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MAXIMAL IDEALS OF THE EXTENDED WEYL ALGEBRA $\mathbb{C}[X, Y, \sqrt{X}, \frac{1}{\sqrt{X}}]\langle\partial_X, \partial_Y\rangle$

Abstract. We prove that, in the algebra

$$\mathbb{C}[x, y, \sqrt{x}, \frac{1}{\sqrt{x}}]\langle\partial_x, \partial_y\rangle,$$

the operator $S = \partial_x + (1 + xy)\partial_y + y$ still generates, as in $A_2(\mathbb{C})$, a principle maximal ideal.

1. Introduction

Let K be the algebraic closure of the field $\mathbb{C}(x, y)$, and let $\xi \in K$. To $\mathbb{C}[x, y]$, we add the elements

$$\partial_x^n \xi, \quad \partial_y^n \xi, \quad \text{where} \quad n \in \mathbb{N},$$

and we ask if the algebra

$$A_2[\xi] = \mathbb{C}[x, y, \partial_x^n \xi, \partial_y^n \xi]\langle\partial_x, \partial_y\rangle$$

still has a maximal principal ideal.

With a direct verification, we will give an affirmative answer in the case

$$\xi^2 - x = 0.$$

Probably, this is true for every $\xi \in K$; but the direct proof seems to be rather involved.

2. Basic Definitions and Notations

Let ξ be an element such that

$$Q(\xi, x, y) = \xi^N + a_{N-1}\xi^{N-1} + \cdots + a_1\xi + a_0 = 0, \quad N \geq 2,$$

where $a_j \in \mathbb{C}[x, y]$ and Q is an irreducible polynomial in $\mathbb{C}[x, y, \xi]$. Then, ξ is locally a function of x and y, and we have

$$[N\xi^{N-1} + (N-1)a_{N-1}\xi^{N-2} + \cdots + a_1]\xi_x = -(a_{N-1})_x\xi^{N-1} - \cdots - (a_1)_x\xi - (a_0)_x.$$

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Since Q is irreducible, $N\xi^{N-1} + (N-1)a_{N-1}\xi^{N-2} + \cdots + a_1 \neq 0$, and we have

$$\xi_x = -\frac{(a_{N-1})_x\xi^{N-1} + \cdots + (a_1)_x\xi + (a_0)_x}{N\xi^{N-1} + (N-1)a_{N-1}\xi^{N-2} + \cdots + a_1}.$$

Similarly, we have

$$\xi_y = -\frac{(a_{N-1})_y\xi^{N-1} + \cdots + (a_1)_y\xi + (a_0)_y}{N\xi^{N-1} + (N-1)a_{N-1}\xi^{N-2} + \cdots + a_1}.$$

If we adopt the notations

$$Q_x = (a_{N-1})_x\xi^{N-1} + \cdots + (a_1)_x\xi + (a_0)_x,$$

$$Q_y = (a_{N-1})_y\xi^{N-1} + \cdots + (a_1)_y\xi + (a_0)_y,$$

and

$$Q_\xi = N\xi^{N-1} + (N-1)a_{N-1}\xi^{N-2} + \cdots + a_1,$$

then we can express ξ_x and ξ_y as follows:

$$\xi_x = -Q_x/Q_\xi, \quad \text{and} \quad \xi_y = -Q_y/Q_\xi.$$

Hence, we define

$$\eta = \frac{1}{Q_\xi},$$

and let $X = \mathbb{C}[x, y, \xi, \eta]$. A typical element of X has a form

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{N-1}\sum_{l=0}^{\infty} a_{ijkl}x^i y^j \xi^k \eta^l, \quad a_{ijkl} \in \mathbb{C},$$

where almost all a_{ijkl} are equal to zero.

The algebraically extended Weyl algebra A by the N-th degree polynomial Q is the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(X)$ generated by the operators x, y, 1, ξ, η, ∂_x and ∂_y. Hence $A = \mathbb{C}[x, y, \xi, \eta]\langle\partial_x, \partial_y\rangle$.

3. Left Maximal Ideals

Let $S = \partial_x + \beta(x, y)\partial_y + \gamma(x, y)$ with $\beta = 1 + xy$. After we prove Theorem 1, which gives a characterization of AS as a maximal ildeal, we prove Theorem 2, which is proved for $\gamma = y$.

In any case, we have the following lemma.

LEMMA 1. *If $\mathcal{I} \subseteq A$ is an ideal, and if $\mathcal{I}$ contains a polynomial p of the form*

$$p = \sum_{k=0}^{M} p_k \xi^k, \quad p_M \neq 0, \quad M \geq 0,$$

then $\mathcal{I}$ contains a polynomial $r(x, y) \in \mathbb{C}[x, y] \setminus \{0\}$.

