Rend. Sem. Mat. Univ. Pol. Torino
Vol. 60, 3 (2002)

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MINIMAL PROJECTIONS IN TENSOR PRODUCT SPACES

Abstract. It is the object of this paper to study the existence and the form of minimal projections in some spaces of tensor products of Banach spaces. We answer a question of Franchetti and Cheney for finitely codimensional subspaces in $C(K)$ [1].

Introduction

The problem of the existence of minimal projections and the computation of the norm of the minimal projection is an important one [1], [5], [6].

The object of this paper is to present new results on the existence of minimal projections in some tensor products of Banach spaces, and to give the form of the minimal projection in certain Banach spaces. It should be remarked that not much is known on the form of minimal projections.

For Banach spaces X and Y , $X \overset{\wedge}{\otimes} Y (X \overset{\vee}{\otimes} Y)$ denotes the completed projective(injective)tensor product of X and Y [5]. For $1 \leq p < \infty$, we set $L^p(I, X)$ denote the space of all $p-$Bochner integrable functions (classes) on the unit interval I with values in the Banach space X. In case X is the set of reals we write $L^p(I)$. For $f \in L^2(I, X)$, $\|f\|_p$ denotes the usual $p-$norm of f [5]. For $p = \infty$, $L^\infty(I, X)$ denotes the essentially bounded functions from I to X, with the usual norm $\|\cdot\|_\infty$. The spaces $\ell^2(X)$, and $\ell^\infty(X)$ are the corresponding sequence spaces. If X is a Banach space, X^* denotes the dual of X, and $L(X, Y)$ the space of all bounded linear operators from X to the Banach space Y.

Throughout this paper, $\Pi(X, Y)$ denotes the set of all projections from X into Y.

1. Existence of Minimal Projections

Let X be Banach space and Y be a closed subspace of X. If Y is finite dimensional then, a minimal projection from X onto Y exists [2]. The problem of existence of minimal projections in tensor product spaces was discussed in [1], [2] and [4]. In this section, we present some general facts on the existence of minimal projections which are not stated explicitly in the literature. We include the proof for completeness.

PROPOSITION 1. *Let X be a dual space. Then for any w^*-closed complemented subspace Y of X there exists a minimal projection onto Y .*

Proof. Let $K = \{P \in L(X, X) : Px = x \text{ for } x \in Y\}$. Since X is a dual space, and Y is w^*- closed in X, it follows that K is w^*-closed. Consequently $\inf\{\|P\| : P \in K\}$ is attained at some P, and such P is a projection. This ends the proof.

□

As a consequence we get

COROLLARY 1. *Let G and H be finite dimensional subspaces of $L^p(I)$. Then there exists a minimal projection from $L^p(I \times I)$ onto $L^p(I, H) + L^p(I, G)$, for $1 < p < \infty$.*

Proof. From theory of tensor products, it is known [5], that

$$L^p(I \times I) = L^p(I) \underset{p}{\otimes} L^p(I), L^p(I, H) = L^p(I) \underset{p}{\otimes} H$$

and

$$L^p(I, G) = G \underset{p}{\otimes} L^p(I),$$

where $X \underset{p}{\otimes} Y$ is the p-nuclear tensor product of X and Y. Since $L^p(I \times I)$ is a dual separable space, and H is finite dimensional, it follows from Proposition 1 above and Proposition 11.2 in [5] that $L^p(I, H) + L^p(I, G)$ is a closed complemented subspace of a reflexive space, and so it is w^*- closed. The result now follows.

□

COROLLARY 2. *If G and H are finite dimensional subspaces of ℓ^1, then there is a minimal projection of $\ell^1 \overset{\wedge}{\otimes} \ell^1$ onto $\ell^1 \overset{\wedge}{\otimes} H + G \overset{\wedge}{\otimes} \ell^1$.*

Proof. The proof follows from the above proposition and the fact $\ell^1 \overset{\wedge}{\otimes} H = \ell^1(H) = c_0(G)$, where $G^* = H$.

□

Another similar result is

PROPOSITION 2. *Let X be a Banach space with separable dual .Then for every complemented weakly sequentially complete subspace W in X, there exists a minimal projection from X onto W.*

Proof. Since W is complemented, $inf\{\|Q\| : Q \in \Pi(X, W)\} = r$, is finite. So there exists a sequence $P_n \in \Pi(X, W)$ such that $\|P_n\| \longrightarrow r$. Thus (P_n^*) is a bounded sequence in $L(X^*, X^*) = (X^* \overset{\wedge}{\otimes} X^*)^*$ (where T^* is the adjoint of the operator T). Since X^* is separable, X is separable, and so $X^* \overset{\wedge}{\otimes} X$ is separable. Hence, using Helly's selection theorem, we can assume that (P_n^*) converges in the weak operator topology. Define $Q : X \longrightarrow W$ by $< Qx, x^* >= lim < x, P_n^* x^* >$. Since Y is weakly sequentially complete then Q is a projection on W, and $\|Q\| \leq \underline{lim} \left\| P_n^* \right\| = r$. So Q is a minimal projections. This ends the proof.

