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A NOTE ON BTT-DEGREES

Abstract. We prove that there are computable enumerable incomplete btt-degrees with the anticupping property. In fact no simple set is btt cuppable in the (not necessarily computable enumerable) btt-degrees. This solves a question of Odifreddi and settles the situation for the one remaining reducibility where the answer was unknown.

1. Introduction

Reducibilities calibrate sets into equivalence classes of equicomputability. Turing reducibility is obtained by adding characteristic functions to Kleene schemes, or alternatively allowing arbitary oracle queries in the procedure. By restricting the types of oracle queries allowable, one obtains finer and finer calibrations of information content. For instance, m-reducibility is obtained by saying that $A \leq_m B$ iff there is a computable function f such that $x \in A$ iff $f(x) \in B$. m-degrees give a very fine classification of sets, and every T-degree splits into infinitely many m-degrees. (Jockusch [8])

While Turing reducibility is the most general type, strong reducibilities are also of considerable interest. Quite aside from the intrinsic interest which stems from asking what happens if we restrict the kinds of access mechanism one has to an oracle, if one considers applications one is naturally drawn to consider strong reducibilities. The point is that in applications of computability theory in classical mathematics, the kinds of reducibilies one usually obtains are, by and large, strong ones. For instance, there are word problems for finitely presented groups (or even cancellation semigroups) in each tt-degree (Collins [2]) but not in each m-degree (nor even in each btt-degree[†] (Jockusch [9])). The natural reduction of the word problem to the conjugancy problem is an m-reduction. One gets an m-reduction in the solution to Hilbert's 10th problem, and many other undecidability results. For computably presented vector spaces one has bases in each wtt-degree below the wtt-degree of the dependence relation, but not each T-degree. (Downey and Remmel [5]) There are many other examples.

In this paper, we will examine btt-reducibility. Recall that $A \leq_{btt} B$ if $A \leq_{tt} B$ via some truth table reduction with bounded norm. (That is, there is a computable function f such that, for all x, $x \in A$ iff $B \models \sigma_{f(x)}$ where σ_y denotes the y-th truth table (so $A \leq_{tt} B$), and furthermore the number of questions is bounded by some fixed n, called the norm. (Post [13], or Odifreddi [14], page 331.) Naturally, if the norm is n then there are at most n distinct elements in the table

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[†]Specifically, Jockusch proved that if a set A represents a word problem in a computably presented group (in the classical sense of having a computable set of generators and relations), it must be *multiplicative* in the sense that there exists a computable function g such that for all x, y $g(x, y) \in A$ iff $g(y, x) \in A$. Jockusch [9] then observed that a simple finite injury proof constructs a c.e. btt-degree containing no multiplicative set, and hence cannot be the btt-degree of the word problem of a computably presented group.

$x_1, ..., x_n$ queried. We use 1-tt, 2-tt, etc with the obvious meaning. Also, we will write $\mathbf{0}'_{btt}$ for the complete computably enumerable set under btt-reducibility.

btt-reducibility is of some interest since it throws considerable light upon both tt- and m-reductions, and points at limits of construction types. For instance, A set is btt-reducible to the halting problem iff it is in the smallest boolean algebra generated by the computably enumerable sets. (See e.g. Odifreddi [14], Proposition III.8.7). Similarly, it is an old elementary result in computability theory that $A \equiv_T B$ iff $A' \equiv_m B'$ (e.g. Soare [15], Ch III 2.3). On the other hand, Mohrherr [12] and, independently, Bickford and Mills [1], constructed computably enumerable sets $A|_T B$ such that $A' \equiv_{tt} B'$. Finally Mohrherr [12] proved that the Mohrherr-Bickford-Mills result cannot be extended to btt reducibility by proving that $A' \leq_{btt} B'$ implies $A \leq_T B$, and hence $A' \leq_{btt} B'$ iff $A' \leq_m B'$.

In the present paper we solve the remaining "anticupping" question of Odifreddi [14]. Results analogous to the following theorem are known for all other major reducibilities.

THEOREM 1. *(i) There exists a computably enumerable btt-degree* $\mathbf{a}$ *strictly below* $\mathbf{0}'_{btt}$ *such that for all (not necessarily computably enumerable) btt degree* $\mathbf{x}$, $\mathbf{x} \cup \mathbf{a} \geq \mathbf{0}'_{btt}$ *implies* $\mathbf{x} \geq \mathbf{0}'_{btt}$. *(That is,* $\mathbf{a}$ *does not* cup *above* $\mathbf{0}'_{btt}$*, in the btt-degrees.)*
(ii) In fact, the btt-degree of a simple set is not btt-cuppable in this strong sense.

