i=1}^{\infty}p_i B_i\sin(p_i x).$$

In such a way the search of A_i and B_i has been driven to detect the coefficients of the Fourier sine expansions of the constant quantities y_0 and v_0, as it will be done soon.

The Fourier series uniform convergency entitles the term by term integration. Let us start from:

$$y(x,0)=\sum_{i=1}^{\infty}A_i\sin(p_i x). \qquad (6)$$

It shall be:

$$\begin{cases} y(x,0) = y_0, & \frac{l}{2} - \frac{s}{2} \le x \le \frac{l}{2} + \frac{s}{2}, \\ y(x,0) = 0, & \text{elsewhere,} \end{cases}$$

where s is the partial length of the rod, centered on $x = \frac{l}{2}$, modeling the rod's mean point, see the attached figure.

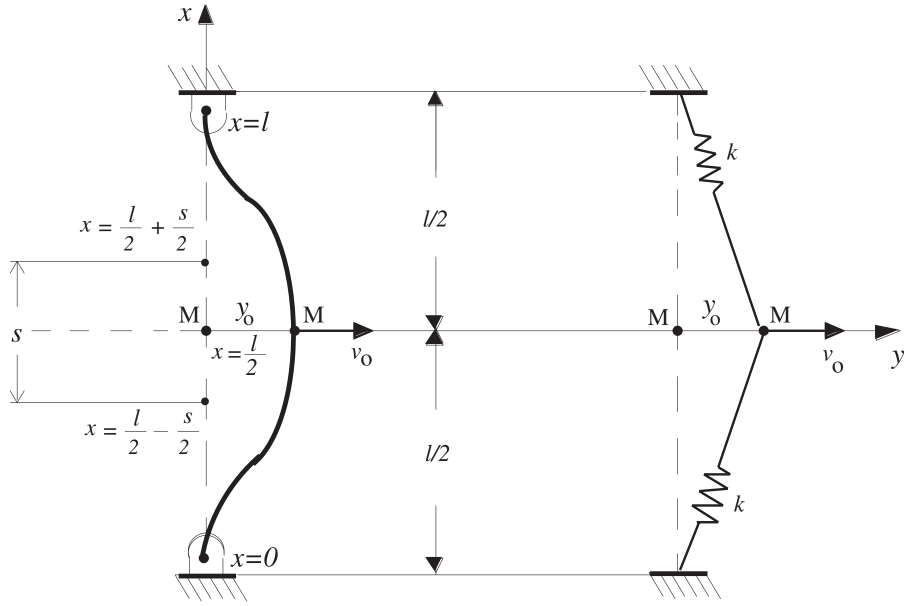

Figure 1: The rod and the oscillator.

Let us multiply both sides of (6) to $\sin\left(\frac{i\pi x}{l}\right)$ and integrate respect to x from 0 to l. One will find:

$$y_0 \int_{\frac{l}{2}-\frac{s}{2}}^{\frac{l}{2}+\frac{s}{2}} \sin\left(\frac{i\pi x}{s}\right) dx = A_i \int_0^l \sin^2\left(\frac{i\pi x}{l}\right) dx,$$

and A_i is then available. Making the same with the v_0 sine expansion, we will find, for any $i = 1, 2, \ldots$:

$$\begin{aligned} A_i &= (-1)^i \frac{4sy_0}{(2i+1)\pi} \sin\left[\frac{(2i+1)\pi l}{2s}\right], \\ B_i &= (-1)^i \frac{4sv_0}{(2i+1)\pi p_i} \sin\left[\frac{(2i+1)\pi l}{2s}\right], \\ p_i &= \frac{i\pi}{l}. \end{aligned}$$

Obviously the rod piece of length s, involved in the initial motion, is a very small portion of the rod. For example, s can be assumed to be $\frac{2l}{100}$, and so on. We are interested in the rod's mean point M motion: then, putting $x = \frac{l}{2}$ in (5), one will get it through the series:

$$y\left(\frac{l}{2}, t\right) = \sum_{i=1}^{\infty} (-1)^{2i} \left[A_i \cos(\omega_i t) + B_i \sin(\omega_i t)\right]. \tag{7}$$

4. From a continuous back to a discrete moving system

At the previous section the problem has been solved of computing by (5) the motion of each point of a continuous body made of a doubly hinged rod, loaded by an imposed motion, namely the *initial* displacement and speed of its middle point. The motion consists of infinite armonics, each having an i^{th} period T_i given by:

$$T_i = \frac{2\pi}{\omega_i} = \frac{2l^2}{\pi i^2}\sqrt{\frac{S\rho}{EJ}}, \tag{8}$$

which is going down for increasing i. But there is no way for appreciating by (8) the *overall* vibration period, and this gives a (quite strong) motivation for trying an approximate model of the continuous body, via a lumped parameter approach.

D. Bernoulli and Euler passed without hesitation from finite systems of particles to continuous systems, by thinking the latter made of infinitely many particles and believing that a proposition, holding for every finite n, would obviously keep its validity for $n \to \infty$.

We are doing the reverse: wishing the motion of one point only of the continuous body, we will focus it, whereas the remaining part of the rod will be simulated through a linear (but inclined) elastic spring without mass.

The mean point is only subjected to an elastic traction T_E given by:

$$T_E = k\,(l - b)$$

where k is the spring elastic (unknown) characteristic and b the initial length of the strained (half) rod:

$$\left(\frac{l}{2}\right)^2 + y_0^2 = b^2. \tag{9}$$

But the Hooke's law:

$$T_E = S\sigma = SE\varepsilon = \frac{SE(l-b)}{b},$$

where σ is the axial pressure, allows to know the equivalent spring characteristic as $k = \frac{SE}{b}$, with b given by (9). The point mass will be taken as that of the whole rod:

$$m = Sl\rho.$$

Then we have translated the previous model in a formulation ruled by (see fig. 2):

$$a = \frac{l}{2}, \qquad b = \sqrt{\left(\frac{l}{2}\right)^2 + y_0^2},$$
$$m = Sl\rho, \qquad k = \frac{SE}{\sqrt{\left(\frac{l}{2}\right)^2 + y_0^2}}.$$

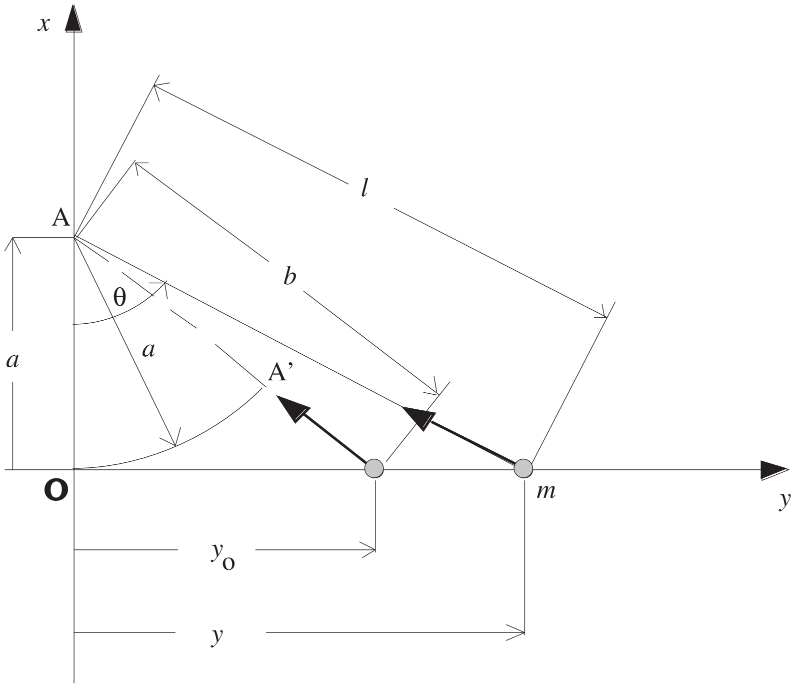

Figure 2: The oscillator's mathematical scheme.

The motion of the $m-$point along the straight line assumed as y axis, is then described by:

$$\begin{cases} m\ddot{y} + T_E \sin\vartheta = 0, \\ y(0) = y_0, \\ \dot{y}(0) = v_0. \end{cases}$$

For any y we have:

$$T_E = k\left(\sqrt{a^2 + y^2} - b\right), \qquad \sin\vartheta = \frac{y}{\sqrt{a^2 + y^2}},$$

then we obtain a second order, nonlinear, initial value problem:

$$\begin{cases} m\ddot{y} + ky\left(1 - \dfrac{b}{\sqrt{a^2+y^2}}\right) = 0, \\ y(0) = y_0, \\ \dot{y}(0) = v_0. \end{cases} \tag{10}$$

The discrete model (10) approaches the continuous phenomenology which, treated as of infinite degrees of freedom, had previously led us to a Fourier series solution.

The discrete model (10) is a continuous one! It shouldn't be confused with those computational methods developed in order to obtain a discretization of a continuous model. They last, properly termed *discretized models*, are designed for being treated numerically, and are out of the authors' interest. The discrete model now introduced, will allow an analytical treatment of the period of the solution (5) .

5. Transverse oscillator equation's integration

The (10) is a second order, nonlinear ODE, modeling *an oscillator* named *transverse* for being its motion directed differently from the elastic force, namely a one-dimensional motion of a particle under a linear elastic force, *but whose direction and intensity are both continuously varying* as the motion itself. Then the problem is a nonlinear one, as a consequence of the nonlinearity in y due to both T_E and $\sin\vartheta$. Of course if $b\,\left(a^2+y^2\right)^{-1/2} << 1$, one would obtain back the elementary armonic case. The structure of (10) enables us to refer to the Weierstrass theorem, [11]. We are faced with a second order autonomous ODE:

$$\begin{cases} \ddot{y} = f(y), \\ y(0) = y_0, \\ \dot{y}(0) = v_0. \end{cases}$$

By the standard Weierstrass method we can write the time equation as:

$$t = \int_{y_0}^{y} \frac{ds}{\sqrt{\Phi(s)}}, \tag{11}$$

with:

$$\Phi(y) = 2\int_{y_0}^{y} f(s)ds + v_0^2,$$

and where the sign either + or - has to be taken, according to the sign of the initial speed v_0, or, if $v_0 = 0$ accordingly with $f(y_0)$ sign, as it is well known, see e.g. [8] page 114 or [1] pages 287-292. The $\Phi = 0$ roots' existence and nature, marks completely the motion, deciding its periodic or aperiodic nature. Furthermore the reality condition $\Phi \geq 0$ must be met. We have:

$$f(s) = -\frac{k}{m}s\left(1 - \frac{\sqrt{a^2+y_0^2}}{\sqrt{a^2+s^2}}\right),$$

and then:

$$\Phi(y) = 2\,\frac{k}{m}\,\sqrt{a^2+y^2}\,\sqrt{a^2+y_0^2} - \frac{k}{m}\,\left(2\,a^2+y^2+y_0^2\right) + v_0^2.$$

Solving the equation $\Phi(y) = 0$, we find the four roots:

$$y_\pm = \pm\sqrt{y_0^2 - \frac{2v_0\,\sqrt{m}\,\sqrt{a^2+y_0^2}}{\sqrt{k}} + \frac{m\,v_0^2}{k}}. \tag{12}$$

$$Y_\pm = \pm\sqrt{y_0^2 + \frac{2\,v_0\sqrt{m}\sqrt{a^2+y_0^2}}{\sqrt{k}} + \frac{m\,v_0^2}{k}}, \tag{13}$$

They will be real and distinct,¶ if:

$$\frac{m\,v_0}{k} + {y_0}^2 - \frac{2\,\sqrt{m}\,\sqrt{v_0}\,\sqrt{a^2+{y_0}^2}}{\sqrt{k}} > 0, \tag{14}$$

then the roots marked by (12) have an absolute value less than (13). Furthermore, if, jointly with the roots' reality condition (14), we have:

$$\sqrt{y_0^2 - \frac{2v_0\,\sqrt{m}\,\sqrt{a^2+y_0^2}}{\sqrt{k}} + \frac{m\,v_0^2}{k}} < y_0 \iff \frac{\sqrt{\dfrac{m\,v_0^2-4\,a^2}{k}}}{2} < y_0, \tag{15}$$

then the particle will oscillate bounded between the limits:

$$\sqrt{y_0^2 - \frac{2v_0\,\sqrt{m}\,\sqrt{a^2+y_0^2}}{\sqrt{k}} + \frac{m\,v_0^2}{k}} \le y \le \sqrt{y_0^2 + \frac{2\,v_0\sqrt{m}\,\sqrt{a^2+y_0^2}}{\sqrt{k}} + \frac{m\,v_0^2}{k}},$$

namely between y_+ and Y_+. If (15) is not true, but (14) is met, then the motion would take place between y_- and y_+. If, finally, (14) is not true, the motion will take place between Y_- and Y_+.