Proof. Without loss of generality, we may assume that $M \leq N - 1$ because $\xi^N = -a_{N-1}\xi^{N-1} - \cdots - a_1\xi - a_0$.

If $M = 0$, then there is nothing to prove.

If $M = 1$, then $p = p_1\xi + p_0$, where $p_1 \neq 0$. Since $Q = 0$, hence $Q \in \mathcal{I}$. Thus, $p_1 Q \in \mathcal{I}$, and

$$p_1 Q = (\xi^{N-1} + \cdots + b_1\xi + b_0)(p_1\xi + p_0) + r(x, y).$$

Since Q is irreducible, thus $r \neq 0$ and $r \in \mathcal{I}$.

By induction, it is enough to prove that $\mathcal{I}$ contains a polynomial of degree less than M in the variable ξ. As before, we have

$$p_M Q = (\xi^{N-M} + \cdots + b_1\xi + b_0)p + c_{M-1}\xi^{M-1} + \cdots + c_1\xi + c_0.$$

Hence $c_{M-1}\xi^{M-1} + \cdots + c_1\xi + c_0 \in \mathcal{I}$, and $c_{M-1}\xi^{M-1} + \cdots + c_1\xi + c_0 \neq 0$ because Q is irreducible.

□

Now, let $R \in \mathbb{C}[x, y, \xi]\langle\partial_y\rangle$. Hence,

$$R = \sum_{k=0}^{M} p_k(x, y, \xi)\partial_y^k, \quad p_M \neq 0, \quad M \geq 0,$$

where $p_k \in \mathbb{C}[x, y, \xi]$. It is immediate to verify that

$$[S, R] \in \mathbb{C}[x, y, \xi, \eta]\langle\partial_y\rangle.$$

We have the following theorem, which is a generalization of Theorem 2.2 in [1].

THEOREM 1. *Let $S = \partial_x + \beta(x, y)\partial_y + \gamma(x, y)$, with $\beta = 1 + xy$. The following statements are equivalent.*

P_1) *AS is maximal in A.*

P_2) *For all $R \in \mathbb{C}[x, y, \xi, \eta]\langle\partial_y\rangle$, where R is not a constant, we have $[S, R] \notin \mathbb{C}[x, y, \xi, \eta]R$.*

Proof. In order to prove that P_1 implies P_2, the same methods as in the case A_2 work (see pages 212–213 in [1].) We only need to modify some. For example, if $\lambda S + \mu R = 1$, and if $\deg_{\partial_x} \lambda = m$, then we have $\deg_{\partial_x} \mu = m + 1$, and we obtain

$$\lambda S + \mu R = \sum_{k=0}^{m} B_k S^{k+1} + \sum_{k=0}^{m+1} C_k S^k R = 1$$

with some B_k and C_k in $\mathbb{C}[x, y, \xi, \eta]\langle\partial_y\rangle$.

We now prove that P_2 implies P_1. Let $R = \sum_{k=0}^{N} p_k \partial_y^k \in \mathbb{C}[x, y, \xi, \eta]\langle\partial_y\rangle$. Since

$$[S, R] = \sum_{k=0}^{N} q_k \partial_y^k,$$

where $q_k \in \mathbb{C}[x, y, \xi, \eta]$, we obtain that $p_N[S, R] - q_N R \neq 0$, and $\deg_{\partial_y}(p_N[S, R] - q_N R) \leq N - 1$ by hypothesis. Together with above lemma, we conclude that $AS + AR$ contains a non-trivial polynomial $p(x, y) \in \mathbb{C}[x, y]$.

Let $p = \sum_{k=0}^{L} r_k(x) y^k$, and $r_L(x) \neq 0$. If $L = 0$, then $AS + AR$ contains a non-trivial polynomial in only x, and $AS + AR = A$. If $L \geq 1$, then, by induction, it is enough to show that $AS + AR$ contains a polynomial of degree less than L in the variable y, with $\beta = 1 + xy$. Now

$$[S, p] = ((r_L)_x + Lxr_L)y^L + \sum_{k=1}^{L-1}((r_k)_x + (k+1)r_{k+1} + kxr_k)y^k + (r_0)_x + r_1.$$

Since $(r_L)_x + Lxr_L$ never vanishes, let us consider $\tilde{p} = ((r_L)_x + Lxr_L)p - r_L[S, p] \in AS + AR$. By assumption, $\tilde{p} \neq 0$ and $\deg_y \tilde{p} \leq L - 1$.