□

PROPOSITION 3. *Let X be a Banach space and W be a complemented subspace of X. If there is a contractive projection from W^{**}onto W, then there is a minimal projection from X onto W.*

Proof. Let $r = inf\{\|P\| : P \in \Pi(X, W)\}$. Let (P_t) be a net of projections onto W such that $\lim_t \|P_t\| = r$. Now $P_t^* \in \Pi(X^*, W^*)$, so $P_t^* \in (X^* \overset{\wedge}{\otimes} X)^*$. So there is a subnet that converges in the weak operator topology. We can assume that $P_t^* \longrightarrow U$ in the operator topology. So U is an element of $\Pi(X^*, W^*)$, and

$$< Ux^*, x >= lim < P_t^* x^*, x >= lim < P_t x, x^* >,$$

for all $x \in X$ and $x^* \in X^*$.

Now, $U^* \in \Pi(X^{**}, W^{**})$, and $U^* x = x$ for all $x \in W$. But $U^* x$ need not be in W for all $x \in X$. For this, we define

$$P : X \longrightarrow W \quad Px = JU^* x,$$

for all $x \in X$, where J is the contractive projection from W^{**} to W. Then P is a projection and

$$\|P\| \leq \|J\| \left\|U^*\right\| \leq \left\|U^*\right\| \leq \underline{lim} \left\|P_t^*\right\| = \underline{lim} \|P_t\| = r.$$

Hence P is minimal. This ends the proof.

$\square$

2. Existence of minimal projections in some function spaces

Let X and Y be Banach spaces ,and G, H be subspaces of X and Y respectively. Let $W = X + Y$, and $V = X \otimes H + G \otimes Y$. Both V and W are subspaces of $X \overset{\eta}{\otimes} Y$ for any uniform cross norm η on $X \otimes Y$. The existence of minimal projections on V and W was discussed in [2] and [5] for $X = L^p(S, \mu) = Y$, where μ is a finite(or σ-finite) measure on S and $1 \leq p < \infty$, and for $X = Y = C(D)$, the space of continuous functions on the compact space D in [3]. In [1], it was asked if there exits a minimal projection from $C(S \times T)$ onto $C(S) \otimes H + G \otimes C(T)$, with G and H finite dimensional. In this section we answer this question for G and H are finite co-dimensional. Some other results are presented.

THEOREM 1. *Let S and T be finite measure spaces and X be any Banach space. Then there is a minimal projection J from $L^p(S \times T, X)$ onto $L^p(S, X) + L^p(T, X)$. Further $\|J\| = 3$.*

Proof. From the theory of tensor product [5], we have: $L^p(M, X) = L^p \overset{n(p)}{\otimes} X$, for any measure space (M, μ), and any Banach space X, where $n(p)$ is the $p-$nuclear cross product norm [5], on $L^p \otimes X$. Hence,

$$L^p(S, X) + L^p(T, X) = L^p \overset{n(p)}{\otimes} X + L^p \overset{n(p)}{\otimes} X = [L^p(S, \mu) + L^p(T, \vartheta)] \overset{n(p)}{\otimes} X.$$

Let P be a minimal projection from $L^p(S \times T)$ onto $L^p(S,\mu) + L^p(T,\vartheta)$ [5]. Define the projection

$$J : L^p(S \times T, X) \longrightarrow [L^p(S,\mu) + L^p(T,\vartheta)] \overset{n(p)}{\otimes} X,$$

with $J = P \otimes I$. We claim J is minimal. Indeed, if J is not minimal then there exists a projection Q such that $\|Q\| < \|J\|$. Let $z \in X$ and $z^* \in X^*$ such that $< z, z^* >= \|z\| = \|z^*\| = 1$. Define $F : L^*(S \times T) \longrightarrow L^*(S,\mu) + L^*(T,\vartheta)$, $F(f) =< Q(f \otimes z), z^* >$. Then F is a projection and $\|F\| \leq \|Q\| < \|J\| =\|P\|$. This contradicts the minimality of P. Thus J is minimal. As for the norm of J, we $\|J\| =\|P\|$. But $\|P\| = 3$, [3]. This ends the proof.