We are going to perform the integration of (10), assuming that the initial conditions y_0, v_0 meet both (14) and (15). With some handling, (11) can be written:

$$t = \int_{y_0}^{y} \frac{ds}{\sqrt{2\,\frac{k}{m}\,\sqrt{a^2+s^2}\,\sqrt{a^2+y_0^2} - \frac{k}{m}\,\left(2\,a^2+s^2+y_0^2\right) + \frac{v_0^2}{m}}}, \tag{16}$$

¶Of course the reality test (14) ensures the reality of both the roots $y_\pm$ of (12), because the roots $Y_\pm$ of (13) are always real.

with $y_+ \leq y \leq Y_+$. The oscillation half-period $\frac{T}{2}$ will follow integrating between the rest points y_+ e Y_+ :

$$\frac{T}{2}=\int_{y_+}^{Y_+}\frac{ds}{\sqrt{2\,\frac{k}{m}\,\sqrt{a^2+s^2}\,\sqrt{a^2+y_0^2}-\frac{k}{m}\,\left(2\,a^2+s^2+y_0^2\right)+\frac{v_0^2}{m}}}. \tag{17}$$

Let us put in (17) a new variable $s=\sqrt{z^2-a^2}$:

$$T=2\sqrt{\frac{m}{k}}\int_{z_1}^{z_2}\frac{z}{\sqrt{\left(z^2-a^2\right)\left(-z^2+2\sqrt{a^2+y_0^2}\,z+\frac{m}{k}v_0^2-y_0^2-a^2\right)}}\,dz, \tag{18}$$

where z_1 and z_2 are the roots, always real, of the polynomial $p(z)=-z^2+2\sqrt{a^2+y_0^2}\,z+\frac{m}{k}v_0^2-y_0^2-a^2$, i.e.:

$$z_1=\frac{\sqrt{m}\,v_0-\sqrt{k}\,\sqrt{a^2+y_0^2}}{\sqrt{k}},\quad z_2=\frac{\sqrt{m}\,v_0+\sqrt{k}\,\sqrt{a^2+y_0^2}}{\sqrt{k}}. \tag{19}$$

The period integral (18) is then of the type:

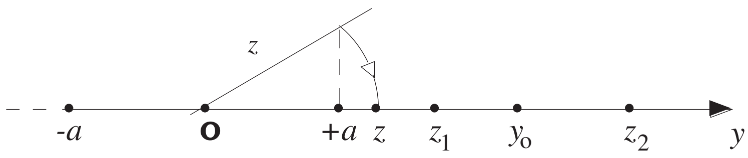

Figure 3: Relative position of z_1 and z_2.

$$\int_{z_1}^{z_2}\frac{z}{\sqrt{-\left(z-z_1\right)\left(z-z_2\right)\left(z^2-a^2\right)}}\,dz.$$

These integrals involve the elliptic functions: e.g. one can see the formulas of [2] page 125, no. 257.11 and page 205 no. 340.01. We will use those numbered as 3.148 6 and 7 of [3] at page 243, coming always from [2]. $F\,(k,\varphi)$ and $\Pi\,(k,\varphi,n)$ are the incomplete elliptic integrals of the first and third kind:

$$F\,(k,\varphi)=\int_0^{\varphi}\frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}},$$
$$\Pi\,(k,\varphi,n)=\int_0^{\varphi}\frac{d\vartheta}{\left(1+n\sin^2\varphi\right)\sqrt{1-k^2\sin^2\vartheta}},$$

being $|k|<1$ the *modulus* and φ the *amplitude* of both F and Π, and $-\infty<n<+\infty$ the *parameter* of Π.

If $\delta < \gamma < \beta < \alpha$, let us define:

$$\lambda = \arcsin\sqrt{\frac{(\alpha-\gamma)(u-\beta)}{(\alpha-\beta)(u-\gamma)}}, \qquad \mu = \arcsin\sqrt{\frac{(\beta-\delta)(\alpha-u)}{(\alpha-\beta)(u-\delta)}},$$

$$r = \sqrt{\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\beta-\delta)}},$$

then, by form. 3.148 6 and 7 [3], we get:

$$\int_\beta^u \frac{x}{\sqrt{(\alpha-x)(x-\beta)(x-\gamma)(x-\delta)}}\,dx = \tag{20}$$
$$= \frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\left\{(\beta-\gamma)\,\Pi\left(\lambda, \frac{\alpha-\beta}{\alpha-\gamma}, r\right) + \gamma F(\lambda, r)\right\},$$

if $\delta < \gamma < \beta \leq u < \alpha$. We will have:

$$\int_u^\alpha \frac{x}{\sqrt{(\alpha-x)(x-\beta)(x-\gamma)(x-\delta)}}\,dx = \tag{21}$$
$$= \frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\left\{(\alpha-\delta)\,\Pi\left(\mu, \frac{\beta-\alpha}{\beta-\delta}, r\right) + \delta F(\mu, r)\right\},$$

if $\delta < \gamma < \beta < u \leq \alpha$.

We supposed here, for simplicity, that $a < z_1 < z_2$ and $z_1 < \sqrt{a^2 + y_0^2} < z_2$. The case $a = z_1$ i.e.:

$$a = \frac{k\,{y_0}^2 - m\,{v_0}^2}{2\sqrt{k}\sqrt{m}\,v_0},$$

is a very special one, and will not be treated here, being quite far from a physical meaning. Then by (20) and (21) we succeeded in integrating the ODE (10) exactly, so that the time equation (16) becomes:

$$t = \sqrt{\frac{k}{m}}\int_{\sqrt{a^2+y_0^2}}^{\sqrt{a^2+y^2}} \frac{z}{\sqrt{(z_2-z)(z-z_1)(z-a)(z+a)}}\,dz,$$

and, evaluating the integral, we finally obtain the time equation:

$$\begin{aligned} t &= \frac{2}{\sqrt{(z_2-a)(z_1+a)}}\sqrt{\frac{m}{k}}\times \\ &\times \left\{(z_2+a)\,\Pi\left(\lambda(y_0), \frac{z_1-z_2}{z_1+a}, r\right) - a\,F(\lambda(y_0), r) + \right. \\ &- \left.(z_1-a)\,\Pi\left(\lambda(y), \frac{z_1-z_2}{z_1+a}, r\right) - a\,F(\lambda(y), r)\right\}, \end{aligned} \tag{22}$$

where:

$$\lambda(y) = \arcsin\sqrt{\frac{(z_1+a)\left(z_2-\sqrt{a^2+y^2}\right)}{(z_2-z_1)\left(\sqrt{a^2+y^2}+a\right)}}, \quad r = \sqrt{\frac{2a\,(z_2-z_1)}{(z_2-a)\,(z_1+a)}},$$

with z_1 and z_2 given by (19) through the system characteristics (m, k, a) and the initial conditions y_0 and v_0.

The authors highlight that their solution (22) is completely new. For istance the excellent and recent treatise [5], page 384, when considering just the same ODE (10), judged that:

> *Exact analytic solutions for the motion of this system are essentially impossible to obtain.*

6. A sample problem

After having solved the nonlinear ODE (10), we wish to provide an example of the behaviour of the solution and its period. This has been done because the time equation (22) holds the displacement y inside the modulus $\lambda(y)$ of both the elliptic integrals, while their amplitudes are constant! So comes an intrinsic computational complexity and then a full discussion, taking into account all the parameters' ranges doesn't worth while. Accordingly, we deemed better to study a sample problem. In any case it is essential to locate carefully the singularities z_1 and z_2 (connected to the rest points) and the other singularities a and $-a$ introduced by the change of variables from s to z.

6.1. The solution

Let it be:

$$a = \frac{1}{2},\ y_0 = \frac{\sqrt{3}}{2},\ b = 1,\ m = 1,\ k = 5,\ v_0 = 1,$$

then we have:

$$f(y) = -5\,y\left(1-\frac{1}{\sqrt{\frac{1}{4}+y^2}}\right), \qquad \Phi(y) = \frac{5+20\,y^2}{2\sqrt{\frac{1}{4}+y^2}} - \frac{21}{4} - 5\,y^2.$$

Replacing our sample values in (22) we obtain the initial value problem:

$$\begin{cases} \ddot{y} = -5y\left(1-\dfrac{1}{\sqrt{\frac{1}{4}+y^2}}\right), \\ y(0) = \frac{\sqrt{3}}{2}, \\ \dot{y}(0) = 1. \end{cases} \tag{23}$$

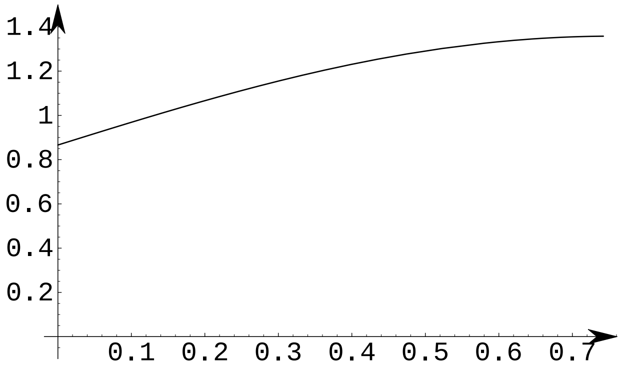

Figure 4: The plot of the exact solution.

We attached the plot of its solution $y = f(t)$ as computed analytically from (22) and inverting. The displacement values are given from the initial datum y_0 to z_2. The solution we found has been found perfectly superimposed with the numerical one as plotted by means of the Mathematica$_{®}$'s packages, VisualDSolve.m, [9] and Ode.m, [4].

To compare the numerical approximate output from (23) with the relevant exact solution (10) provided by the elliptic integrals and the inversion, it would require an analysis on the special functions software quality and implementation.

But this assessment goes outside the authors' interest, who are struggling to detect new integrable systems of Classical and Celestial Mechanics, finding their exact solutions, in terms of elementary and/or special functions.

6.2. The oscillation period

To obtain the oscillation period of our sample problem, the integration range including the poles of the integrand function, has been subdivided: in such a way both the formulas (21) and (20) have been used, with:

$$\begin{aligned}
\lambda &= \arcsin\sqrt{\frac{1}{2}+\frac{1}{\sqrt{5}}}, & \mu &= \arcsin\sqrt{\frac{1}{2}-\frac{1}{3\sqrt{5}}},\\
r &= \sqrt{\frac{8\sqrt{5}}{11+4\sqrt{5}}}, & \delta &= -a = -\frac{1}{4},\\
\gamma &= a = \frac{1}{4}, & \beta &= 1-\frac{1}{\sqrt{5}},\\
\alpha &= 1+\frac{1}{\sqrt{5}}, & u &= 1.
\end{aligned}$$

As a consequence of the time equation (22), the oscillation period T will be given by:

$$
\begin{aligned}
5\sqrt{\frac{11}{20}+\frac{1}{\sqrt{5}}}\; T &= -2\sqrt{5}\,F(\mu,r)+2\sqrt{5}\,F(\lambda,r)+\\
&+ \left(4+6\sqrt{5}\right)\Pi\left(\mu,\frac{4\left(2+3\sqrt{5}\right)}{41},r\right)+\\
&+ \left(2\sqrt{5}-4\right)\Pi\left(\lambda,4\left(\sqrt{5}-2\right),r\right).
\end{aligned}
\tag{24}
$$

The tables of the elliptic integrals allow to know them up to 12 exact decimal digits. We found $T \simeq 3.891179181311$.

Among the elliptic functions' tables it will be recalled that till in 1931 the Wittwer Verlag (Stuttgart) reproduced the Legendre tables (1825) at nine digits for F and E.

But the common reference is actually given by the Jahnke-Emde tables (1948) at four digits. Higher precision computations can be performed via the Spenceley tables (12 digits) issued in 1947 by the Smithsonian Institute of Washington, D.C. Then (24) holds a symbolic formulation that can be be practically accomplished with 12 digits, an accuracy going beyond each practical necessity.

Alternatively, the period could be numerically obtained from (23) by the integral:

$$
T=\frac{2}{\sqrt{5}}\int_{1-\frac{1}{\sqrt{5}}}^{1+\frac{1}{\sqrt{5}}}\frac{x}{\sqrt{-\left[x-\left(1-\frac{1}{\sqrt{5}}\right)\right]\left[x-\left(1+\frac{1}{\sqrt{5}}\right)\right]}\sqrt{x^2-\frac{1}{4}}}\,dx,
\tag{25}
$$

which is quite troublesome holding an unbounded integrand.