□

The following lemma is used in the proof of Theorem 2. We recall that

$$Q_x = (a_{N-1})_x \xi^{N-1} + \cdots + (a_1)_x \xi + (a_0)_x,$$

$$Q_y = (a_{N-1})_y \xi^{N-1} + \cdots + (a_1)_y \xi + (a_0)_y,$$

and

$$Q_\xi = N\xi^{N-1} + (N-1)a_{N-1}\xi^{N-2} + \cdots + a_1.$$

Similarly, if $q = q_{N-1}(x, y)\xi^{N-1} + \cdots + q_1(x, y)\xi + q_0(x, y)$, then q_x, q_y and q_ξ indicate the following expressions:

$$q_x = (q_{N-1})_x \xi^{N-1} + \cdots + (q_1)_x \xi + (q_0)_x,$$

$$q_y = (q_{N-1})_y \xi^{N-1} + \cdots + (q_1)_y \xi + (q_0)_y,$$

and

$$q_\xi = N\xi^{N-1} + (N-1)q_{N-1}\xi^{N-2} + \cdots + q_1.$$

LEMMA 2. *Assume that the differential equation,*

$$Q_\xi \left(q_x + (1+xy)q_y\right) + q_\xi \left(Q_x + (1+xy)Q_y\right) = qr, \tag{1}$$

does not have any solution q in $\mathbb{C}[x, y, \xi]\backslash\mathbb{C}$ for any $r \in \mathbb{C}[x, y, \xi]$. Then the following statements hold.

(1) The differential equation

$$(\partial_x + (1+xy)\partial_y)p = pr$$

does not have any solution p in $\mathbb{C}[x, y, \xi]\backslash\mathbb{C}$ for any $r \in \mathbb{C}[x, y, \xi]$. (Cf. Lemma 4)

(2) The differential equation

$$\left(\partial_x + (1+xy)\partial_y + x\right) f = 1$$

does not have any solution in $\mathbb{C}(x, y, \xi)$.

Proof. If p were a solution of (1), then we would have

$$p_x + (1+xy)p_y + p_\xi(\xi_x + (1+xy)\xi_y) = pr.$$

Multiplying the both sides by Q_ξ, we would obtain the contradiction:

$$Q_\xi(p_x + (1+xy)p_y) + p_\xi(Q_x + (1+xy)Q_y) = prQ_\xi.$$

If p/q were a solution of (2), where p and q are mutually prime, then we would have

$$\begin{aligned} q[Q_\xi p_x + Q_x p_\xi + (1+xy)Q_\xi p_y + (1+xy)Q_y p_\xi + xQ_\xi p - Q_\xi q] = \\ p[Q_\xi q_x + Q_x q_\xi + (1+xy)Q_\xi q_y + (1+xy)Q_y q_\xi] \end{aligned}$$

Hence we obtain

$$Q_\xi p_x + Q_x p_\xi + (1+xy)Q_\xi p_y + (1+xy)Q_y p_\xi + xQ_\xi p - Q_\xi q = pr$$

and

$$Q_\xi q_x + Q_x q_\xi + (1+xy)Q_\xi q_y + (1+xy)Q_y q_\xi = qr$$

for some $r \in \mathbb{C}[x, y, \xi] \setminus \{0\}$. From the last equality, we obtain

$$Q_\xi \left(q_x + (1+xy)q_y\right) + \left(Q_x + (1+xy)Q_y\right) q_\xi = qr.$$

□

THEOREM 2. *Let $S = \partial_x + \beta\partial_y + \gamma \in A$, where $\beta = 1 + xy$, and $\gamma = y$. The following statements are equivalent.*

P_1) *AS is maximal in A.*

P_2) *The equation (1)*

$$Q_\xi\left(q_x + (1+xy)q_y\right) + q_\xi\left(Q_x + (1+xy)Q_y\right) = qr,$$

and the equation

$$\partial_x\left(\frac{r}{s}\right) - x\left(\frac{r}{s}\right) = -1 \tag{2}$$

do not have any solutions respectively in $\mathbb{C}[x, y, \xi]\setminus\mathbb{C}$ for any $r \in [x, y, \xi]\setminus\{0\}$ and in $\mathbb{C}(x)$.

Proof. We, first, prove that P_1 implies that P_2. If there were a solution q of (1), then q would satisfy the differential equation $(\partial_x + (1+xy)\partial_y)q = qr\eta$. Hence, $[S, q] = qr\eta$, and it contradicts P_2 in Theorem 1.

Similarly, if there were a solution of (2), then letting $p = r(x) + s(x)y$, we would have, with $\beta = 1 + xy$,

$$s[S, p] = (s_x + xs)p.$$

We, now, prove that P_2 implies that P_1. Without loss of generality, it is enough to show that, for all $R \in \mathbb{C}[x, y, \xi]\langle\partial_y\rangle$ (instead of $R \in \mathbb{C}[x, y, \xi, \eta]\langle\partial_y\rangle$,) we have $AS + AR = A$. Hence, let $R = \sum_{k=0}^{n} p_k\partial_y^k$ with $p_k \in \mathbb{C}[x, y, \xi]$ and $p_n \neq 0$.