We remark that the proof technique is nonuniform and ought to be applicable to other btt-questions. None of the other techniques such as Lachlan [10] and Downey [3] seem directly applicable, although the very idea of a nonuniform method comes from Lachlan [10], and we use "good" and "bad" elements along the lines of the proof of Lachlan's theorem from Soare [15], page 145, Ex 4.16. However, the strategies of Lachlan and here are quite different.

2. The proof of Theorem 1

Let A be a simple computably enumerable set. We will build an m-complete computable enumerable set B to meet the requirements

$$\mathcal{R}_e : (B \leq_{btt} A \oplus X \text{ via } \alpha_e) \to B \leq_{btt} X.$$

Here by abuse of notation, we regard the α_e'th truth table as having the norm in question.

2.1. The case of m-reducibility

To make the argument clear, we first consider the easy case that α_e is simply the e-th possible m-reduction. Thus we will prove *no simple set is m-cuppable,* generalizing the well-known theorem of Post [13], that no simple set is m-complete.

So here we can regard α_e as simply saying

$$x \in B \text{ iff } \varphi_e(x) \in A \oplus X,$$

with φ_e the e-th partial computable function. Now B will have 0-th column $B^{(0)} = \{\langle x, 0\rangle : x \in K\}$. We will devote column $e+1$ to the satisfaction of $\mathcal{R}_e$. We will say that an element y is *bad* if $\varphi(y) \in 2 \cdot \omega$, (i.e. $\varphi(y)$ is even). The reason we regard y as bad, is because its entry into B is more or less decided by the even part, $A \oplus \emptyset$ rather than X. This conflicts with our idea which is to try to code K into $B^{(e+1)}$ in such a way that X can recover this fact.

We will pursue this idea via coding markers, $c(e, z)$ which will denote the coding location for $z \in K$? for requirement $\mathcal{R}_e$. At any stage s we will have a least as yet unassigned $z = z(e, s)$ which initially is 0.

All the requirements work independently. While $\mathcal{R}_e$ is not, as yet, declared met, we do the following.

Let $\ell(e, s)$ denote the length of convergence of α_e. That is, $\ell(e, s) = \max\{z : \forall z' \leq z(\varphi_{e,s}(z') \downarrow\}$. We will also build an auxiliary computably enumerable set C_e for $\mathcal{R}_e$. This set is used to ensure that there cannot be too many bad elements around, and is used to challenge the simplicity of A.

Let s be the least stage where $\ell(e, s) > \langle y, e+1\rangle$. First see if there is some $\langle m, e+1\rangle$ which is bad, and yet $\varphi_e(\langle m, e + 1\rangle) \in A_s \oplus \emptyset$. If this is the case, then declare $\mathcal{R}_e$ as met (since it will be the case that $\langle m, e + 1\rangle \notin B$).

We need to figure out whether $\langle y, e + 1\rangle$ should be the position of a coding marker or not.

Case 1. $\langle y, e + 1\rangle$ is good.

Action: Define $c(z) = \langle y, e + 1\rangle$. For all stages $s' \geq s$, while $\mathcal{R}_e$ is not declared met, put $c(z)$ into $B_{s'}$ whenever $z \in K_{s'}$.

Case 2. $\langle y, e + 1\rangle$ is bad. There are two subcases.

Case 2a. For some $y' \leq y$, $\varphi_e(\langle y', e + 1\rangle) = \varphi_e(\langle y, e + 1\rangle)$. (Necessarily, $\langle y', e + 1\rangle$ is bad.)

Action: Put $\langle y', e + 1\rangle$ into B. Declare $\mathcal{R}_e$ as met.

Case 2b. Otherwise.

Action: Put n into C_e, where $2n = \varphi_e(\langle y, e + 1\rangle)$.

Verification for the m-degree case. First note that if $\mathcal{R}_e$ is ever declared met then for some m, $\langle m, e + 1\rangle \in B$ iff $\varphi(\langle m, e + 1\rangle \notin A$. This is because bad elements are never used for coding purposes, and are never put into B except for diagonalization . In Case 2a, one of y or y' will take the role of m. Therefore if for some set X $A \oplus X \geq_m B$ via α_e, we can suppose that $\mathcal{R}_e$ is never declared satisfied.

As a consequence, we claim that Case 2b can pertain only finitely often. Otherwise C_e would be an infinite computably enumerable set. Since A is simple, we see that for some n, $n \in C_e \cap A$. So, for some bad $\langle y, e + 1\rangle$, $\varphi_e(\langle y, e + 1\rangle) = 2n \in A \oplus \emptyset$ so that $\mathcal{R}_e$ would be met, assuming that $\lim_s \ell(e, s) = \infty$.