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AMS Subject Classification: 34A05, 34C25.

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Lavoro pervenuto in redazione il 01.10.2002 e, in forma definitiva, il 01.09.2003.

Rend. Sem. Mat. Univ. Pol. Torino
Vol. 61, 1 (2003)

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MAXIMAL IDEALS OF THE EXTENDED WEYL ALGEBRA $\mathbb{C}[X, Y, \sqrt{X}, \frac{1}{\sqrt{X}}]\langle\partial_X, \partial_Y\rangle$

Abstract. We prove that, in the algebra

$$\mathbb{C}[x, y, \sqrt{x}, \frac{1}{\sqrt{x}}]\langle\partial_x, \partial_y\rangle,$$

the operator $S = \partial_x + (1 + xy)\partial_y + y$ still generates, as in $A_2(\mathbb{C})$, a principle maximal ideal.

1. Introduction

Let K be the algebraic closure of the field $\mathbb{C}(x, y)$, and let $\xi \in K$. To $\mathbb{C}[x, y]$, we add the elements

$$\partial_x^n \xi, \quad \partial_y^n \xi, \quad \text{where} \quad n \in \mathbb{N},$$

and we ask if the algebra

$$A_2[\xi] = \mathbb{C}[x, y, \partial_x^n \xi, \partial_y^n \xi]\langle\partial_x, \partial_y\rangle$$

still has a maximal principal ideal.

With a direct verification, we will give an affirmative answer in the case

$$\xi^2 - x = 0.$$

Probably, this is true for every $\xi \in K$; but the direct proof seems to be rather involved.

2. Basic Definitions and Notations

Let ξ be an element such that

$$Q(\xi, x, y) = \xi^N + a_{N-1}\xi^{N-1} + \cdots + a_1\xi + a_0 = 0, \quad N \geq 2,$$

where $a_j \in \mathbb{C}[x, y]$ and Q is an irreducible polynomial in $\mathbb{C}[x, y, \xi]$. Then, ξ is locally a function of x and y, and we have

$$[N\xi^{N-1} + (N-1)a_{N-1}\xi^{N-2} + \cdots + a_1]\xi_x = -(a_{N-1})_x\xi^{N-1} - \cdots - (a_1)_x\xi - (a_0)_x.$$

*This article was prepared, under the supervision of Prof. G. Bratti, while the author was a research fellow at the Dipartimento di Matematica Pura ed Applicata, Università di Padova, Italy.

Since Q is irreducible, $N\xi^{N-1} + (N-1)a_{N-1}\xi^{N-2} + \cdots + a_1 \neq 0$, and we have

$$\xi_x = -\frac{(a_{N-1})_x\xi^{N-1} + \cdots + (a_1)_x\xi + (a_0)_x}{N\xi^{N-1} + (N-1)a_{N-1}\xi^{N-2} + \cdots + a_1}.$$

Similarly, we have

$$\xi_y = -\frac{(a_{N-1})_y\xi^{N-1} + \cdots + (a_1)_y\xi + (a_0)_y}{N\xi^{N-1} + (N-1)a_{N-1}\xi^{N-2} + \cdots + a_1}.$$

If we adopt the notations

$$Q_x = (a_{N-1})_x\xi^{N-1} + \cdots + (a_1)_x\xi + (a_0)_x,$$

$$Q_y = (a_{N-1})_y\xi^{N-1} + \cdots + (a_1)_y\xi + (a_0)_y,$$

and

$$Q_\xi = N\xi^{N-1} + (N-1)a_{N-1}\xi^{N-2} + \cdots + a_1,$$

then we can express ξ_x and ξ_y as follows:

$$\xi_x = -Q_x/Q_\xi, \quad \text{and} \quad \xi_y = -Q_y/Q_\xi.$$

Hence, we define

$$\eta = \frac{1}{Q_\xi},$$

and let $X = \mathbb{C}[x, y, \xi, \eta]$. A typical element of X has a form

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{N-1}\sum_{l=0}^{\infty} a_{ijkl}x^i y^j \xi^k \eta^l, \quad a_{ijkl} \in \mathbb{C},$$

where almost all a_{ijkl} are equal to zero.

The algebraically extended Weyl algebra A by the N-th degree polynomial Q is the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(X)$ generated by the operators x, y, 1, ξ, η, ∂_x and ∂_y. Hence $A = \mathbb{C}[x, y, \xi, \eta]\langle\partial_x, \partial_y\rangle$.

3. Left Maximal Ideals

Let $S = \partial_x + \beta(x, y)\partial_y + \gamma(x, y)$ with $\beta = 1 + xy$. After we prove Theorem 1, which gives a characterization of AS as a maximal ildeal, we prove Theorem 2, which is proved for $\gamma = y$.

In any case, we have the following lemma.

LEMMA 1. *If $\mathcal{I} \subseteq A$ is an ideal, and if $\mathcal{I}$ contains a polynomial p of the form*

$$p = \sum_{k=0}^{M} p_k \xi^k, \quad p_M \neq 0, \quad M \geq 0,$$

then $\mathcal{I}$ contains a polynomial $r(x, y) \in \mathbb{C}[x, y] \setminus \{0\}$.

Proof. Without loss of generality, we may assume that $M \leq N - 1$ because $\xi^N = -a_{N-1}\xi^{N-1} - \cdots - a_1\xi - a_0$.

If $M = 0$, then there is nothing to prove.

If $M = 1$, then $p = p_1\xi + p_0$, where $p_1 \neq 0$. Since $Q = 0$, hence $Q \in \mathcal{I}$. Thus, $p_1 Q \in \mathcal{I}$, and

$$p_1 Q = (\xi^{N-1} + \cdots + b_1\xi + b_0)(p_1\xi + p_0) + r(x, y).$$

Since Q is irreducible, thus $r \neq 0$ and $r \in \mathcal{I}$.

By induction, it is enough to prove that $\mathcal{I}$ contains a polynomial of degree less than M in the variable ξ. As before, we have

$$p_M Q = (\xi^{N-M} + \cdots + b_1\xi + b_0)p + c_{M-1}\xi^{M-1} + \cdots + c_1\xi + c_0.$$

Hence $c_{M-1}\xi^{M-1} + \cdots + c_1\xi + c_0 \in \mathcal{I}$, and $c_{M-1}\xi^{M-1} + \cdots + c_1\xi + c_0 \neq 0$ because Q is irreducible.

□

Now, let $R \in \mathbb{C}[x, y, \xi]\langle\partial_y\rangle$. Hence,

$$R = \sum_{k=0}^{M} p_k(x, y, \xi)\partial_y^k, \quad p_M \neq 0, \quad M \geq 0,$$

where $p_k \in \mathbb{C}[x, y, \xi]$. It is immediate to verify that

$$[S, R] \in \mathbb{C}[x, y, \xi, \eta]\langle\partial_y\rangle.$$

We have the following theorem, which is a generalization of Theorem 2.2 in [1].

THEOREM 1. *Let $S = \partial_x + \beta(x, y)\partial_y + \gamma(x, y)$, with $\beta = 1 + xy$. The following statements are equivalent.*

P_1) *AS is maximal in A.*

P_2) *For all $R \in \mathbb{C}[x, y, \xi, \eta]\langle\partial_y\rangle$, where R is not a constant, we have $[S, R] \notin \mathbb{C}[x, y, \xi, \eta]R$.*

Proof. In order to prove that P_1 implies P_2, the same methods as in the case A_2 work (see pages 212–213 in [1].) We only need to modify some. For example, if $\lambda S + \mu R = 1$, and if $\deg_{\partial_x} \lambda = m$, then we have $\deg_{\partial_x} \mu = m + 1$, and we obtain

$$\lambda S + \mu R = \sum_{k=0}^{m} B_k S^{k+1} + \sum_{k=0}^{m+1} C_k S^k R = 1$$

with some B_k and C_k in $\mathbb{C}[x, y, \xi, \eta]\langle \partial_y \rangle$.

We now prove that P_2 implies P_1. Let $R = \sum_{k=0}^{N} p_k \partial_y^k \in \mathbb{C}[x, y, \xi, \eta]\langle \partial_y \rangle$. Since

$$[S, R] = \sum_{k=0}^{N} q_k \partial_y^k,$$

where $q_k \in \mathbb{C}[x, y, \xi, \eta]$, we obtain that $p_N[S, R] - q_N R \neq 0$, and $\deg_{\partial_y}(p_N[S, R] - q_N R) \leq N - 1$ by hypothesis. Together with above lemma, we conclude that $AS + AR$ contains a non-trivial polynomial $p(x, y) \in \mathbb{C}[x, y]$.

Let $p = \sum_{k=0}^{L} r_k(x) y^k$, and $r_L(x) \neq 0$. If $L = 0$, then $AS + AR$ contains a non-trivial polynomial in only x, and $AS + AR = A$. If $L \geq 1$, then, by induction, it is enough to show that $AS + AR$ contains a polynomial of degree less than L in the variable y, with $\beta = 1 + xy$. Now

$$[S, p] = ((r_L)_x + Lxr_L)y^L + \sum_{k=1}^{L-1}((r_k)_x + (k+1)r_{k+1} + kxr_k)y^k + (r_0)_x + r_1.$$

Since $(r_L)_x + Lxr_L$ never vanishes, let us consider $\tilde{p} = ((r_L)_x + Lxr_L)p - r_L[S, p] \in AS + AR$. By assumption, $\tilde{p} \neq 0$ and $\deg_y \tilde{p} \leq L - 1$.

□

The following lemma is used in the proof of Theorem 2. We recall that

$$Q_x = (a_{N-1})_x \xi^{N-1} + \cdots + (a_1)_x \xi + (a_0)_x,$$

$$Q_y = (a_{N-1})_y \xi^{N-1} + \cdots + (a_1)_y \xi + (a_0)_y,$$

and

$$Q_\xi = N\xi^{N-1} + (N-1)a_{N-1}\xi^{N-2} + \cdots + a_1.$$

Similarly, if $q = q_{N-1}(x, y)\xi^{N-1} + \cdots + q_1(x, y)\xi + q_0(x, y)$, then q_x, q_y and q_ξ indicate the following expressions:

$$q_x = (q_{N-1})_x \xi^{N-1} + \cdots + (q_1)_x \xi + (q_0)_x,$$

$$q_y = (q_{N-1})_y \xi^{N-1} + \cdots + (q_1)_y \xi + (q_0)_y,$$

and

$$q_\xi = N\xi^{N-1} + (N-1)q_{N-1}\xi^{N-2} + \cdots + q_1.$$

LEMMA 2. *Assume that the differential equation,*

$$Q_\xi \left(q_x + (1+xy)q_y\right) + q_\xi \left(Q_x + (1+xy)Q_y\right) = qr, \tag{1}$$

does not have any solution q in $\mathbb{C}[x, y, \xi]\backslash\mathbb{C}$ for any $r \in \mathbb{C}[x, y, \xi]$. Then the following statements hold.

(1) The differential equation

$$(\partial_x + (1+xy)\partial_y)p = pr$$

does not have any solution p in $\mathbb{C}[x, y, \xi]\backslash\mathbb{C}$ for any $r \in \mathbb{C}[x, y, \xi]$. (Cf. Lemma 4)

(2) The differential equation

$$\left(\partial_x + (1+xy)\partial_y + x\right) f = 1$$

does not have any solution in $\mathbb{C}(x, y, \xi)$.