First, we prove that $AS + AR$ contains a non-trivial polynomial only in x and y. If $n = 0$, then it is already shown. Assume that $n \geq 1$. By induction, it is enough to show that $AS + AR$ contains an element $\tilde{R}$ such that $\tilde{R} \neq 0$ and $\deg_{\partial_y} = n - 1$. Since

$$[S, p_n\partial_y^n] = ((p_n)_x + \xi_x(p_n)_\xi + \beta(p_n)_y + \beta(p_n)_\xi\xi_y - np_n\beta_y)\partial_y^n - np_n\gamma_y\partial_y^{n-1},$$

by Lemma 2, it is confirmed that $(p_n)_x + \xi_x(p_n)_\xi + \beta(p_n)_y + \beta(p_n)_\xi\xi_y - np_n\beta_y \neq 0$. Hence $\deg_{\partial_y}[S, R] = \deg_{\partial_y} R = n$, and we consider the element

$$\tilde{R} = p_n[S, R] - ((p_n)_x + \xi_x(p_n)_\xi + \beta(p_n)_y + \beta(p_n)_\xi\xi_y - np_n\beta_y)R.$$

Then, $\tilde{R}$ is of the form $\sum_{k=0}^{n-1} q_k\partial_y^k$, and a simple calculation shows that

$$q_{n-1} = np_n^2\left(\frac{p_n(p_{n-1})_x - p_{n-1}(p_n)_x}{np_n^2} + \frac{p_n\xi_x(p_{n-1})_\xi - p_{n-1}\xi_x(p_n)_\xi}{np_n^2} +\right.$$
$$\left.\beta\frac{p_n(p_{n-1})_y - p_{n-1}(p_n)_y}{np_n^2} + \beta\frac{p_n\xi_y(p_{n-1})_\xi - p_{n-1}\xi_y(p_n)_\xi}{np_n^2} + \beta_y\frac{p_{n-1}}{np_n} - \gamma_y\right).$$

Hence

$$q_{n-1} = np_n^2\left(\partial_x\left(\frac{p_{n-1}}{np_n}\right) + \beta\partial_y\left(\frac{p_{n-1}}{np_n}\right) + \beta_y\left(\frac{p_{n-1}}{np_n}\right) - \gamma_y\right).$$

By Lemma 2, $q_{n-1} \neq 0$, and $\deg_{\partial_y}\tilde{R} = n - 1$.

Thus, there is an element in $(AS + AR) \cap (\mathbb{C}[x, y, \xi]\setminus\{0\})$. By the lemma above, we obtain that $(AS + AR) \cap (\mathbb{C}[x, y]\setminus\{0\}) \neq \emptyset$. Similarly, the equation (2) serves to guarantee the existence of an element in $(AS + AR) \cap (\mathbb{C}[x]\setminus\{0\})$.

$\square$

4. Application of Theorem 2 with $Q = \xi^2 - x$

In this section, we show an application of Theorem 2 with $Q = \xi^2 - x$ and $S = \partial_x + (1 + xy)\partial_y + y$. In order to show that AS is maximal in A, we prove the following preposition through various lemmas and propositions. The difficulty is caused by the fact that, when we solve the differential equation in the following proposition, we cannot equate the coefficient of ξ^2 because $\xi^2 = x$. Please compare Lemma 3 with Lemma 4.

PROPOSITION 1. *If $Q = \xi^2 - x$, then no elements $q \in \mathbb{C}[x, y, \xi] \setminus \mathbb{C}$ satisfy the equation (1), namely*

$$Q_\xi \left(q_x + (1 + xy)q_y\right) + \left(Q_x + (1 + xy)Q_y\right) q_\xi = qr,$$

for any $r \in \mathbb{C}[x, y, \xi]$

Hence, we prove various lemmas and propositions in order to prove the above proposition. Let

$$L(q) = Q_\xi(q_x + (1 + xy)q_y) + (Q_x + (1 + xy)Q_y)q_\xi.$$

Since

$$L(x^n y^m) = 2(nx^{n-1}y^m + mx^n y^{m-1} + mx^{n+1}y^m)\xi,$$

and

$$L(x^n y^m \xi) = 2nx^n y^m + 2mx^{n+1}y^{m-1} + 2mx^{n+2}y^m - x^n y^m,$$

where $n \geq 1$ and $m \geq 1$, we can observe that r must be of the form

$$r = 2(c_1 x + c_0)\xi + d_1 x + d_0$$

where c_1, c_0, d_1, and d_0 are constants.