□

We should remark that the same result holds if L^p is replaced by $C(K)$, the space of continuous functions on a compact space K. Now we prove

THEOREM 2. *Let G be a finite dimensional subspace of* $X = C(T)$ (*or* $L^1(T)$), *and P be a minimal projection onto G. Then* $I - P$ *is a minimal projection onto* $ker(P)$.

Proof. Let H =ker(P), and $r = inf\{\|Q\| : Q \in \Pi(X, H)\}$.Then there is a sequence of projections Q_n in $\Pi(X, H)$, such that $\|Q_n\| \longrightarrow r$. So $X = H \oplus G_n$. Put $P_n = I - Q_n$. Hence $H \subseteq ker(P_n)$ for all n. Define $T_n : X \longrightarrow X/H$ by $T_n(x) = [P_n x]$, where$[z]$ denotes the coset of z in X/H. Then T_n is well defined. Since X/H is finite dimensional, it follows that $X/H \simeq (X/H)^{**} \simeq (H^\perp)^*$, where $H^\perp$ denotes the annihilator of H. Hence from theory of tensor products of Banach spaces [5], we get $T_n \in (X \overset{\wedge}{\otimes} H^\perp)^*$. Since X is separable and $H^\perp$ is finite dimensional, there is a subsequence of T_n that converges in the $w*$-topology. Assume that T_n itself converges to T. Thus, $< T_n x, x^* > \longrightarrow < Tx, x^* >$, for all x in X and x^* in $H^\perp$. But $< [P_n x], x^* >=< P_n x, x^* >$ for all x in X and x^* in $H^\perp$, since $P_n x$ is not in H. Since(ker(Q_n))*is isomorphic to $H^\perp$, it follows that $< P_n x, x^* >$ converges in the weak operator topology. Let P be the limit of P_n, so $< P_n x, x^* > \longrightarrow < Px, x^* >$ for all x in X and x^* in X^*. Since $P_n + Q_n = I$, it follows that Q_n converges to some Q in the weak operator topology. Further, $P + Q = I$. Since for each x in H, $P_n x = 0$, it follows that $Qx = x$ for all x in H, and Q is a projection. Being the weak operator limit of Q_n , we have $\|Q\| \leq lim \|Q_n\| = r$. From the definition of r we get $\|Q\| = r$, and Q is minimal. From the Daugavit property of $C(T)$ (and of $L^1(T)$), it follows that P is minimal on ker(Q). This ends the proof.

□

REMARK 1. The existence of the minimal projection Q in Theorem 2 is independent of $C(T)$ and $L^1(T)$, since only separability of the space X was used.

Now we are ready to answer the question of Cheney and Franchetti ([1]) for finite co- dimensional subspaces.

THEOREM 3. *If G and H are finite co-dimensional subspaces of $C(K_1)$ and $C(K_2)$ respectively, then there exists a minimal projection from $C(K_1 \times K_2)$ onto $W = C(K_1) \overset{\vee}{\otimes} H + G \overset{\vee}{\otimes} C(K_2)$. Further, the minimal projection onto W is a Boolean sum of two minimal projections.*

Proof. It follows from the proof of Theorem 2 that there exists finite dimensional supplements $\overline{G}$ and $\overline{H}$ of G and H respectively and minimal projections P and Q on $\overline{G}$ and $\overline{H}$ such that $I - P$ is minimal onto G and $I - Q$ is minimal onto H. From Theorem 3.1 of [4], $P \otimes Q$ is a minimal projection onto $\overline{G} \otimes \overline{H}$. Now $C(K_1 \times K_2) = (\overline{G} \otimes \overline{H}) \oplus W$. Further $W = ker(P \otimes Q)$. Again, by Theorem 2.2 $I \otimes I - P \otimes Q$ is minimal. But $I \otimes I - P \otimes Q = I \otimes (I - Q) + (I - P) \otimes Q - P \otimes Q$. This ends the proof. □

It should be remarked that Theorem 3 holds true if $C(K)$ is replaced by $L^1(S, \mu)$ for some finite(or $\sigma-$finite) measure space(S, μ).

3. The form of minimal

Let X and Y be Banach spaces, and G and H be subspaces of X and Y respectively. Not much is known in general about the form of the minimal projection of $X \overset{\eta}{\otimes} Y$ onto any of the subspaces $W_1 = X \otimes H$, $W_2 = G \otimes Y$ and $W_1 + W_2$. In this section we will discuss the form of minimal projections in certain classes of tensor product spaces.