Proof. If p were a solution of (1), then we would have

$$p_x + (1+xy)p_y + p_\xi(\xi_x + (1+xy)\xi_y) = pr.$$

Multiplying the both sides by Q_ξ, we would obtain the contradiction:

$$Q_\xi(p_x + (1+xy)p_y) + p_\xi(Q_x + (1+xy)Q_y) = prQ_\xi.$$

If p/q were a solution of (2), where p and q are mutually prime, then we would have

$$\begin{aligned} q[Q_\xi p_x + Q_x p_\xi + (1+xy)Q_\xi p_y + (1+xy)Q_y p_\xi + xQ_\xi p - Q_\xi q] = \\ p[Q_\xi q_x + Q_x q_\xi + (1+xy)Q_\xi q_y + (1+xy)Q_y q_\xi] \end{aligned}$$

Hence we obtain

$$Q_\xi p_x + Q_x p_\xi + (1+xy)Q_\xi p_y + (1+xy)Q_y p_\xi + xQ_\xi p - Q_\xi q = pr$$

and

$$Q_\xi q_x + Q_x q_\xi + (1+xy)Q_\xi q_y + (1+xy)Q_y q_\xi = qr$$

for some $r \in \mathbb{C}[x, y, \xi] \setminus \{0\}$. From the last equality, we obtain

$$Q_\xi \left(q_x + (1+xy)q_y\right) + \left(Q_x + (1+xy)Q_y\right) q_\xi = qr.$$

□

THEOREM 2. *Let $S = \partial_x + \beta\partial_y + \gamma \in A$, where $\beta = 1 + xy$, and $\gamma = y$. The following statements are equivalent.*

P_1) *AS is maximal in A.*

P_2) *The equation (1)*

$$Q_\xi \left(q_x + (1+xy)q_y\right) + q_\xi \left(Q_x + (1+xy)Q_y\right) = qr,$$

and the equation

$$\partial_x \left(\frac{r}{s}\right) - x\left(\frac{r}{s}\right) = -1 \tag{2}$$

do not have any solutions respectively in $\mathbb{C}[x, y, \xi]\setminus\mathbb{C}$ *for any* $r \in [x, y, \xi]\setminus\{0\}$ *and in* $\mathbb{C}(x)$.

Proof. We, first, prove that P_1 implies that P_2. If there were a solution q of (1), then q would satisfy the differential equation $(\partial_x + (1+xy)\partial_y)q = qr\eta$. Hence, $[S, q] = qr\eta$, and it contradicts P_2 in Theorem 1.

Similarly, if there were a solution of (2), then letting $p = r(x) + s(x)y$, we would have, with $\beta = 1 + xy$,

$$s[S, p] = (s_x + xs)p.$$

We, now, prove that P_2 implies that P_1. Without loss of generality, it is enough to show that, for all $R \in \mathbb{C}[x, y, \xi]\langle\partial_y\rangle$ (instead of $R \in \mathbb{C}[x, y, \xi, \eta]\langle\partial_y\rangle$,) we have $AS + AR = A$. Hence, let $R = \sum_{k=0}^{n} p_k\partial_y^k$ with $p_k \in \mathbb{C}[x, y, \xi]$ and $p_n \neq 0$.

First, we prove that $AS + AR$ contains a non-trivial polynomial only in x and y. If $n = 0$, then it is already shown. Assume that $n \geq 1$. By induction, it is enough to show that $AS + AR$ contains an element $\tilde{R}$ such that $\tilde{R} \neq 0$ and $\deg_{\partial_y} = n - 1$. Since

$$[S, p_n\partial_y^n] = ((p_n)_x + \xi_x(p_n)_\xi + \beta(p_n)_y + \beta(p_n)_\xi\xi_y - np_n\beta_y)\partial_y^n - np_n\gamma_y\partial_y^{n-1},$$

by Lemma 2, it is confirmed that $(p_n)_x + \xi_x(p_n)_\xi + \beta(p_n)_y + \beta(p_n)_\xi\xi_y - np_n\beta_y \neq 0$. Hence $\deg_{\partial_y}[S, R] = \deg_{\partial_y} R = n$, and we consider the element

$$\tilde{R} = p_n[S, R] - ((p_n)_x + \xi_x(p_n)_\xi + \beta(p_n)_y + \beta(p_n)_\xi\xi_y - np_n\beta_y)R.$$

Then, $\tilde{R}$ is of the form $\sum_{k=0}^{n-1} q_k\partial_y^k$, and a simple calculation shows that

$$q_{n-1} = np_n^2\Bigg(\frac{p_n(p_{n-1})_x - p_{n-1}(p_n)_x}{np_n^2} + \frac{p_n\xi_x(p_{n-1})_\xi - p_{n-1}\xi_x(p_n)_\xi}{np_n^2} +$$

$$\beta\frac{p_n(p_{n-1})_y - p_{n-1}(p_n)_y}{np_n^2} + \beta\frac{p_n\xi_y(p_{n-1})_\xi - p_{n-1}\xi_y(p_n)_\xi}{np_n^2} + \beta_y\frac{p_{n-1}}{np_n} - \gamma_y\Bigg).$$

Hence

$$q_{n-1} = np_n^2\left(\partial_x\left(\frac{p_{n-1}}{np_n}\right) + \beta\partial_y\left(\frac{p_{n-1}}{np_n}\right) + \beta_y\left(\frac{p_{n-1}}{np_n}\right) - \gamma_y\right).$$

By Lemma 2, $q_{n-1} \neq 0$, and $\deg_{\partial_y}\tilde{R} = n - 1$.

Thus, there is an element in $(AS + AR) \cap (\mathbb{C}[x, y, \xi]\setminus\{0\})$. By the lemma above, we obtain that $(AS + AR) \cap (\mathbb{C}[x, y]\setminus\{0\}) \neq \emptyset$. Similarly, the equation (2) serves to guarantee the existence of an element in $(AS + AR) \cap (\mathbb{C}[x]\setminus\{0\})$.

□

4. Application of Theorem 2 with $Q = \xi^2 - x$

In this section, we show an application of Theorem 2 with $Q = \xi^2 - x$ and $S = \partial_x + (1 + xy)\partial_y + y$. In order to show that AS is maximal in A, we prove the following preposition through various lemmas and propositions. The difficulty is caused by the fact that, when we solve the differential equation in the following proposition, we cannot equate the coefficient of ξ^2 because $\xi^2 = x$. Please compare Lemma 3 with Lemma 4.

PROPOSITION 1. *If $Q = \xi^2 - x$, then no elements $q \in \mathbb{C}[x, y, \xi] \setminus \mathbb{C}$ satisfy the equation (1), namely*

$$Q_\xi \left(q_x + (1 + xy)q_y\right) + \left(Q_x + (1 + xy)Q_y\right) q_\xi = qr,$$

for any $r \in \mathbb{C}[x, y, \xi]$

Hence, we prove various lemmas and propositions in order to prove the above proposition. Let

$$L(q) = Q_\xi (q_x + (1 + xy)q_y) + (Q_x + (1 + xy)Q_y)q_\xi.$$

Since

$$L(x^n y^m) = 2(nx^{n-1}y^m + mx^n y^{m-1} + mx^{n+1}y^m)\xi,$$

and

$$L(x^n y^m \xi) = 2nx^n y^m + 2mx^{n+1}y^{m-1} + 2mx^{n+2}y^m - x^n y^m,$$

where $n \geq 1$ and $m \geq 1$, we can observe that r must be of the form

$$r = 2(c_1 x + c_0)\xi + d_1 x + d_0$$

where c_1, c_0, d_1, and d_0 are constants.

LEMMA 3. *Let $q = b_1(x, y)\xi + b_0(x, y)$, where b_1 and b_0 are elements in $\mathbb{C}[x, y]$. If $L(q) = qr$ for some $r \in \mathbb{C}[x, y, \xi] \setminus \{0\}$, then b_1 and b_0 must satisfy the following differential equations:*

(i) $2x((\partial_x + (1 + xy)\partial_y)b_1 - b_1(c_1 x + c_0)) = b_1 + b_0(d_1 x + d_0)$

(ii) $2((\partial_x + (1 + xy)\partial_y)b_0 - b_0(c_1 x + c_0)) = b_1(d_1 x + d_0)$

Proof. Direct calculations show that

$$L(b_1(x, y)\xi + b_0(x, y)) = 2x(b_{1x} + (1 + xy)b_{1y}) - b_1 + 2(b_{0x} + (1 + xy)b_{0y})\xi,$$

and

$$qr = 2b_1 x(c_1 x + c_0) + b_0(d_1 x + d_0) + (2b_0(c_1 x + c_0) + b_1(d_1 x + d_0))\xi.$$

Confronting the coefficients of terms with ξ and those without ξ, we obtain the above equations.

□

The following lemma, which was already proved in [1], is useful.

LEMMA 4. *If p in $\mathbb{C}[x, y]$ satisfies the differential equation*

$$(\partial_x + (1+xy)\partial_y)p = pr$$

for some $r \in \mathbb{C}[x, y]$, then p is a constant and $r = 0$.

Proof. See [1], page 216.

□

COROLLARY 1. *If $q = b_1(x, y)\xi + b_0(x, y)$ and $L(q) = qr$ for some $r \in \mathbb{C}[c, y, \xi] \setminus \{0\}$, then $b_1 \neq 0$ and $b_0 \neq 0$.*

Proof. If $b_1 = 0$ and $b_0 \neq 0$, then $q = b_0$. From the equation (ii), we have

$$(\partial_x + (1+xy)\partial_y)b_0 - b_0(c_1x + c_0) = 0.$$

By lemma (4), this is possible only when $c_1x + c_0 = 0$, and b_0 is a constant. Hence, from the equation (i) in lemma (3), we have $0 = b_0(d_1x + d_0)$. Thus $d_1x + d_0 = 0$ and $r = 0$.

If $b_1 \neq 0$ and $b_0 = 0$, then from the equation (i), we have

$$2x((\partial_x + (1+xy)\partial_y)b_1 - b_1(c_1x + c_0)) = b_1$$

Hence b_1 is divisible by x. Let us write $b_1 = x^n\tilde{b}_1$, where $n \geq 1$ and $\tilde{b}_1$ is not divisible by x. Then, simplifying the above equation, we obtain

$$2n\tilde{b}_1 + 2x\left(\left(\partial_x + (1+xy)\partial_y\right)\tilde{b}_1 - \tilde{b_1}(c_1x + c_0)\right) = \tilde{b}_1.$$

Since $(2n-1)\tilde{b}_1$ is not divisible by x, it is zero. Hence

$$2x\left(\left(\partial_x + (1+xy)\partial_y\right)\tilde{b}_1 - \tilde{b_1}(c_1x + c_0)\right) = 0;$$

by lemma 4, $\tilde{b}_1$ is a constant and $c_1x + c_0 = 0$. Then, by equation (ii), we have $0 = x^n\tilde{b}_1(d_1x + d_0) = 0$. Hence $d_1x + d_0 = 0$ and $r = 0$.

□

LEMMA 5. $q = \xi + \sum_{k=0}^{m} b_{0k}y^k$ *does not satisfy* $L(q) = qr$ *for any* $r \in \mathbb{C}[x, y, \xi] \setminus \{0\}$.

Proof. We have already proved the case $\sum_{k=0}^{m} b_{0k}y^k = 0$ in Corollary 1.

We now consider it when b_0 is a constant. Then

$$L(q) = 0 - 1 = c_1x^2 + (c_0 + b_0d_1)x + b_0d_0 + (d_1 + b_0c_1)x\xi + (d_0 + b_0c_0)\xi.$$

Hence $c_1 = 0$, and we obtain

$$-1 = (c_0 + b_0 d_1)x + b_0 d_0 + d_1 x\xi + (d_0 + b_0 c_0)\xi.$$

From the coefficient of $x\xi$, we obtain $d_1 = 0$, and we have

$$-1 = c_0 x + b_0 d_0 + (d_0 + b_0 c_0)\xi.$$

Thus, $c_0 = 0$ and $-1 = b_0 d_0 + d_0\xi$. Thus $d_0 = 0$ and we get a contradiction $-1 = 0$.

We now assume that $m \geq 1$ and $b_{0m} \neq 0$. We have the equality

$$c_1 x^2 + (c_0 + \sum_{k=0}^{m} d_1 b_{0k} y^k)x + \sum_{k=0}^{m} d_0 b_{0k} y^k = -1.$$

Hence $c_1 = 0$, $c_0 + \sum_{k=0}^{m} d_1 b_{0k} y^k = 0$ and $\sum_{k=0}^{m} d_0 b_{0k} y^k = -1$. Since $d_0 b_{0m} = 0$ and $b_{0m} \neq 0$, we have $d_0 = 0$ and we obtain the contradiction: $0 = -1$.