LEMMA 3. *Let $q = b_1(x, y)\xi + b_0(x, y)$, where b_1 and b_0 are elements in $\mathbb{C}[x, y]$. If $L(q) = qr$ for some $r \in \mathbb{C}[x, y, \xi] \setminus \{0\}$, then b_1 and b_0 must satisfy the following differential equations:*

(i) $2x((\partial_x + (1 + xy)\partial_y)b_1 - b_1(c_1 x + c_0)) = b_1 + b_0(d_1 x + d_0)$

(ii) $2((\partial_x + (1 + xy)\partial_y)b_0 - b_0(c_1 x + c_0)) = b_1(d_1 x + d_0)$

Proof. Direct calculations show that

$$L(b_1(x, y)\xi + b_0(x, y)) = 2x(b_{1x} + (1 + xy)b_{1y}) - b_1 + 2(b_{0x} + (1 + xy)b_{0y})\xi,$$

and

$$qr = 2b_1 x(c_1 x + c_0) + b_0(d_1 x + d_0) + (2b_0(c_1 x + c_0) + b_1(d_1 x + d_0))\xi.$$

Confronting the coefficients of terms with ξ and those without ξ, we obtain the above equations.

□

The following lemma, which was already proved in [1], is useful.

LEMMA 4. *If p in $\mathbb{C}[x, y]$ satisfies the differential equation*

$$(\partial_x + (1 + xy)\partial_y)p = pr$$

for some $r \in \mathbb{C}[x, y]$, then p is a constant and $r = 0$.

Proof. See [1], page 216. □

COROLLARY 1. *If $q = b_1(x, y)\xi + b_0(x, y)$ and $L(q) = qr$ for some $r \in \mathbb{C}[c, y, \xi] \setminus \{0\}$, then $b_1 \neq 0$ and $b_0 \neq 0$.*

Proof. If $b_1 = 0$ and $b_0 \neq 0$, then $q = b_0$. From the equation (ii), we have

$$(\partial_x + (1 + xy)\partial_y)b_0 - b_0(c_1 x + c_0) = 0.$$

By lemma (4), this is possible only when $c_1 x + c_0 = 0$, and b_0 is a constant. Hence, from the equation (i) in lemma (3), we have $0 = b_0(d_1 x + d_0)$. Thus $d_1 x + d_0 = 0$ and $r = 0$.

If $b_1 \neq 0$ and $b_0 = 0$, then from the equation (i), we have

$$2x((\partial_x + (1 + xy)\partial_y)b_1 - b_1(c_1 x + c_0)) = b_1$$

Hence b_1 is divisible by x. Let us write $b_1 = x^n \tilde{b}_1$, where $n \geq 1$ and $\tilde{b}_1$ is not divisible by x. Then, simplifying the above equation, we obtain

$$2n\tilde{b}_1 + 2x\left(\left(\partial_x + (1 + xy)\partial_y\right)\tilde{b}_1 - \tilde{b_1}(c_1 x + c_0)\right) = \tilde{b}_1.$$

Since $(2n - 1)\tilde{b}_1$ is not divisible by x, it is zero. Hence

$$2x\left(\left(\partial_x + (1 + xy)\partial_y\right)\tilde{b}_1 - \tilde{b_1}(c_1 x + c_0)\right) = 0;$$

by lemma 4, $\tilde{b}_1$ is a constant and $c_1 x + c_0 = 0$. Then, by equation (ii), we have $0 = x^n \tilde{b}_1(d_1 x + d_0) = 0$. Hence $d_1 x + d_0 = 0$ and $r = 0$. □

LEMMA 5. *$q = \xi + \sum_{k=0}^{m} b_{0k} y^k$ does not satisfy $L(q) = qr$ for any $r \in \mathbb{C}[x, y, \xi] \setminus \{0\}$.*

Proof. We have already proved the case $\sum_{k=0}^{m} b_{0k} y^k = 0$ in Corollary 1.

We now consider it when b_0 is a constant. Then

$$L(q) = 0 - 1 = c_1 x^2 + (c_0 + b_0 d_1)x + b_0 d_0 + (d_1 + b_0 c_1)x\xi + (d_0 + b_0 c_0)\xi.$$

Hence $c_1 = 0$, and we obtain

$$-1 = (c_0 + b_0 d_1)x + b_0 d_0 + d_1 x\xi + (d_0 + b_0 c_0)\xi.$$

From the coefficient of $x\xi$, we obtain $d_1 = 0$, and we have

$$-1 = c_0 x + b_0 d_0 + (d_0 + b_0 c_0)\xi.$$

Thus, $c_0 = 0$ and $-1 = b_0 d_0 + d_0 \xi$. Thus $d_0 = 0$ and we get a contradiction $-1 = 0$.