THEOREM 4. *Let G be a closed subspace of a Banach space X . The following are equivalent:*

(i) There is a unique minimal projection P from X onto G .

(ii) There is a unique minimal projection J from $\ell^1(X)$ onto $\ell^1(G)$ and $J = I \otimes P$.

Proof. $(i) \Rightarrow (ii)$. Let P be a minimal projection from X onto G, and assume P is unique. Consider the projection $J : \ell^1(X) \longrightarrow \ell^1(G)$, defined by $J(f \otimes x) = f \otimes Px$, noting that $\ell^1(X) = \ell^1 \overset{\wedge}{\otimes} X$. Then $\|J\| = \|P\|$. If J is not minimal then there is a projection L onto $\ell^1(G)$ and $\|L\| < \|J\|$. Define $Q : X \longrightarrow G$ defined by, $Qx = P_1 L(\delta_1 \otimes x)$, where P_1 is the first coordinate projection. Then, Q is a projection X onto G and $\|Q\| \leq \|L\| < \|J\| = \|P\|$. This contradicts the minimality of P. Hence J is minimal. Further J is unique, for otherwise $P_1 J$ will be minimal projections on G, contradicting uniqueness of P .

$(ii) \Rightarrow (i)$. Let J be a unique minimal projection of $\ell^1(X)$ onto $\ell^1(G)$. Define $P : X \longrightarrow G$, by $P(x) = < J(\delta_1 \otimes x), \delta_1 >$. Then P is a projection and $\|P\| \leq \|J\|$. If P were not minimal, there is a projection $Q : X \longrightarrow G$ such that $\|Q\| < \|P\|$. But then $L = I \otimes Q$ is a projection of $\ell^1(X)$ onto $\ell^1(G)$, with $\|L\| \leq \|Q\| < \|P\| \leq \|J\|$, which contradicts the minimality of J. Thus P is minimal. Now, since P is minimal, the projection $L = I \otimes P$ is minimal. Since J is unique, we get $J = L$. This ends the proof.

□

We should remark that if uniqueness is not assumed, then we have

THEOREM 5. *A projection J from $L^1(S, X)$ onto $L^1(S, G)$ is minimal if and only if the projection $P : X \longrightarrow G$, $Px = J(f \otimes x), f^* >$ is minimal, where $f \in L^1(S, \mu)$ and $f^* \in L^\infty(S, \mu)$ and $< f, f^* >= 1$, for any $\sigma-$finite measure space(S, μ).*

The proof follows the same line as that of Theorem 4 and will be omitted.

Now let $X = \ell_2^1 = \{(x, y), \|(x, y)\| = |x| + |y|, x, y \in R.\}$, and $G = [z]$, the span of z in ℓ_2^1. Then

THEOREM 6. *Every minimal projection from $\ell_2^1(X)$ onto $\ell_2^1(G)$ has the form $J = I \otimes P$ for some minimal projection P from X onto G.*

Proof. Let us write V for $\ell_2^1(G)$ and W for $\ell_2^1(X)$. The space ℓ_2^1 has a basis $\{\delta_1, \delta_2\}$, where $\delta_1 = (1, 0)$, and $\delta_2 = (0, 1)$. Then V has $\{\delta_1 \otimes z, \delta_2 \otimes z\}$. Hence any projection from W onto V has the form $L = f_1^* \otimes (\delta_1 \otimes z) + f_2^* \otimes (\delta_2 \otimes z)$ where $f_i^* \in \ell_2^\infty(X)$, and

$$< f_i^*, \delta_j \otimes z >= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1}$$

Now, $f_i^* \in \ell_2^\infty(\ell_2^\infty)$. So $f_i^* = \delta_1 \otimes x_i^* + \delta_2 \otimes y_i^*$, where x_i^*, and y_i^* are in ℓ_2^∞. It follows from (1) that

$$< x_1^*, z >=< y_2^*, z >= 1, \text{ and } < y_1^*, z >=< x_2^*, z >= 0.$$

Thus

$$L = (\delta_1 \otimes z_1^* + \delta_2 \otimes a_1^*) \otimes (\delta_1 \otimes z) + (\delta_1 \otimes a_2^* + \delta_2 \otimes z_2^*) \otimes (\delta_2 \otimes z),$$

where $< z_i^*, z >= 1$, and a_i^* are in the annihilator of G, and $\|z_i^*\| = 1$. Now, $\|L\| = sup\{\|L(\delta_{i\otimes}x)\| : x \in X, \|x\| = 1, i = 1, 2\}$. If we calculate $\|L(\delta_1 \otimes x)\|$ we find :

$$\begin{aligned} \|L(\delta_1 \otimes x)\| &= \|(< z_1^*, x > \delta_1 + < a_2^*, x > \delta_2) \otimes z\| \\ &= |< z_1^*, x >| + |< a_2^*, x >|, \end{aligned} \tag{2}$$

we are taking $\|z\| = 1$.