□

LEMMA 6. *$q = \sum_{k=0}^{n} b_{1k} y^k \xi + \sum_{k=0}^{m} b_{0k} y^k$, where $b_{1n} \neq 0$ and $b_{0m} \neq 0$, does not satisfy $L(q) = qr$ for any $r \in \mathbb{C}[x, y, \xi] \setminus \{0\}$.*

Proof. We have already proved for $n = 0$ in Lemma 5. If $n \geq 1$, then we obtain the following equality:

$$\begin{aligned} &2x^2 \sum_{k=0}^{n} k b_{1k} y^k + 2x \sum_{k=0}^{n} k b_{1k} y^{k-1} - \sum_{k=0}^{n} b_{1k} y^k \\ &= 2c_1 \sum_{k=0}^{n} b_{1k} y^k x^2 + (2c_0 \sum_{k=0}^{n} b_{1k} y^k + d_1 \sum_{k=0}^{m} b_{0k} y^k)x + d_0 \sum_{k=0}^{m} b_{0k} y^k. \end{aligned}$$

Comparing the coefficient of terms only in y, we obtain

$$-\sum_{k=0}^{n} b_{1k} y^k = d_0 \sum_{k=0}^{m} b_{0k} y^k.$$

Hence we may take $n = m$, and we have

$$d_0 b_{0k} = b_{1k}, \quad k = 0, ..., n.$$

Comparing the coefficient of x^2, we have

$$2 \sum_{k=0}^{n} k b_{1k} y^k = 2c_1 \sum_{k=0}^{n} b_{1k} y^k.$$

Since $b_{1n} \neq 0$, we obtain that $c_1 = n$ and $b_{1k} = 0$ if $k < n$. Since $d_0 b_{0n} = b_{1n} \neq 0$, we have $d_0 \neq 0$, and therefore $b_{0k} = 0$ if $k < n$. Comparing the coefficient of x, we have

$$2c_0 \sum_{k=0}^{n} b_{1k} y^k + d_1 \sum_{k=0}^{n} b_{0k} y^k = 2 \sum_{k=1}^{n-1} (k+1) b_{1,k+1} y^k.$$

Hence

$$2c_0 b_{1n} y^n + d_1 b_{0n} y^n = 2n b_{1n} y^{n-1}.$$

From here, we obtain the contradiction that $2nb_{1n} = 0$.

□

We will frequently use the following lemma:

LEMMA 7. *If $f \in \mathbb{C}[y] \setminus \{0\}$ satisfies the differential equation,*

$$yf'(y) - af(y) = 0, \quad \textit{where} \quad a \in \mathbb{C},$$

then $f = cy^m$ for some $c \in \mathbb{C} \setminus \{0\}$, for some $m \in \mathbb{N} \cup \{0\}$, and $a = m$.

Proof. If we write $f = \sum_{k=0}^{n} f_k y^k$, then the equation becomes

$$\sum_{k=1}^{n} k f_k y^k = \sum_{k=0}^{n} a f_k y^k.$$

Hence, we have the n equations $kf_k = af_k$, where $k = 0, ..., n$. Thus, there exists $m \in \{0, ..., n\}$ such that $mf_m = af_m$, where $f_m \neq 0$, and $f_k = 0$ if $k \neq m$. We write $c = f_m$, and we obtain $f = cy^m$ with $a = m$.

□

Let $b(x, y) = f(y)x^n + g(y)x^{n-1} + \cdots$, where $n \geq 1$. The following formulas, which specify only the two terms of the highest degree and the second highest degree in x, will be useful in the poof of the lemmas which follow.

$$\begin{aligned}&(\partial_x + (1 + xy)\partial_y)b - b(c_1 x + c_0)\\&= (yf'(y) - c_1 f(y))x^{n+1} + (f'(y) - c_0 f(y) + yg'(y) - c_1 g(y))x^n + \cdots,\end{aligned}$$

and

$$b(d_1 x + d_0) = d_1 f(y)x^{n+1} + (d_0 f(y) + d_1 g(y))x^n + \cdots,$$

where "$\cdots$" indicates terms whose degree of x is lower than n.

PROPOSITION 2. *No b_1 and b_0 in $\mathbb{C}[x, y, \xi] \setminus \{0\}$ satisfy the differential equations (i) and (ii) in Lemma 3.*

Proof. Let us put that $\deg_x b_0 = n$, and put $B_0 = 2((\partial_x + (1+xy)\partial_y)b_0 - b_0(c_1 x + c_0))$. By Lemma 6, it is enough to prove the case $n \geq 1$. By Lemma 4, we see that $B_0 \neq 0$, and (1) $\deg_x B_0 = n + 1$ or (2) $\deg_x B_0 < n + 1$.

(1) If $\deg_x B_0 = n + 1$, then from the equation (ii) in Lemma 3, $\deg_x b_1(d_1 x + d_0) = n + 1$. Thus, (1-1) $\deg_x b_1 = n + 1$ with $d_1 = 0$ or (1-2) $\deg_x b_1 = n$ with $d_1 \neq 0$.

(1-1) If $\deg_x b_1 = n + 1$ and $d_1 = 0$, then we have the following:

(i') $2x((\partial_x + (1 + xy)\partial_y)b_1 - b_1(c_1 x + c_0)) = b_1 + b_0 d_0$

(ii') $2((\partial_x + (1+xy)\partial_y)b_0 - b_0(c_1x + c_0)) = b_1d_0$

Now, let $B_1 = 2((\partial_x + (1+xy)\partial_y)b_1 - b_1(c_1x + c_0))$. By Lemma 4, $B_1 \neq 0$. From (i'), $\deg_x xB_1 = n+1$. Thus, $\deg_x B_1 = n$. Hence if b_1 is of the form $f(y)x^{n+1} + g(y)x^n + \cdots$, then

$$\begin{aligned}\frac{1}{2}B_1 =&(yf'(y) - c_1f(y))x^{n+2} + (f'(y) - c_0f(y) + yg'(y) - c_1g(y))x^{n+1} + \cdots\\ =&f(y)x^{n+1} + \cdots,\end{aligned}$$

From $yf'(y) - c_1f(y) = 0$, we have $f = cy^m$ for some constant c and $m \in \mathbb{N} \cup \{0\}$. If $m = 0$, then $c_1 = 0$ and (i') becomes $2x(b_{1x} + (1+xy)b_{1y} - b_1c_0) = b_1 + b_0d_0$. From the coefficient of x^{n+1}, we have $-c_0c + yg'(y) = 0$. Thus, $c_0 = 0$ and $y = \tilde{c}$, a constant. Hence b_1 is of the form $cx^{n+1} + \tilde{c}x^n + \cdots$, and the equation (i') gives

$$2x((n+1)cx^n + n\tilde{c}x^{n-1} + \cdots) = cx^{n+1} + \cdots.$$

Thus we arrive to the contradiction: $2(n+1) = 1$. Hence $m \neq 0$ and $c_1 = m \neq 0$. Now, from the coefficient of x^{n+1}, we have $cmy^{m-1} - c_0cy^m + yg'(y) - mg(y) = 0$. If we write $g = \sum_{k=0}^{l} g_ky^k$, we have $cmy^{m-1} - c_0cy^m + \sum_{k=1}^{l} kg_ky^k - \sum_{k=0}^{l} mg_ky^k = 0$. Thus $kg_k = mg_k$ if $k \neq m-1$, and if $k \neq m$. Hence $g_k = 0$ if $k < m-1$. Moreover, $cm + (m-1)g_{m-1} - mg_{m-1} = 0$. Thus $g_{m-1} = cm$ and therefore, $g = my^m + cmy^{m-1}$. Hence b_1 must have the form

$$b_1 = cy^mx^{n+1} + (my^m + cy^{m-1})x^n + \cdots.$$

Now let $b_0 = h(y)x^n + \cdots$. Then $\frac{1}{2}B_1 = (yh'(y) - mh(y))x^{n+1} + \cdots$. Thus, by (ii'), we have the equation,

$$\frac{d_0}{2}cy^m = yh'(y) - mh(y).$$

If we write $h = \sum_{k=0}^{l} h_ky^k$, then

$$\frac{d_0}{2}cy^m = \sum_{k=1}^{l} kh_ky^k - \sum_{k=0}^{l} mh_ky^k.$$

Hence $\frac{d_0}{2}c = mh_k - mh_k = 0$. This is a contradiction because $d_0 \neq 0$ by Lemma 4, and $c \neq 0$ by assumption $\deg_x b_1 = n$.

(1-2)Now, let us consider the case $\deg_x b_1 = n$ and $d_1 \neq 0$. By Lemma 5, $n \neq 0$. Let us write $b_1 = f(y)x^n + g(y)x^{n-1} + \cdots$, and $b_0 = h(y)x^n + l(y)x^{n-1} + \cdots$. Then the equation (i) becomes

$$\begin{aligned}B_1x =&2(yf'(y) - c_1f(y))x^{n+2} + 2(f'(y) - c_0f(y) + yg'(y) - c_1g(y))x^{n+1}\cdots\\ =&h(y)d_1x^{n+1} + \cdots\end{aligned}$$

Hence $yf'(y) - c_1 f(y) = 0$, and we have $f = cy^m$, and $c_1 = m$. From the equation (ii), we have

$$2(yh'(y) - mh(y))x^{n+1} + \cdots = cd_1 y^m x^{n+1} + \cdots .$$

Thus, $2(yh'(y) - mh(y)) = cd_1 y^m$. If we write $h(y) = \sum_{k=0}^{l} h_k y^k$, then we have $2(\sum_{k=1}^{l} kh_k y^k - \sum_{k=0}^{l} mh_k y^k) = cd_1 y^m$, and we obtain the contradiction that $2(mh_m - mh_m) = cd_1 = 0$.

(2) We now consider the case $\deg_x B_0 < n + 1$. Let us write $\deg_x B_0 = K$. Hence $K \leq n$. Then $\deg_x b_1(d_1 x + d_0) = K$. Hence (2-1) $\deg_x b_1 = K$ with $d_1 = 0$, or (2-2) $\deg_x b_1 = K - 1$ with $d_1 \neq 0$.

(2-1) If $\deg_x b_1 = K$ with $d_1 = 0$, then we write $b_1 = f(y)x^K + g(y)x^{K-1} + \cdots$. By the equation (i), we have

$$\begin{aligned}&2(yf(y) - c_1 f(y))x^{K+2} + 2(f'(y) - c_0 f(y) + yg'(y) - c_1 g(y))x^{K+1}\\&= f(y)x^K + g(y)x^{K-1} + \cdots + d_0 h(y)x^K + d_0 l(y)x^{K-1} + \cdots\end{aligned}$$

As before, from $yf'(y) - c_1 f(y) = 0$, we conclude that $f = cy^m$ and $c_1 = m$. From the coefficient of x^{K+1}, we have

$$mcy^{m-1} - c_0 cy^m + yg'(y) - mg(y) = 0.$$

If we write $g = \sum_{k=0}^{t} g_k y^k$, then the above equation becomes

$$mcy^{m-1} + \sum_{k=1}^{t} kg_k y^k - \sum_{k=0}^{t} mg_k y^k - c_0 cy^m = 0.$$

Hence $mg_m - mg_m - c_0 c = 0$. Thus, $c_0 = 0$.

Now, since $b_0 = h(y)x^n + l(y)x^{n-1} \cdots$, the equation (ii) becomes

$$\begin{aligned}&2(yh'(y) - mh(y))x^{n+1} + 2(h'(y) - mh(y) + yl'(y) - ml(y))x^n + \cdots\\&= d_0 cy^m x^K + d_0 g(y)x^{K-1} + \cdots\end{aligned}$$

From the coefficient of x^{n+1}, we have $h(y) = \tilde{c}y^m$. From the coefficient of x^n, we have $2m\tilde{c}y^{m-1} - 2mh(y) + 2yl'(y) - 2ml(y) = d_0 cy^m$ if $K = n$. As before, we obtain $d_0 c = 0$, and therefore we obtain the contradiction $d_0 = 0$. If $K < n$, then $m\tilde{c}y^{m-1} - m\tilde{c}y^m + yl'(y) - ml(y) = 0$. Hence if we write $l(y) = \sum_{k=0}^{t} l_k y^k$, we obtain

$$m\tilde{c}y^{m-1} - m\tilde{c}y^m + \sum_{k=1}^{t} kl_k y^k - \sum_{k=0}^{t} ml_k y^k = 0.$$

Therefore, $-m\tilde{c} + ml_m - ml_m = 0$, and we obtain the contradiction that $m\tilde{c} = 0$.

(2-2) We, now, consider the case $\deg_x b_1 = K - 1$ with $d_1 \neq 0$, where $K \leq n$. We write $b_1 = f(y)x^{K-1} \cdots$ and $b_0 = h(y)x^n + l(y)x^{n-1} + \cdots$. From the equation (ii),

$$2((yh'(y) - c_1 h(y))x^{n+1} + \cdots) = d_1 f(y)x^K + \cdots$$

Hence as before, $yh'(y) - c_1 h(y) = 0$ implies that $c_1 = m$ and $h(y) = cy^m$. From the equation (i),

$$2((yf'(y) - mf(y))x^{K+1} + \cdots) = d_1 c y^m x^{n+1} + \cdots .$$

Thus, $yf'(y) - mf(y) = d_1 c y^m$ if $K = n$ and $0 = d_1 c$ if $K < n$. In any case we obtain $d_1 c = 0$, and this is a contradiction because $d_1 \neq 0$ and $c \neq 0$.