We now assume that $m \geq 1$ and $b_{0m} \neq 0$. We have the equality

$$c_1 x^2 + (c_0 + \sum_{k=0}^{m} d_1 b_{0k} y^k)x + \sum_{k=0}^{m} d_0 b_{0k} y^k = -1.$$

Hence $c_1 = 0$, $c_0 + \sum_{k=0}^{m} d_1 b_{0k} y^k = 0$ and $\sum_{k=0}^{m} d_0 b_{0k} y^k = -1$. Since $d_0 b_{0m} = 0$ and $b_{0m} \neq 0$, we have $d_0 = 0$ and we obtain the contradiction: $0 = -1$.

□

LEMMA 6. *$q = \sum_{k=0}^{n} b_{1k} y^k \xi + \sum_{k=0}^{m} b_{0k} y^k$, where $b_{1n} \neq 0$ and $b_{0m} \neq 0$, does not satisfy $L(q) = qr$ for any $r \in \mathbb{C}[x, y, \xi] \setminus \{0\}$.*

Proof. We have already proved for $n = 0$ in Lemma 5. If $n \geq 1$, then we obtain the following equality:

$$\begin{aligned} &2x^2 \sum_{k=0}^{n} k b_{1k} y^k + 2x \sum_{k=0}^{n} k b_{1k} y^{k-1} - \sum_{k=0}^{n} b_{1k} y^k \\ &= 2c_1 \sum_{k=0}^{n} b_{1k} y^k x^2 + (2c_0 \sum_{k=0}^{n} b_{1k} y^k + d_1 \sum_{k=0}^{m} b_{0k} y^k)x + d_0 \sum_{k=0}^{m} b_{0k} y^k. \end{aligned}$$

Comparing the coefficient of terms only in y, we obtain

$$-\sum_{k=0}^{n} b_{1k} y^k = d_0 \sum_{k=0}^{m} b_{0k} y^k.$$

Hence we may take $n = m$, and we have

$$d_0 b_{0k} = b_{1k}, \quad k = 0, ..., n.$$

Comparing the coefficient of x^2, we have

$$2 \sum_{k=0}^{n} k b_{1k} y^k = 2c_1 \sum_{k=0}^{n} b_{1k} y^k.$$

Since $b_{1n} \neq 0$, we obtain that $c_1 = n$ and $b_{1k} = 0$ if $k < n$. Since $d_0 b_{0n} = b_{1n} \neq 0$, we have $d_0 \neq 0$, and therefore $b_{0k} = 0$ if $k < n$. Comparing the coefficient of x, we have

$$2c_0 \sum_{k=0}^{n} b_{1k} y^k + d_1 \sum_{k=0}^{n} b_{0k} y^k = 2 \sum_{k=1}^{n-1} (k+1) b_{1,k+1} y^k.$$

Hence

$$2c_0 b_{1n} y^n + d_1 b_{0n} y^n = 2nb_{1n} y^{n-1}.$$

From here, we obtain the contradiction that $2nb_{1n} = 0$.

□

We will frequently use the following lemma:

LEMMA 7. *If* $f \in \mathbb{C}[y] \setminus \{0\}$ *satisfies the differential equation,*

$$yf'(y) - af(y) = 0, \quad \textit{where} \quad a \in \mathbb{C},$$

then $f = cy^m$ *for some* $c \in \mathbb{C} \setminus \{0\}$*, for some* $m \in \mathbb{N} \cup \{0\}$*, and* $a = m$*.*

Proof. If we write $f = \sum_{k=0}^{n} f_k y^k$, then the equation becomes

$$\sum_{k=1}^{n} kf_k y^k = \sum_{k=0}^{n} af_k y^k.$$

Hence, we have the n equations $kf_k = af_k$, where $k = 0, ..., n$. Thus, there exists $m \in \{0, ..., n\}$ such that $mf_m = af_m$, where $f_m \neq 0$, and $f_k = 0$ if $k \neq m$. We write $c = f_m$, and we obtain $f = cy^m$ with $a = m$.

□

Let $b(x, y) = f(y)x^n + g(y)x^{n-1} + \cdots$, where $n \geq 1$. The following formulas, which specify only the two terms of the highest degree and the second highest degree in x, will be useful in the poof of the lemmas which follow.

$$\begin{aligned}&(\partial_x + (1 + xy)\partial_y)b - b(c_1 x + c_0)\\&= (yf'(y) - c_1 f(y))x^{n+1} + (f'(y) - c_0 f(y) + yg'(y) - c_1 g(y))x^n + \cdots,\end{aligned}$$

and

$$b(d_1 x + d_0) = d_1 f(y)x^{n+1} + (d_0 f(y) + d_1 g(y))x^n + \cdots,$$

where "$\cdots$" indicates terms whose degree of x is lower than n.