Since Case 2b can pertain at most finitely often, we see that for almost all y, $\langle y, e + 1\rangle$ is good. To decide if $z \in K$, find the y with $c(e, z) = \langle y, e + 1\rangle$. Then, assuming that $A \oplus X$ is m-above B via α_e, $z \in K$ iff $1/2(\varphi_e(\langle y, e + 1\rangle) - 1 \in X$.

2.2. The 1-tt case

Note that the above can easily be adapted to the 1-tt case. For the 1-tt case, The computably enumerable set C_e can be thought of as the union of two c.e. sets C_e^+ and C_e^-. We will have a similar notion of an element being bad. An element being bad will mean that the query is to the A-side. If the query is negative, rather than positive, as in the m-case, then we can put the element $\langle y, e + 1\rangle$ into B and $2n$ into C_e^- a set built on the assumption that there are infinitely many reductions of the form $\langle y, e + 1\rangle \in B$ iff $\varphi(\langle y, e + 1\rangle) \notin A \oplus \emptyset$.

If the query is positive, as in the m case, we put in C_e^+. Note that if we assume that $A \oplus X \geq_{1-tt} B$ via α_e, both of these sets are finite by the same argument, that Case 2b pertains only finitely often in the m-degree case. Hence $C_e = C_e^+ \cup C_e^-$ is finite. Thus, again, almost all elements are good, and hence their fates are decided by X.

2.3. The 2-tt degree case

In the case that the norm is bigger than one we proceed similarly, except that the number of cases grows. The 2-tt case is enough for the problems of a general norm, and we will prove this in detail. So suppose now that $\alpha_e(x)$ is a 2-tt reduction concerning two elements x_1, x_2. We say that x *projects* to x_1 and x_2. Again we devote $B^{(e+1)}$ to $\mathcal{R}_e$. This time the coding of K into X is nonuniform depending on the outcome. We assume in the following that α_e is total, and the module acts at α_e expansionary stages. (That is where the length of convergence increases.) We will have coding markers $c(x)$ (dropping the "e"). for the good x. (A *good* x is one that projects purely into X as we see below.) If we fail for some x to define this, then we will try for back up markers $c'(x, p)$ for $p \in \omega$. These elements correspond to elements which are "somewhat good" in the sense that part of them projects into X and part into A. Finally if we fail for some x and as well for some p, then we will have additional back-up markers $c''(x, p, d)$. Again the requirements do not interfere with each other since they work on differing rows of B. The formal details are as follows.

Construction, stage s.

At the beginning of stage s we will have a currently active z, (this is the "primary" coding marker not yet assigned, so that $c(z) \uparrow$) a least unassigned corresponding $c'(z, p) \uparrow$, and a least unassigned $c''(c, p, r)$. There will also be associated auxiliary sets C_e, $C_{e,z}$. Elements can be 1-reasonable, 1-active, or 2-reasonable. Finally associated with the 1-reasonable elements, will be a restraint $r(1, s)$.

Find the least y such that y has not yet been considered, and $\ell(e, s) > \langle y, e+1\rangle$ for the first time.

Case 1. $\langle y, e+1\rangle$ projects to two (or fewer) elements both of which are odd. That is, y is *good.*

Action: Find the least z with $c(z) \uparrow$. Assign $c(z) = \langle y, e+1\rangle$ and initialize all $c'(z, p)$ and $c'(z, p, d)$ which have been defined up to this point, in the sense that they will be no longer coding markers. Also initialize all 1-reasonable, 1-active, and 2-reasonable elements. We declare $c(z+1, 0)$ to be *active.*

Case 2. y is not good.

Case 2a. See if y is 1-*reasonable*. We say that y is 1-reasonable, in the case that $\langle y, e+1\rangle$ projects to two elements, one, y_1, of which is even and one odd.

Action: In this case we put $1/2 \cdot y_1$ into the auxiliary set $C_{e,z}$.

Subcase 1. There is some 1-reasonable element $y' \leq y$, such that y''s even projection is in $A \oplus \emptyset$. and y''s even projection is bigger than $r(1, s)$.

Action: For the least such y' declare that y' is $1-$*active,* and for the least $c'(z, p)$ not yet assigned, assign $c'(z, p)$ to $\langle y', e+1\rangle$. Declare that *while* $c(z)$ *is not assigned,* $p \in K$ iff $c'(z, p) \in B$. Reset $r(1, s+1) = s+1$.