If $z = (s, t)$ with $s + t = 1$, and $s \neq 0$, $t \neq 0$, then z is a smooth point of $B_1(\ell^1(G))$, and so there is only one z^* in $B_1(\ell^\infty(G^*))$ such that $< z^*, z >= 1$. Thus $z_1 = z_2$. Further, since $< a_i^*, z >= 0$, then one of the two coordinates of a_i^* must be positive, say the first coordinate, which we denote by α_{i1}. Now choose $x = \delta_1$. Then from (2) we get

$$\|L(\delta_1 \otimes x)\| \geq 1 + \alpha_{i1} > \|z_1^*\|.$$

So $\|L\| > \|z_1^*\|$. Similarly $\|L\| > \|z_2^*\|$. Define $J : \ell^1(X) \longrightarrow \ell^1(G)$, as $J = (\delta_1 \otimes \delta_1 + \delta_2 \otimes \delta_2) \otimes (z_1^* \otimes z_2^*) = I \otimes P$, where $P = z_1^* \otimes z$. Then since $\|z\| = 1$, it follows that $\|J\| = \|z_1^*\|$. Thus $\|J\| < \|L\|$. This contradicts the minimality of L. So minimal projections on $\ell^1(G)$ are of the form $I \otimes P$. This ends the proof.

□

THEOREM 7. *Let G be a (closed) complemented subspace in the Banach space Y. Let X be a Banach space such that $X \overset{\wedge}{\otimes} G$ is a closed subspace of $X \overset{\wedge}{\otimes} Y$. Then if there is a minimal projection L from $X \overset{\wedge}{\otimes} Y$ onto $X \overset{\wedge}{\otimes} G$, then there is a minimal projection of the form $I \otimes P$, for some minimal projection from Y onto G.*

Proof. Let L be a minimal projection from $X \overset{\wedge}{\otimes} Y$ onto $X \overset{\wedge}{\otimes} G$. Let $x \in X$ be a fixed element in X. Choose $\phi \in X^*$ such that $\varphi(x) = \|\varphi\| = 1$. Define:

$$T : Y \longrightarrow X \otimes Y, \quad T(y) = x \otimes y.$$

$$B : X \otimes G \longrightarrow G, \quad B(\sum_{i=1}^{n} x_i \otimes g_i) = \sum_{i=1}^{n} \varphi(x_i) g_i.$$

Using T and B we define $P : Y \longrightarrow G$ by $P = BLT$. Since $\|T\| = \|B\| = 1$, it follows that $\|P\| \leq \|L\|$. Further, if $g \in G$ then $P(g) = BLT(g) = BL(x \otimes g) = B(x \otimes g) = g$. Thus P is a projection. Define $J : X \otimes Y \longrightarrow X \otimes G$, by $J = I \otimes P$. Then $\|J\| = \|P\| \leq \|L\|$. Since L is minimal we have $\|J\| = \|L\|$, and J is minimal. That P is minimal is immediate. This ends the proof.

□

A consequence of Theorem 7 is

LEMMA 1. *Let G be a finite dimensional subspace of $C(T)$, with T a compact metric space. If J is a minimal projection from $C(T \times T)$ onto $C(T) \otimes G$, then $\|I - J\| = 1 + \|J\|$.*

Proof. By Theorem 7, there is a minimal projection L onto $C(T) \otimes G$ of the form $I \otimes P$, for some minimal projection P onto G. Thus $I - L = I \otimes (I - P)$. Since G is finite dimensional in $C(T)$, it follows that $\|I - P\| = 1 + \|P\|$ .Thus $I - L$ is minimal. Hence,

$$\|I - J\| \geq \|I - L\| = 1 + \|P\| = 1 + \|L\| = 1 + \|J\| \tag{3}$$

But $1 + \|J\| \geq \|I - J\|$. It follows from (3) above that $\|I - J\| = 1 + \|J\|$. This ends the proof.

□

It should be remarked that Lemma 1 holds true if $C(T \times T)$ is replaced by $L^1(T \times T)$.

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AMS Subject Classification: 41A65, 41A35.

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Lavoro pervenuto in redazione il 06.02.2002 e, in forma definitiva, il 10.12.2002.