PROPOSITION 2. *No* b_1 *and* b_0 *in* $\mathbb{C}[x, y, \xi] \setminus \{0\}$ *satisfy the differential equations (i) and (ii) in Lemma 3.*

Proof. Let us put that $\deg_x b_0 = n$, and put $B_0 = 2((\partial_x + (1+xy)\partial_y)b_0 - b_0(c_1 x + c_0))$. By Lemma 6, it is enough to prove the case $n \geq 1$. By Lemma 4, we see that $B_0 \neq 0$, and (1) $\deg_x B_0 = n + 1$ or (2) $\deg_x B_0 < n + 1$.

(1) If $\deg_x B_0 = n + 1$, then from the equation (ii) in Lemma 3, $\deg_x b_1(d_1 x + d_0) = n + 1$. Thus, (1-1) $\deg_x b_1 = n + 1$ with $d_1 = 0$ or (1-2) $\deg_x b_1 = n$ with $d_1 \neq 0$.

(1-1) If $\deg_x b_1 = n + 1$ and $d_1 = 0$, then we have the following:

(i') $2x((\partial_x + (1 + xy)\partial_y)b_1 - b_1(c_1 x + c_0)) = b_1 + b_0 d_0$

(ii') $2((\partial_x + (1+xy)\partial_y)b_0 - b_0(c_1x + c_0)) = b_1 d_0$

Now, let $B_1 = 2((\partial_x + (1+xy)\partial_y)b_1 - b_1(c_1x + c_0))$. By Lemma 4, $B_1 \neq 0$. From (i'), $\deg_x xB_1 = n+1$. Thus, $\deg_x B_1 = n$. Hence if b_1 is of the form $f(y)x^{n+1} + g(y)x^n + \cdots$, then

$$\begin{aligned}\frac{1}{2}B_1 =&(yf'(y) - c_1 f(y))x^{n+2} + (f'(y) - c_0 f(y) + yg'(y) - c_1 g(y))x^{n+1} + \cdots\\ =&f(y)x^{n+1} + \cdots,\end{aligned}$$

From $yf'(y) - c_1 f(y) = 0$, we have $f = cy^m$ for some constant c and $m \in \mathbb{N} \cup \{0\}$. If $m = 0$, then $c_1 = 0$ and (i') becomes $2x(b_{1x} + (1+xy)b_{1y} - b_1 c_0) = b_1 + b_0 d_0$. From the coefficient of x^{n+1}, we have $-c_0 c + yg'(y) = 0$. Thus, $c_0 = 0$ and $y = \tilde{c}$, a constant. Hence b_1 is of the form $cx^{n+1} + \tilde{c}x^n + \cdots$, and the equation (i') gives

$$2x((n+1)cx^n + n\tilde{c}x^{n-1} + \cdots) = cx^{n+1} + \cdots.$$

Thus we arrive to the contradiction: $2(n+1) = 1$. Hence $m \neq 0$ and $c_1 = m \neq 0$. Now, from the coefficient of x^{n+1}, we have $cmy^{m-1} - c_0 cy^m + yg'(y) - mg(y) = 0$. If we write $g = \sum_{k=0}^{l} g_k y^k$, we have $cmy^{m-1} - c_0 cy^m + \sum_{k=1}^{l} kg_k y^k - \sum_{k=0}^{l} mg_k y^k = 0$. Thus $kg_k = mg_k$ if $k \neq m-1$, and if $k \neq m$. Hence $g_k = 0$ if $k < m-1$. Moreover, $cm + (m-1)g_{m-1} - mg_{m-1} = 0$. Thus $g_{m-1} = cm$ and therefore, $g = my^m + cmy^{m-1}$. Hence b_1 must have the form

$$b_1 = cy^m x^{n+1} + (my^m + cy^{m-1})x^n + \cdots.$$

Now let $b_0 = h(y)x^n + \cdots$. Then $\frac{1}{2}B_1 = (yh'(y) - mh(y))x^{n+1} + \cdots$. Thus, by (ii'), we have the equation,

$$\frac{d_0}{2}cy^m = yh'(y) - mh(y).$$

If we write $h = \sum_{k=0}^{l} h_k y^k$, then

$$\frac{d_0}{2}cy^m = \sum_{k=1}^{l} kh_k y^k - \sum_{k=0}^{l} mh_k y^k.$$

Hence $\frac{d_0}{2}c = mh_k - mh_k = 0$. This is a contradiction because $d_0 \neq 0$ by Lemma 4, and $c \neq 0$ by assumption $\deg_x b_1 = n$.