Initialise all $c''(p, q, r)$ and 2-reasonable elements.

Subcase 2. Although $\langle y, e+1\rangle$ has an even projection, this projection is below $r(1, s)$. In this case we say that y is 2-*reasonable.* Note that this is the first stage that y is considered, and hence y is not, nor has it been 1-active.

Action: For the least unassigned $c''(z, p, d)$ assign $c''(z, p, d)$ to $\langle y, e+1\rangle$. The promise is that *provided that* $c'(z, p)$ *is not defined,* $d \in K$ *iff* $c''(z, p, d) \in B$.

Case 3. $\langle y, e+1\rangle$ projects purely into 2ω.

Case 3a. There exists an element y' also purely projecting into A such that $\langle y', e+1\rangle \notin B_s$ and y and y' project to the *same* elements.

Action: Put $\langle y', e+1\rangle$ into B.

Case 3b. For some $\langle y', e+1\rangle$ not yet B, we have its projection in $2A$.

Action: Assuming that the length of agreement exceeds $\langle y', e+1\rangle$, which it does by assumption, we diagonalize by enumerating $\langle y', e+1\rangle$ into B.

Case 3c. There exist elements y', y'', y''' projecting purely into 2ω, with $\langle y', e+1\rangle$, $\langle y'', e+1\rangle$, $\langle y''', e+1\rangle \notin B_s$, such that they all project to a pair of elements one of which is a fixed $a \notin 2A$, and the other is in $2A$.

Action: Put $\langle y', e+1\rangle$ into B. (Note that this will diagonalize).

Case 3d. Otherwise.

Action: If there is some x in the projection of y not yet in C_e put x into $C_{e,s}$.

End of Construction.

Verification.

LEMMA 1. *An element can never be more than one of good, 1-active, or 2-reasonable.*

Proof. If an element is good, then it projects solely into the odds, so it cannot be either of the other categories. To be one of the other categories it must be 1-reasonable. When it first becomes 1-resonable if it is below $r(1, s)$ then it can never be 1-active, because of the monotonicity of $r(1, s)$. Similarly if it is below $r(1, s)$ and not yet 1-active then it can never become that way. If it is one active then by construction, we make sure that 1-cative elements never become anything else, except cancelled.

□

LEMMA 2. *If Case 1 pertains infinitely often, then* $K \leq_{btt} X$.

Proof. If Case 1 pertains infinitely often then $c(z)$ is assigned for every z. If $A \oplus X = B$ via α_e then to decide if $z \in K$, we compute $c(z)$, and use the fact that $c(z)$ projects purely into $\emptyset \oplus X$ to construct a 2-tt reduction computing B from X.

□

For the remainder, we suppose that case 1 only pertains finitely often. Hence there some fixed least z such that $c(z) \uparrow$.

LEMMA 3. *Suppose that there are infinitely many* 1*-active elements. Then* $K \leq_{btt} X$.

Proof. Since there are infinitely many 1-active elements, $c'(z, p)$ will be defined for all p. Notice that $c'(z, p)$ is defined only if its even projection lies already in $A \oplus \emptyset$. (That is what 1-active means.) Therefore, we can conclude that $K \leq_{1-tt} X$ via the odd half of the projection as in the 1-tt case.

□

LEMMA 4. *If there are infinitely many* 1*-reasonable elements* $\lim_s \ell(e, s) = \infty$ *but only finitely many* 1*-active elements, then* $C_{e,z}$ *is finite.*

Proof. If $C_{e,z}$ is infinite, then, by simplicity of A, $A \cap C_{e,z}$ would be infinite. Hence infinitely many elements would exceed $r(1, s)$ for any fixed s. This would give rise to infinitely many 1-active elements.

□

LEMMA 5. *If there are infinitely many* 1*-reasonable elements* $\lim_s \ell(e, s) = \infty$ *but only finitely many* 1*-active elements, then* $K \leq_{1-tt} X$.

Proof. By the above, we know that $C_{e,z}$ is finite, and $\lim_s r(1, s) = r$ exists. Hence there is a fixed p least, such that $c'(z, p) \uparrow$, and a stage s_0 after which $r(1, s)$ does not increase. Let $t > s_0$ be a stage where $A_t \upharpoonright r = A \upharpoonright r$. We know that after stage s_0 every even projection will be below r, and after stage t we know precisely what the correct answer is, via the $A_t \upharpoonright r$ lookup table. Since there are infinitely many 1-reasonable elements, almost all of them are 2-reasonable, and hence, for almost all d, $d \in K$ iff $c''(z, p, d) \in B$ iff $A \upharpoonright r \oplus X \models \alpha_e(\langle y, e + 1\rangle)$ where $c''(z, p, d)$ is assigned to y. This is a 1-reduction.