$\square$

Acknowledgement. The author would like to thank Prof. Giuliano Bratti for his helpful guidance and advice throughout the preparation of this paper.

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AMS Subject Classification: 35A27, 16D30.

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Lavoro pervenuto in redazione il 15.04.2003.

Rend. Sem. Mat. Univ. Pol. Torino
Vol. 61, 1 (2003)

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QUASICLASSICAL ANALYSIS OF HYPOELLIPTIC OPERATORS

Abstract. In this paper we take into account the hypoelliptic operators introduced by Hörmander in [3] and we develop their quasiclassical analysis, obtaining an asymptotic formula for their counting function, as the Planck constant ϵ goes to 0, under somewhat more general hypotheses than Hörmander's ones.

1. Introduction

At the beginning of our work we give a brief sketch of the physical motivations underlying the mathematical branch known as *quasiclassical analysis*, without having the pretension of being totally exhaustive. For further details we refer to [7], [8] and [9] among others.

In Classical Mechanics it is well known that a particle of mass m can be described by a curve $t \to (x(t), \xi(t))$ (called the *classical trajectory*) such that, in the case the particle is subject to a force field $F = -\operatorname{grad} V$, the so called *Hamiltonian equations* are fulfilled:

$$\begin{cases} \dot{\xi}(t) = -\operatorname{grad} V(x(t)) \\ \dot{x}(t) = \frac{1}{m}\xi(t), \end{cases}$$

where $x(t)$ is the *position* of the particle at any time t and $\xi(t) = m\dot{x}(t)$ is the *momentum* of the particle.

A *classical observable* is any real smooth function $h(x, \xi)$ defined on the phase space $\mathbb{R}^n \times \mathbb{R}^n$: its value at the point $(x(t), \xi(t))$ gives an information on the position of the particle at time t. An example of classical observable is the *total energy* $\frac{1}{2m}\xi^2 + V(x)$.

In Quantum Mechanics a particle is instead described by a function $\psi(t, x)$ (called the *wave function*) such that for all $t \in \mathbb{R}$

$$\int_{\mathbb{R}^n} |\psi(t, x)|^2 \, dx = 1,$$

that is $\psi_t : x \mapsto \psi(t, x)$ belongs to $L^2(\mathbb{R}^n)$ and $\|\psi_t\|_{L^2(\mathbb{R}^n)} = 1$. The function ψ_t is called the *state* of the particle at time t.

The classical quantities of position x and momentum ξ can be investigated in Quantum Mechanics by considering the following two operators:

1. the *multiplication operator*
$$x : \psi \to x\psi$$
for the position;

2. the *derivation operator*
$$\epsilon D_x : \psi \to -i\epsilon \operatorname{grad} \psi$$
for the momentum,

where ϵ is the *Planck constant*. *

Note that these operators are selfadjoint with respect to the L^2 scalar product. More generally, any selfadjoint operator on L^2 is called a *quantum observable*.

The question about the correspondence between classical observable $h(x, \xi)$ and quantum observable $h(x, \epsilon D_x)$ is called the problem of *quantization* and one of the purposes of pseudodifferential calculus is just to provide a correspondence between the space of all classical observables endowed with the usual multiplication and the space of quantum observables endowed with the composition of operators.

For example, to the classical total energy $\frac{1}{2m}\xi^2 + V(x)$ we associate the operator

$$-\frac{\epsilon^2}{2m}\Delta + V(x)$$

(where Δ is the standard Laplacian) which is the *Schrödinger operator*.

Actually, it is convenient to consider the so called *Weyl quantization*:

$$h^w(x, \epsilon^2\xi)u(x) = \int e^{i(x-y)\cdot\xi} h\left(\frac{x+y}{2}, \epsilon^2\xi\right) u(y)\, dy\, đ\xi \tag{1}$$

or the unitarily equivalent form

$$h^w(\epsilon x, \epsilon\xi)u(x) = \int e^{i(x-y)\cdot\xi} h\left(\epsilon\frac{x+y}{2}, \epsilon\xi\right) u(y)\, dy\, đ\xi. \tag{2}$$

(for suitable functions u and h) since these operators are symplectically invariant, as remarked at the end of the paper, and are selfadjoint if and only if the symbol h is real valued.

Since Classical Mechanics (which is much simpler than Quantum Mechanics) describes quite well most of common elementary physical phenomena, we expect that Quantum Mechanics is a kind of generalization of Classical Mechanics, in the sense that one could recapture the classical properties of a system by making some approximation of its quantum properties. This is the so called *Bohr correspondence principle*: Classical Mechanics is nothing but the limits of Quantum Mechanics as the Planck constant ϵ tends to 0. This motivates an interest in the asymptotic properties of Weyl

*In order to be consistent with the notations used in [13] here we denote the Planck constant by ϵ and not by h, since we use h to denote our operators.

operators (1) or (2) when $\epsilon \to 0$. The corresponding asymptotic analysis is called the *quasiclassical analysis*.

In our work we consider the Weyl operators already considered by Hörmander in [3] for which he obtained an asymptotic formula for their counting function $\mathcal{N}(\tau)$ as $\tau \to +\infty$ (see Section 2). Then we consider the corresponding quantum observables $h^w(\epsilon x, \epsilon\xi)$ and we obtain a quasiclassical asymptotic formula for the associated counting function $\mathcal{N}_\epsilon(\tau)$ as $\epsilon \to 0$, under somewhat more general hypotheses than Hörmander's ones.

We employ the following notation: given two functions $f, g : X \to \mathbb{R}$, and a subset $A \subset X$, we write

$$f(x) \prec g(x), \quad \forall x \in A,$$

if there exists a constant C such that

$$f(x) \leq Cg(x), \quad \forall x \in A.$$

I would like to acknowledge professor Buzano for his precious suggestions while writing this paper.

2. Basic definitions and results

First we recall some definitions and results from [2], [4] and [13].

Let us begin with a brief review of Weyl-Hörmander calculus.

Let

$$\phi(x, \xi; y, \eta) = \sum_{j=1}^{n} \xi_j y_j - x_j \eta_j$$

be the standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$.

DEFINITION 1. *A Riemannian metric $g_{x,\xi}(y, \eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is* slowly varying *if there exist a positive real number r such that*

$$g_{x,\xi}(y, \eta) \prec g_{t,\tau}(y, \eta) \prec g_{x,\xi}(y, \eta),$$

for all $(x, \xi), (t, \tau), (y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$g_{x,\xi}(x - t, \xi - \tau) \leq r.$$

DEFINITION 2. *A positive function $m(x, \xi)$ is said to be g* continuous *if there is a positive real number r such that*

$$m(y, \eta) \prec m(x, \xi) \prec m(y, \eta),$$

for all $(x, \xi), (y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$g_{x,\xi}(x - y, \xi - \eta) \leq r.$$

DEFINITION 3. *A Riemannian metric* $g_{x,\xi}(y,\eta)$ *on* $\mathbb{R}^n \times \mathbb{R}^n$ *is* locally ϕ temperate *if it is slowly varying and there exist two positive real numbers* r *and* N *and a slowly varying metric* $G_x(y)$ *on* $\mathbb{R}^n$ *such that*

$$G_x(y) \le g_{x,\xi}(y,\eta),$$

for all $(x,\xi), (y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$, *and*

$$g_{x,\xi}(z,\zeta) \prec g_{y,\eta}(z,\zeta)(1+g^{\phi}_{x,\xi}(x-y,\xi-\eta))^N, \tag{3}$$

for all $(x,\xi), (y,\eta), (z,\zeta) \in \mathbb{R}^n \times \mathbb{R}^n$ *such that*

$$G_x(x-y) \le r.$$

The quadratic form $g^{\phi}_{x,\xi}(y,\eta)$ appearing in (3) is the *dual metric*

$$g^{\phi}_{x,\xi}(y,\eta) = \sup\{(\phi(y,\eta;z,\zeta))^2 : g_{x,\xi}(z,\zeta) = 1\}.$$

DEFINITION 4. *A positive function* $m(x,\xi)$ *is* locally ϕ, g temperate *if it is* g *continuous and there exist two positive real numbers* r *and* N *such that*

$$m(x,\xi) \prec m(y,\eta)(1+g^{\phi}_{x,\xi}(x-y,\xi-\eta))^N,$$

for all $(x,\xi), (y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$ *such that*

$$G_x(x-y) \le r.$$

Now we recall the definition of the class of symbols of Weyl-Hörmander $S(m,g)$.

DEFINITION 5. *Let* g *be a slowly varying Riemannian metric. Let* m *be a* g *continuous function. Then we say that a function* $h \in S(m,g)$ *if it is smooth and*

$$\sup_{x,\xi} \frac{|h|^g_k(x,\xi)}{m(x,\xi)} < +\infty, \quad \forall k \in \mathbb{N},$$

where

$$|h|^g_k(x,\xi) = \sup_{y_j,\eta_j} \frac{|h^{(k)}((x,\xi);(y_1,\eta_1),\ldots,(y_k,\eta_k))|}{\prod_{j=1}^k g_{x,\xi}(y_j,\eta_j)^{1/2}}$$

and $h^{(k)}((x,\xi);(y_1,\eta_1),\ldots,(y_k,\eta_k))$ *is the* k*-th differential of* h *at* (x,ξ).

We can introduce here the operators we deal with in the next section.

DEFINITION 6. *A differential operator* h^w *is* formally hypoelliptic† *if its Weyl symbol* $h(x,\xi)$ *satisfies the following conditions:*

†This definition is not to be confused with Definition 2.3 of Chapter III of [10]

1. *h is a smooth function such that*

$$h(x,\xi) \neq 0, \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n;$$

2. *there exists a locally ϕ temperate metric $g_{x,\xi}(y,\eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that $|h|$ is locally ϕ, g temperate and*

$$h \in S(|h|, g).$$

In the proof of our main theorem (see Theorem 5) we will make use of the following two results.

THEOREM 1. *Consider a positive and formally hypoelliptic symbol $h \in S(h,g)$ and assume that there exists a positive real number γ such that*

$$g_{x,\xi}(y,\eta) \prec h(x,\xi)^{-\gamma} g^{\phi}_{x,\xi}(y,\eta),$$

for all (x,ξ), $(y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$, then h^w is semi-bounded from below and essentially self-adjoint in $L^2(\mathbb{R}^n)$.

Moreover, if

$$h(x,\xi) \to +\infty, \qquad \text{as } |x| + |\xi| \to +\infty, \tag{4}$$

then the closure H of h^w in $L^2(\mathbb{R}^n)$ has discrete spectrum diverging to $+\infty$.

Proof. This is Theorem 1 of [13]. Actually we must observe that in the proof of Theorem 1 of [13] we did not assume that the metric g should satisfy the so called *principle of indetermination*, that is

$$\sup_{x,\xi} \frac{g_{x,\xi}}{g^{\phi}_{x,\xi}} \leq 1.$$

In our case, from the fact that there exists $\gamma > 0$ such that

$$g_{x,\xi}(y,\eta) \prec h(x,\xi)^{-\gamma} g^{\phi}_{x,\xi}(y,\eta)$$

and that $h(x,\xi) \to +\infty$ as $|x| + |\xi| \to +\infty$, it follows that

$$\sup_{x,\xi} \frac{g_{x,\xi}}{g^{\phi}_{x,\xi}} \leq C$$

for a suitable constant $C > 0$. If $0 < C \leq 1$ then the principle of indetermination is trivially satisfied. If $C > 1$ it is sufficient to replace g with the following new metric

$$\tilde{g} = \frac{g}{\sqrt{C}}.$$

It is easy to show that $\tilde{g}^{\phi} = \sqrt{C} g^{\phi}$. Therefore

$$\sup_{x,\xi} \frac{\tilde{g}_{x,\xi}}{\tilde{g}^{\phi}_{x,\xi}} = \sup_{x,\xi} \frac{\frac{g_{x,\xi}}{\sqrt{C}}}{\sqrt{C} g^{\phi}_{x,\xi}} = \frac{1}{C} \sup_{x,\xi} \frac{g_{x,\xi}}{g^{\phi}_{x,\xi}} \leq 1.$$

□

Thanks to Theorem 1, we can define the *counting function* of the operator H:

(5) $\mathcal{N}(\tau) =$ number of eigenfunctions of H, corresponding to eigenvalues less than or equal to τ.