(1-2)Now, let us consider the case $\deg_x b_1 = n$ and $d_1 \neq 0$. By Lemma 5, $n \neq 0$. Let us write $b_1 = f(y)x^n + g(y)x^{n-1} + \cdots$, and $b_0 = h(y)x^n + l(y)x^{n-1} + \cdots$. Then the equation (i) becomes

$$\begin{aligned}B_1 x =&2(yf'(y) - c_1 f(y))x^{n+2} + 2(f'(y) - c_0 f(y) + yg'(y) - c_1 g(y))x^{n+1} \cdots\\ =&h(y)d_1 x^{n+1} + \cdots\end{aligned}$$

Hence $yf'(y) - c_1 f(y) = 0$, and we have $f = cy^m$, and $c_1 = m$. From the equation (ii), we have

$$2(yh'(y) - mh(y))x^{n+1} + \cdots = cd_1 y^m x^{n+1} + \cdots .$$

Thus, $2(yh'(y) - mh(y)) = cd_1 y^m$. If we write $h(y) = \sum_{k=0}^{l} h_k y^k$, then we have $2(\sum_{k=1}^{l} kh_k y^k - \sum_{k=0}^{l} mh_k y^k) = cd_1 y^m$, and we obtain the contradiction that $2(mh_m - mh_m) = cd_1 = 0$.

(2) We now consider the case $\deg_x B_0 < n + 1$. Let us write $\deg_x B_0 = K$. Hence $K \leq n$. Then $\deg_x b_1(d_1 x + d_0) = K$. Hence (2-1) $\deg_x b_1 = K$ with $d_1 = 0$, or (2-2) $\deg_x b_1 = K - 1$ with $d_1 \neq 0$.

(2-1) If $\deg_x b_1 = K$ with $d_1 = 0$, then we write $b_1 = f(y)x^K + g(y)x^{K-1} + \cdots$. By the equation (i), we have

$$\begin{aligned} &2(yf(y) - c_1 f(y))x^{K+2} + 2(f'(y) - c_0 f(y) + yg'(y) - c_1 g(y))x^{K+1} \\ &= f(y)x^K + g(y)x^{K-1} + \cdots + d_0 h(y)x^K + d_0 l(y)x^{K-1} + \cdots \end{aligned}$$

As before, from $yf'(y) - c_1 f(y) = 0$, we conclude that $f = cy^m$ and $c_1 = m$. From the coefficient of x^{K+1}, we have

$$mcy^{m-1} - c_0 cy^m + yg'(y) - mg(y) = 0.$$

If we write $g = \sum_{k=0}^{t} g_k y^k$, then the above equation becomes

$$mcy^{m-1} + \sum_{k=1}^{t} kg_k y^k - \sum_{k=0}^{t} mg_k y^k - c_0 cy^m = 0.$$

Hence $mg_m - mg_m - c_0 c = 0$. Thus, $c_0 = 0$.

Now, since $b_0 = h(y)x^n + l(y)x^{n-1} \cdots$, the equation (ii) becomes

$$\begin{aligned} &2(yh'(y) - mh(y))x^{n+1} + 2(h'(y) - mh(y) + yl'(y) - ml(y))x^n + \cdots \\ &= d_0 cy^m x^K + d_0 g(y)x^{K-1} + \cdots \end{aligned}$$

From the coefficient of x^{n+1}, we have $h(y) = \tilde{c}y^m$. From the coefficient of x^n, we have $2m\tilde{c}y^{m-1} - 2mh(y) + 2yl'(y) - 2ml(y) = d_0 cy^m$ if $K = n$. As before, we obtain $d_0 c = 0$, and therefore we obtain the contradiction $d_0 = 0$. If $K < n$, then $m\tilde{c}y^{m-1} - m\tilde{c}y^m + yl'(y) - ml(y) = 0$. Hence if we write $l(y) = \sum_{k=0}^{t} l_k y^k$, we obtain

$$m\tilde{c}y^{m-1} - m\tilde{c}y^m + \sum_{k=1}^{t} kl_k y^k - \sum_{k=0}^{t} ml_k y^k = 0.$$

Therefore, $-m\tilde{c} + ml_m - ml_m = 0$, and we obtain the contradiction that $m\tilde{c} = 0$.

(2-2) We, now, consider the case $\deg_x b_1 = K - 1$ with $d_1 \neq 0$, where $K \leq n$. We write $b_1 = f(y)x^{K-1} \cdots$ and $b_0 = h(y)x^n + l(y)x^{n-1} + \cdots$. From the equation (ii),

$$2((yh'(y) - c_1 h(y))x^{n+1} + \cdots) = d_1 f(y)x^K + \cdots$$

Hence as before, $yh'(y) - c_1 h(y) = 0$ implies that $c_1 = m$ and $h(y) = cy^m$. From the equation (i),

$$2((yf'(y) - mf(y))x^{K+1} + \cdots) = d_1 c y^m x^{n+1} + \cdots .$$

Thus, $yf'(y) - mf(y) = d_1 c y^m$ if $K = n$ and $0 = d_1 c$ if $K < n$. In any case we obtain $d_1 c = 0$, and this is a contradiction because $d_1 \neq 0$ and $c \neq 0$.

$\square$

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