□

LEMMA 6. *Suppose that there are only finitely many* 1*-reasonable elements. Then* $B \not\leq_{btt} A \oplus X$ *via* α_e.

Proof. If $\ell(e, s) \to \infty$, then we will consider infinitely many y in the course of the construction. Each time we see another y, if it is not good or 1-reasonable it's even part must be above $r(1, s)$, or it must project purely into the evens. If there are infinitely many elements of the former case, then we put infinitely many elements into $C_{e,z}$ causing infinitely many to enter A by simplicity, making infinitely many 1-active elements. In the case that almost all of the projections are purely evens, consider the set C consisting of all the purely even projections of $\langle y, e + 1\rangle$'s. If this set is finite, then there will be many elements with the same projections, and we will diagonalize in case 3a. If the set is infinite then its intersection with A is infinite. Now *either* there are infinitely many elements with *both* of their projection in A, in which case we diagonalize in case 3b, *or* there are a finite set of elements not in A common to all the projections. (A simple set meets a bounded disjoint strong array.) Then we will apply case 3c to infinitely many such elements. This wins either directly, or by a reduction to case 3a.

□

The reader should note that we could have simplified the case where $\langle y, e + 1\rangle$ projects purely into 2ω by assigning coding markers $k(d)$ for $d \in K$ for each such y based on the assumption that almost all elements project into the evens. The result or this outcome would be that $K \leq_{btt} A$, then we could use Post's result [13] that no simple set is btt-complete. The method we use basically proves this and keeps the paper self contained.

2.4. The general case

The general case follows the pattern above and uses induction on the norm of the reductions. So suppose that we are considering α_e's of norm $n + 1$. The analog of case 1 is that the element $\langle y, e + 1\rangle$ projects purely into the odds. This makes progress on building a reduction from K to X, because X alone decides the fate of such elements. At the other extreme, the analog of case 3 is that almost all elemets projects purely into the evens. In that case we have the choice of building a reduction from k to A and appealing to Post's theorem that no simple set is btt-complete, or using the strategy above to diagonalize. That is observe that either the projection is into a finite set (generated by some number r), in which case diagonalization is easy (infinitely many elements tied to the same finite set); or, the range of the projection is infinite, in which case we can wait for infinitely many of the elemants from the projection to enter A. If an element $\langle y, e + 1\rangle$ projects into the evens via a btt-reduction with norm $n + 1$, and one of its table enters A, then the reduction is in fact equivalent to a btt-reduction with norm n. So, in this way we can

inductively argue that either we diagonalize (because the projection is finite) or get $K \leq_{btt} A \oplus X$ via a reduction with norm n.

If these extreme possibilities do not occur, then infinitely many elements must project into both the evens and the odds. Again either the even part is bounded, or it is not. In the latter case, the simplicity of A will ensure that there is a stream of "1-active" elements: elements where at least one of their even projection enters A, allowing us to build a reduction with norm n from K to $A \oplus X$. In the former case, the range of the even projection is finite, and bounded by r. Then once $A \upharpoonright r = A_s \upharpoonright r$, we can build a reduction from K to $A \oplus X$ with smaller norm, since we know the answer on the evens.

3. Open Questions

There is a general pattern here. This pattern concerns the relationship between the position of a c.e. set in the partial ordering of c.e. set and cupping properties.

Recall that a set is called hypersimple if it is noncomputable and meets all strong arrays. Also a set is called η-maximal if it is maximal relative to some c.e. equivalence relation. Finally a set A is called semicomputable if there is a computable function f such that for all x, y, $f(x, y) \in \{x, y\}$ and if $f(x, y) = z$, then $z \in A$ implies that both x and y are in A.

No simple set is m- or btt-cuppable. Also Downey and Jockusch [4] proved that no hypersimple set is wtt-cuppable. And certainly no hypersimple set is tt-complete (Friedberg and Rogers [6]). Finally Marchenkov [11] solved Post's problem in the manner of Post by showing that a η-maximal semicomputable set is not T-complete. Odifreddi has asked the following questions.

Question 1. Is a hypersimple set ever tt-cuppable?

Question 2. Is a η-maximal semicomputable set ever T-cuppable?

Finally, Harrington and Soare [7] introduced an elementarily definable property $Q(A)$ of c.e. sets which guarantees incompleteness. Andre Nies has asked the following.

Question 3. If A satisfies $Q(A)$, does this mean that A is not T-cuppable?

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