THEOREM 2. *Under the hypotheses of Theorem 1 assume there exists* $\kappa > 0$ *such that*

$$h^{-\kappa} \in L^1. \tag{6}$$

Then for all $0 < \delta < \frac{\gamma}{3}$ *we have*

$$\mathcal{N}(\tau) = \mathcal{W}(\tau)\left\{1 + O\big(\mathcal{R}(\tau)\big)\right\}, \qquad \textit{as } \tau \to +\infty, \tag{7}$$

where

$$\mathcal{W}(\tau) = (2\pi)^{-n} \int\limits_{h \le \tau} dx\, d\xi, \tag{8}$$

and

$$\mathcal{R}(\tau) = \frac{\mathcal{W}\left(\tau + \tau^{1-\delta}\right) - \mathcal{W}\left(\tau - \tau^{1-\delta}\right)}{\mathcal{W}(\tau)}. \tag{9}$$

REMARK 1. Estimate (7) is known as *Weyl formula.*

Proof. This is Theorem 2 of [13]. □

3. Quasiclassical Analysis of Hypoelliptic Operators

Consider a formally hypoelliptic operator h^w.

Let us introduce the operator h^w_ϵ whose Weyl symbol is

$$h_\epsilon(x, \xi) = h(\epsilon x, \epsilon \xi),$$

where ϵ is a real parameter, such that $0 < \epsilon \le 1$.

Then we define a new Riemannian metric ${}^\epsilon g_{x,\xi}(y, \eta)$ in this way:

$${}^\epsilon g_{x,\xi}(y, \eta) = g_{\epsilon x, \epsilon \xi}(\epsilon y, \epsilon \eta) = \epsilon^2 g_{\epsilon x, \epsilon \xi}(y, \eta),$$

where $g_{x,\xi}(y, \eta)$ is the one appearing in Definition 6. We start by giving some results concerning this new metric ${}^\epsilon g$.

PROPOSITION 1. *The metric* ${}^\epsilon g_{x,\xi}(y, \eta)$ *is slowly varying and locally* ϕ *temperate, for all* $0 < \epsilon \le 1$.

Proof. We know that g is slowly varying. Then there exists a positive real number r such that

$$g_{x,\xi}(t,\tau) \prec g_{y,\eta}(t,\tau) \prec g_{x,\xi}(t,\tau)$$

for all (x,ξ), (y,η), $(t,\tau) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $g_{x,\xi}(x-y,\xi-\eta) \leq r$. Therefore

$$g_{\epsilon x,\epsilon\xi}(\epsilon t,\epsilon\tau) \prec g_{\epsilon y,\epsilon\eta}(\epsilon t,\epsilon\tau) \prec g_{\epsilon x,\epsilon\xi}(\epsilon t,\epsilon\tau)$$

when $g_{\epsilon x,\epsilon\xi}(\epsilon(x-y),\epsilon(\xi-\eta)) \leq r$, that is

$$^{\epsilon}g_{x,\xi}(t,\tau) \prec {}^{\epsilon}g_{y,\eta}(t,\tau) \prec {}^{\epsilon}g_{x,\xi}(t,\tau)$$

when $^{\epsilon}g_{x,\xi}(x-y,\xi-\eta) \leq r$ and so we have proved that $^{\epsilon}g$ is slowly varying, for all $0 < \epsilon \leq 1$.

Let us define a new metric

$$^{\epsilon}G_x(y) = G_{\epsilon x}(\epsilon y).$$

Therefore it is obvious that

$$^{\epsilon}G_x \leq {}^{\epsilon}g_{x,\xi}.$$

By hypothesis we know that the metric g is locally ϕ temperate. Therefore there exist two positive real numbers $r > 0$ and N such that

$$g_{x,\xi}(t,\tau) \prec g_{y,\eta}(t,\tau)(1 + g^{\phi}_{x,\xi}(x-y,\xi-\eta))^N,$$

when $G_x(x-y) \leq r$. Therefore

$$g_{\epsilon x,\epsilon\xi}(t,\tau) \prec g_{\epsilon y,\epsilon\eta}(t,\tau)(1 + g^{\phi}_{\epsilon x,\epsilon\xi}(\epsilon(x-y),\epsilon(\xi-\eta)))^N,$$

when $G_{\epsilon x}(\epsilon(x-y)) \leq r$, for all $0 < \epsilon \leq 1$, that is when $^{\epsilon}G_x(x-y) \leq r$. Now, we have that

$$\begin{aligned}
g^{\phi}_{\epsilon x,\epsilon\xi}(\epsilon t,\epsilon\tau) &= \sup_{(y,\eta)\neq 0} \frac{\phi(\epsilon t,\epsilon\tau;y,\eta)^2}{g_{\epsilon x,\epsilon\xi}(y,\eta)} = \\
&= \sup_{(y,\eta)\neq 0} \frac{\epsilon^2\phi(t,\tau;y,\eta)^2}{g_{\epsilon x,\epsilon\xi}(y,\eta)} = \\
&= \epsilon^4 \sup_{(y,\eta)\neq 0} \frac{\phi(t,\tau;y,\eta)^2}{\epsilon^2 g_{\epsilon x,\epsilon\xi}(y,\eta)} = \\
&= \epsilon^4 \sup_{(y,\eta)\neq 0} \frac{\phi(t,\tau;y,\eta)^2}{g_{\epsilon x,\epsilon\xi}(\epsilon y,\epsilon\eta)} = \\
&= \epsilon^4 ({}^{\epsilon}g)^{\phi}_{x,\xi}(t,\tau).
\end{aligned}$$

Thus we can conclude that

$$\begin{aligned}
\epsilon^2 g_{\epsilon x,\epsilon\xi}(t,\tau) &\prec \epsilon^2 g_{\epsilon y,\epsilon\eta}(t,\tau)(1 + g^{\phi}_{\epsilon x,\epsilon\xi}(\epsilon x-\epsilon y,\epsilon\xi-\epsilon\eta))^N \prec \\
&\prec \epsilon^2 g_{\epsilon y,\epsilon\eta}(t,\tau)(1 + \frac{1}{\epsilon^4}\cdot g^{\phi}_{\epsilon x,\epsilon\xi}(\epsilon x-\epsilon y,\epsilon\xi-\epsilon\eta))^N = \\
&= \epsilon^2 g_{\epsilon y,\epsilon\eta}(t,\tau)(1 + \frac{1}{\epsilon^4}\cdot \epsilon^4({}^{\epsilon}g)^{\phi}_{x,\xi}(x-y,\xi-\eta))^N,
\end{aligned}$$

when ${}^{\epsilon}G_x(x-y) \le r$, that is

$$ {}^{\epsilon}g_{x,\xi}(t,\tau) \prec {}^{\epsilon}g_{y,\eta}(t,\tau)(1+({}^{\epsilon}g)^{\phi}_{x,\xi}(x-y,\xi-\eta))^N, $$

when ${}^{\epsilon}G_x(x-y) \le r$, for all $0<\epsilon\le 1$.

$\square$

PROPOSITION 2. *We have that*

$$ h_\epsilon \in S(|h_\epsilon|, {}^{\epsilon}g), $$

uniformly with respect to $0<\epsilon\le 1$.

Proof. Using Definition 5 we have that, for all $k\in\mathbb{N}$,

$$ \begin{aligned} \sup_{x,\xi}\frac{|h_\epsilon|_k^{{}^{\epsilon}g}(x,\xi)}{|h_\epsilon(x,\xi)|} &= \sup_{x,\xi}\frac{\sup_{y_j,\eta_j}\frac{|h_\epsilon^{(k)}((x,\xi);(y_1,\eta_1),\dots,(y_k,\eta_k))|}{\prod_{j=1}^k {}^{\epsilon}g_{x,\xi}(y_j,\eta_j)^{1/2}}}{|h_\epsilon(x,\xi)|} = \\ &= \sup_{x,\xi}\frac{\sup_{y_j,\eta_j}\frac{|h^{(k)}((\epsilon x,\epsilon\xi);(y_1,\eta_1),\dots,(y_k,\eta_k))|}{\prod_{j=1}^k g_{\epsilon x,\epsilon\xi}(\epsilon y_j,\epsilon\eta_j)^{1/2}}}{|h_\epsilon(x,\xi)|} = \\ &= \sup_{x,\xi}\frac{\sup_{y_j,\eta_j}\frac{\epsilon^k|h^{(k)}((\epsilon x,\epsilon\xi);(y_1,\eta_1),\dots,(y_k,\eta_k))|}{\epsilon^k\prod_{j=1}^k g_{\epsilon x,\epsilon\xi}(y_j,\eta_j)^{1/2}}}{|h_\epsilon(x,\xi)|} = \\ &= \sup_{x,\xi}\frac{|h|_k^g(\epsilon x,\epsilon\xi)}{|h_\epsilon(x,\xi)|} = \\ &= \sup_{x,\xi}\frac{|h|_k^g(\epsilon x,\epsilon\xi)}{|h(\epsilon x,\epsilon\xi)|} = \sup_{x,\xi}\frac{|h|_k^g(x,\xi)}{|h(x,\xi)|} < +\infty, \end{aligned} $$

uniformly with respect to $0<\epsilon\le 1$, since $h\in S(|h|,g)$ by hypothesis.

$\square$

We can now state the following proposition:

PROPOSITION 3. *Let us suppose that there exists a real number* $\gamma>0$ *such that*

$$ g_{x,\xi}(y,\eta) \prec h(x,\xi)^{-\gamma} g^{\phi}_{x,\xi}(y,\eta), \tag{10} $$

for all $(x,\xi),(y,\eta)\in\mathbb{R}^n\times\mathbb{R}^n$. *Then we have that*

$$ {}^{\epsilon}g_{x,\xi}(y,\eta) \prec (h_\epsilon(x,\xi)\epsilon^{-\frac{4}{\gamma}})^{-\gamma}({}^{\epsilon}g)^{\phi}_{x,\xi}(y,\eta), $$

for all $(x,\xi),(y,\eta)\in\mathbb{R}^n\times\mathbb{R}^n$, *for all* $0<\epsilon\le 1$.

Proof. Recalling that

$$({}^{\epsilon}g)^{\phi}_{x,\xi}(y,\eta) = \epsilon^{-2} g^{\phi}_{\epsilon x,\epsilon\xi}(y,\eta),$$

and using (10) we have:

$$\frac{{}^{\epsilon}g_{x,\xi}(y,\eta)}{({}^{\epsilon}g)^{\phi}_{x,\xi}(y,\eta)} = \frac{\epsilon^2 g_{\epsilon x,\epsilon\xi}(y,\eta)}{\epsilon^{-2} g^{\phi}_{\epsilon x,\epsilon\xi}(y,\eta)} \prec$$
$$\prec h(\epsilon x,\epsilon\xi)^{-\gamma}\epsilon^4 =$$
$$= (h_\epsilon(x,\xi)\cdot\epsilon^{-\frac{4}{\gamma}})^{-\gamma}.$$

□

Due to this proposition, from now on we will work with the following symbol:

$$H_\epsilon(x,\xi) = \epsilon^{-\frac{4}{\gamma}} h_\epsilon(x,\xi).$$

Now we formulate Theorem 1 in this new context:

THEOREM 3. *Consider a positive and formally hypoelliptic symbol $h \in S(h,g)$ and assume that there exists a positive real number γ such that* (10) *is satisfied and $h(x,\xi) \to +\infty$ as $|x|+|\xi| \to +\infty$.*

Then the operator $H^w_\epsilon(x,\xi)$, corresponding to the new symbol $H_\epsilon(x,\xi)$, is semibounded from below and essentially self-adjoint in $L^2(\mathbb{R}^n)$ and its closure has discrete spectrum diverging to $+\infty$.

Proof. This is an immediate consequence of Theorem 1 .

□

REMARK 2. Thanks to Theorem 3 we can define the *counting function* of the closure of the operator H^w_ϵ:

$\mathcal{N}_{H_\epsilon}(\tau) =$ number of eigenfunctions of the closure of H^w_ϵ, corresponding to eigenvalues less than or equal to τ.

Before claiming our results, we have to state the following theorem, which is a direct consequence of Theorem 2.

THEOREM 4. *Under the hypotheses of Theorem 1, assume that there exists $k > 0$ such that*

$$h^{-k} \in L^1(\mathbb{R}^{2n}).$$

Then for all $0 < \delta < \frac{\gamma}{3}$ we have

$$\mathcal{N}_{H_\epsilon}(\tau) = \mathcal{W}_\epsilon(\tau)\{1 + O(\mathcal{R}_\epsilon(\tau))\},$$

as $\tau \to +\infty$*, uniformly with respect to* $0 < \epsilon \le 1$*, where*

$$\mathcal{W}_\epsilon(\tau) = (2\pi)^{-n} \iint\limits_{H_\epsilon(x,\xi)\le\tau} dx\, d\xi$$

and

$$\mathcal{R}_\epsilon(\tau) = \frac{\mathcal{W}_\epsilon(\tau+\tau^{1-\delta}) - \mathcal{W}_\epsilon(\tau-\tau^{1-\delta})}{\mathcal{W}_\epsilon(\tau)}.$$

Proof. By means of a change of coordinates we immediately obtain that

$$\begin{aligned} \|H_\epsilon^{-k}\|_{L^1} &= \epsilon^{\frac{4k}{\gamma}} \int |h_\epsilon(x,\xi)|^{-k}\, dx\, d\xi = \\ &= \epsilon^{\frac{4k}{\gamma}} \int h(\epsilon x, \epsilon\xi)^{-k}\, dx\, d\xi = \\ &= \epsilon^{\frac{4k}{\gamma}-2n} \int h(x,\xi)^{-k}\, dx\, d\xi \le \\ &\le \int h(x,\xi)^{-k}\, dx\, d\xi = \|h^{-k}\|_{L^1}, \end{aligned}$$

if we take $\frac{4k}{\gamma} - 2n \ge 0$, that is

$$\gamma \le \frac{2k}{n}.$$

Therefore we obtain that the integrability of the symbol $h(x,\xi)$ implies the integrability of the new symbol $H_\epsilon(x,\xi)$ and that the L^1 norm of H_ϵ^{-k} is uniformly bounded with respect to $0 < \epsilon \le 1$. The rest of the proof is an immediate consequence of Theorem 2, Proposition 1 and Proposition 2.

□

Now we can state and prove our main result.

THEOREM 5. *Let* $\mathcal{N}_\epsilon(\tau)$ *be the counting function associated to the operator* h_ϵ^w*. Let*

$$\mathcal{W}(\tau) = (2\pi)^{-n} \iint\limits_{h(x,\xi)\le\tau} dx\, d\xi.$$

Let τ *be not a critical value of* $h(x,\xi)$*, that is* $\operatorname{grad} h(x,\xi) \neq 0$ *on the surface* $\{(x,\xi) : h(x,\xi) = \tau\}$*. Under the hypotheses of Theorem 4, we have the following asymptotic formula for* $\mathcal{N}_\epsilon(\tau)$*:*

$$\mathcal{N}_\epsilon(\tau) = \epsilon^{-2n}(\mathcal{W}(\tau) + O(\epsilon^\theta)), \tag{11}$$

as $\epsilon \to 0$*, for all* $0 < \theta < \frac{4}{3}$*.*

COROLLARY 1. *The asymptotic formula* (11) *is valid when* τ *belongs to the complementary of a set of zero measure.*

Proof of Corollary 1. We know from Theorem 5 that in the asymptotic formula (11) we must exclude the critical values of $h(x,\xi)$. Corollary 1 is an immediate consequence of Sard Theorem, according to which the set of the critical values of $h(x,\xi)$ has zero measure.

□

Proof of Theorem 5. Since $\mathcal{N}_\epsilon(\tau)$ is the counting function associated to the operator h_ϵ^w, then it is clear that H_ϵ^w has exactly $\mathcal{N}_\epsilon(\epsilon^{\frac{4}{\gamma}}\tau)$ eigenvalues less than or equal to τ and that

$$(2\pi)^{-n} \iint\limits_{h(\epsilon x,\epsilon\xi)\epsilon^{-\frac{4}{\gamma}}\le\tau} dx\, d\xi = \epsilon^{-2n}\mathcal{W}(\epsilon^{\frac{4}{\gamma}}\tau).$$

Thanks to Theorem 4, we obtain that for all $0<\delta<\frac{\gamma}{3}$ there exists a real number $C_\delta>0$ such that

$$\begin{aligned}&|\mathcal{N}_\epsilon(\epsilon^{\frac{4}{\gamma}}\tau)-\epsilon^{-2n}\mathcal{W}(\epsilon^{\frac{4}{\gamma}}\tau)|\le\\ &\le C_\delta\epsilon^{-2n}(\mathcal{W}(\epsilon^{\frac{4}{\gamma}}(\tau+\tau^{1-\delta})-\mathcal{W}(\epsilon^{\frac{4}{\gamma}}(\tau-\tau^{1-\delta})),\end{aligned}$$

as $\tau\to+\infty$, for all $0<\epsilon\le 1$.

Letting $\epsilon^{\frac{4}{\gamma}}\tau=\lambda$ we obtain:

$$\begin{aligned}&|\mathcal{N}_\epsilon(\lambda)-\epsilon^{-2n}\mathcal{W}(\lambda)|\le\\ &\le C_\delta\epsilon^{-2n}(\mathcal{W}(\lambda(1+\epsilon^{\frac{4\delta}{\gamma}}\lambda^{-\delta}))-\mathcal{W}(\lambda(1-\epsilon^{\frac{4\delta}{\gamma}}\lambda^{-\delta}))\end{aligned} \tag{12}$$

for $0<\delta<\frac{\gamma}{3}$ and $\epsilon\to 0$.

From [9], 28.7, we know that when τ is not a critical value of $h(x,\xi)$ then $\mathcal{W}(\tau)$ is differentiable with

$$\mathcal{W}'(\tau)=(2\pi)^{-n}\int\limits_{V(\tau)}\frac{dS}{|\operatorname{grad} h|}$$

where

$$V(\tau)=\{(x,\xi):h(x,\xi)=\tau\}$$

and dS is the area element of the surface $V(\tau)$. Therefore when λ is not a critical value of h, using Taylor's formula, we can state that

$$\mathcal{W}(\lambda+\lambda^{1-\delta}\epsilon^{\frac{4\delta}{\gamma}})=\mathcal{W}(\lambda)+\lambda^{1-\delta}\epsilon^{\frac{4\delta}{\gamma}}(\mathcal{W}'(\lambda)+o(1))$$

and

$$\mathcal{W}(\lambda-\lambda^{1-\delta}\epsilon^{\frac{4\delta}{\gamma}})=\mathcal{W}(\lambda)-\lambda^{1-\delta}\epsilon^{\frac{4\delta}{\gamma}}(\mathcal{W}'(\lambda)+o(1)),$$

as $\epsilon\to 0$.

Thus

$$\mathcal{W}(\lambda(1+\epsilon^{\frac{4\delta}{\gamma}}\lambda^{-\delta}))-\mathcal{W}(\lambda(1-\epsilon^{\frac{4\delta}{\gamma}}\lambda^{-\delta}))=2\lambda^{1-\delta}\epsilon^{\frac{4\delta}{\gamma}}(\mathcal{W}'(\lambda)+o(1)),$$

as $\epsilon \to 0$, and then

$$\mathcal{W}(\lambda(1+\epsilon^{\frac{4\delta}{\gamma}}\lambda^{-\delta})) - \mathcal{W}(\lambda(1-\epsilon^{\frac{4\delta}{\gamma}}\lambda^{-\delta})) = O(\epsilon^{\frac{4\delta}{\gamma}}), \tag{13}$$

as $\epsilon \to 0$.

Since $0 < \delta < \frac{\gamma}{3}$, using (12) and (13) we conclude that

$$\mathcal{N}_\epsilon(\lambda) = \epsilon^{-2n}(\mathcal{W}(\lambda) + O(\epsilon^\theta)),$$

as $\epsilon \to 0$, for all $0 < \theta < \frac{4}{3}$.

□

REMARK 3. As pointed out in the introduction, quasiclassical operators could be also in the form $h^w(x, \epsilon^2\xi)$. In this case they are also called *semiclassical operators* and ϵ^2 is the Planck constant. Everything we obtained about quasiclassical operators is true also for semiclassical operators. Indeed the property of being ϕ temperate is invariant under *symplectic* changes of coordinates and the two transformations $(x, \xi) \to (\epsilon x, \epsilon\xi)$ and $(x, \xi) \to (x, \epsilon^2\xi)$ are symplectically equivalent. It is enough to consider the map

$$\begin{aligned} T : \mathbb{R}^{2n} &\to \mathbb{R}^{2n} \\ (x, \xi) &\mapsto T(x, \xi) = (\epsilon x, \epsilon^{-1}\xi). \end{aligned}$$

For further details see [6], Lemma 18.5.8. In the case of semiclassical operators we can actually keep the old metric: ${}^\epsilon G_x(y) = G_x(y)$.

4. An example

At the end of our paper we give an example of hypoelliptic operator satisfying our hypotheses.

PROPOSITION 4. *Let us consider*

- *a real polynomial $p(\xi)$ vanishing at the origin and hypoelliptic, i. e. such that*

$$\lim_{|\xi|\to+\infty} p(\xi) = +\infty$$

and

$$\nabla p(\xi) \prec p(\xi)^{1-\rho}, \qquad \textit{for } p(\xi) \geq 1,$$

with $0 < \rho \leq 1$,

- *a positive smooth function $q(x)$ such that*
  - $h(x, \xi) = p(\xi) + q(x) > 0$, $\forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$,
  - *there exists $0 \leq \nu < \rho$ such that for all $\alpha \in \mathbb{N}^n$ we have $D^\alpha q(x) \prec q(x)^{1+\nu|\alpha|}$, $\forall x \in \mathbb{R}^n$,*

– *there exists* $k_1 > 0$ *such that* $q(x)^{-k_1} \in L^1(\mathbb{R}^n)$.

Then $h^w_\epsilon(x, \xi) = p^w_\epsilon(\xi) + q(\epsilon x)$ *satisfies the hypotheses of Theorem 5.*

Proof. We give a brief sketch of the proof. First we prove that there exists $k > 0$ such that

$$h^{-k} \in L^1(\mathbb{R}^{2n}). \tag{14}$$

From hypotheses of Proposition 4, it is clear that there exists $R > 0$ such that

$$\sqrt{p(\xi)q(x)} \le p(\xi) + q(x), \quad \forall \xi \notin B(R), \quad \forall x \in \mathbb{R}^n,$$

where $B(R)$ is the open ball with center at the origin and radius R. Therefore, in order to verify (14), it suffices to show that there exists $k_2 > 0$ such that $p^{-k_2} \in L^1(\mathbb{R}^n \setminus B(R))$. We have that $p \to +\infty$, as $|\xi| \to +\infty$. Then, because p is a polynomial, it follows from Tarski-Seidenberg Theorem that there exists $k_0 > 0$ such that

$$p(\xi) \succ |\xi|^{k_0}, \qquad \forall\, |\xi| \ge R. \tag{15}$$

In fact,

$$E = \left\{(s, t, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n : s = |\xi|^2,\ p(\xi) = t\right\}$$

is algebraic in $\mathbb{R}^{2+n}$, and therefore

$$f(s) = \inf_{|\xi|^2 = s} p(\xi) = \inf\left\{t : \exists s\ \exists \xi\ \left((s, t, \xi) \in E\right)\right\} < +\infty$$

is semi-algebraic. Moreover $f(s) \to +\infty$, as $s \to +\infty$; hence (15) follows from Corollary A.2.6 of [5].

It is clear that estimate (15) implies the existence of κ_2.

Then it suffices to verify the hypotheses of Theorem 3. Consider the metric

$$g_{x,\xi}(y, \eta) = h(x, \xi)^{2\nu} |y|^2 + h(x, \xi)^{-2\rho} |\eta|^2.$$

Then one verifies that g and h are locally ϕ temperate, for example see [12]. Moreover one can show that $h \to +\infty$, as $|x| + |\xi| \to +\infty$, $h \in S(h, g)$ and (10) is verified. We omit the details.

□

For example we can take (see [1], section 1.1)

$$p(\xi) = \left(\xi_1^{2k} - \xi_2^{2k-1}\right)^2 + \xi_1^{2k} \xi_2^{2k-2},$$
$$q(x) = \exp\left(x_1^{2m_1} + x_2^{2m_2}\right),$$

where k is an integer greater than 1 and m_1 and m_2 are positive integers.

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AMS Subject Classification: 35P20, 47B06.

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Lavoro pervenuto in redazione il 15.09.2002 e, in forma definitiva, il 28.02.2003.

Rendiconti del Seminario Matematico
dell'Università e del Politecnico di Torino

Volume 61, N. 1 (2003)

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ISSN 0373-1243EDITRICE UNIVERSITARIA LEVROTTO & BELLA - TORINO