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AN ALTERNATIVE INTERPRETATION OF THE BEHAVIOR LAW OF MATTER BY MEANS OF A GENERALIZED STOCHASTIC PROCESS

Abstract. Discrepancy between discrete models and continuous theoretical ones is a common concern with the behavior laws of matter. We propose an alternative frame in which the transition from a discrete to a continuous model becomes very natural. A statistical description of matter laws is given in this contexte.

1. Introduction

The aim of this paper is to present a new mathematical background which allows to modelize in a natural way the mechanical behavior of matter. In such a model the transition from the discrete microscopic structure to the macroscopic behavior is very easy. This point of view, involves an alternative mathematical theory called "Radically Frequentist Statistics" (RFS), based on an idealized concept of *very large finite sequence of outcomes*. Such an idealization cannot be performed within the classical mathematical frame based on Zermelo-Fraenkel's set theory. Therefore we use an extended frame based on a conservative extensions with a double scale of magnitude order of ZF, where the concepts of large numbers and small fluctuations can be formalized in a way which is very close to the statistical language. The use of a conservative extentions of ZF with one scale of magnitude order as a mathematical background for modelisation is not news in mechanics (see [2], [3], [4], [5]).

The structure of the paper is the following: first we present roughly an intuitive statistical description of matter; then we present the mathematical model considered . In section 3 we introduced the RFS theory; we end in section 4 with the description of matter laws within RFS.

2. Intuitive statistical description

Consider a number s of macroscopically identical samples of some matter, say plates of concrete or a generalized composite. Divide each sample into adjacent cells of the same size (this last hypothesis is not essential).

Let $X_{i,j,k}$ be a mechanical parameter of interest (Young modulus, shear modulus...).and $x^a_{i,j,k}$ its value for the a^{th} sample. The range of $X_{i,j,k}$ is discretized into finite numbers of little intervals. Thus we may suppose that $X_{i,j,k}$ takes its values in a finite set E. For $a \in E$, we define the frequency

$$fr\left(X_{i,j,k} = a\right) = \frac{1}{s} card\left\{a \leqslant s, x^a_{i,j,k} = a\right\}$$

and the conditional frequency

$$fr\left(X_{i,j,k} = a \mid X_{i_1,j_1,k_1} = a_1, X_{i_2,j_2,k_2} = a_2, \cdots, X_{i_r,j_r,k_r} = a_r\right) =$$
$$= \frac{fr\left(X_{i,j,k} = a X_{i_1,j_1,k_1} = a_1, X_{i_2,j_2,k_2} = a_2, \cdots, X_{i_r,j_r,k_r} = a_r\right)}{fr\left(X_{i_1,j_1,k_1} = a_1, X_{i_2,j_2,k_2} = a_2, \cdots, X_{i_r,j_r,k_r} = a_r\right)}$$

which measures the dependence of the mechanical properties of one cell with respect to the r other cells.

The knowledge of all these frequencies gives an approximative description of the statistical behavior law of this material.

A first way to transform this description into a mathematical model is to introduce a family of random variables $X^{\varepsilon}_{i,j,k}$ where ε is the size of the cells and $f^{\varepsilon}_{i,j,k}$ denotes the associated conditional probabilities relying each cell to the others. For $\varepsilon \to 0$ the corresponding continuous model is rather to manage since we must handle the intricacies of continuous stochastic processes with moderate parameter space. In the present work we introduce an alternative mathematical model, which remains in the ream of finite combinatorics but replaced limit procedures by perfect approximations. This is possible in a slight extension of classical mathematics where absolute orders of magnitude are formalized.

3. The mathematical framework ZFL$_2$

Scientists often deal in an intuitive way with orders of magnitude, large, little, very large, near, very near... and they manipulate informally these fuzzy concepts in order to support their reasoning about real integers. But these concepts have no counterpart within classical mathematics. This is a fundamental weakness of mathematics, in particular as concerns modelisation where the link between micro et macroscopic levels has to be described. Fortunately there are now, since the emergence of A.Robinson's *Non Standard Analysis* [10] in the sixties, various conservative extensions of ZF where absolute orders of magnitude can be introduced in a very natural way. The most famous is E.Nelson's *Internal Set Theory* (IST) [8], an axiomatic setting of *Non Standard Analysis*. Weaker extensions may also be useful to the probabilist, as Nelson showed in his book on *Radically Elementary Probability Theor* y [9] (see also [1]).

In the present paper we introduce an elementary conservative extension of ZF with a double hierarchy of orders of magnitude in which we develop the theory of Radically Frequentist Statistics. Classical mathematics may be formalized in the context of Zermelo-Freankel's set theory (called here ZF). To get the extension ZFL$_2$ (second order Leibniz extension of ZF),we call internal the formulas of ZF we add to the language of ZF the two unary external predicates moderate and weakly moderate and the following axiom rule:

1) 1 is moderate

2) every integer which is lower than a moderate integer is moderate;

3) every integer which is lower than a weakly moderate integer is weakly moderate;

4) if n and m are moderate integers, so are $n + m$, nm and nm;

5) all moderate integers are weakly moderate;

6) there exists a weakly moderate integer which is not moderate;

7) there exists an integer which is not weakly moderate;

It is possible to prove (see [1]) that ZFL$_2$ is a conservative extension of ZF. This means that :

(i) every internal statement which is a theorem in ZFL_2 is also a theorem in ZF;

(ii) all theorems of ZF are theorems within ZFL_2.

Thus ZFL_2 enriches classical mathematics with new concepts, but does not alter the status of internal statements: they are neither more nor less theorems in ZFL_2 than in ZF. This legitimates, from the logical point of view the use of ZFL_2 as a basis for the mathematical practice. But we have external theorems which may be of help in modelisation procedures. Among all external concepts expressible in the language of ZFL_2 we have the following:

DEFINITION 1. *A real number is called moderate (resp. weakly moderate) if and only if its absolute value has a moderate (resp. weakly moderate) integral part. A positive real number which is not moderate (resp weakly moderate) is called large (write $\sim \infty$) (resp. very large (write $\approx \infty$)). A real number is called small (resp. very small) if and only if it is* 0 *or its inverse is not moderate (resp. weakly moderate). Two real numbers x and y are called close (write $x \sim y$) (resp. very close (write $x \approx y$)) if their difference is small (resp. very small)*

The orders of magnitude satisfy the following generalized Leibniz rules:
Concerning the first scale:

THEOREM 1. *Moderate + moderate = moderate,*
moderate × moderate =moderate,
small + small = small,
small × moderate = small.

Concerning the second scale:

THEOREM 2. *Weakly moderate + weakly moderate = weakly moderate,*
weakly moderate × weakly moderate = weakly moderate,
very small + very small = very small,
very small × weakly moderate = very small.

The two scales are linked by the following relations:

THEOREM 3. *Very small ⇒ small,*
very large ⇒ large,
moderate ⇒ weakly moderate.

The proofs are easy consequences of the external axioms (see [1]).
Notice that a good model for the macroscopic continuous is a finite set $x_1 < x_2 < \cdots < x_n$ with $x_{1\approx}x_{2\approx \ldots \approx}x_n$ where $x_1 \approx -\infty$ and $x_n \approx +\infty$. If we use the weak scale, we have an intermediate near-continuous where a very large numbers of x_i remain at a small distance.

4. The theory RFS

In a ZFL_2 context we introduce the Radically Frequentist Statistics (RFS) theory which can be considered as an alternative mathematical foundation of statistics (see [6], [7]). The mean fact about RFS is that all results of Probability Theory which are relevant in statistics have a more general counterpart in RFS. Moreover, these probabilistic statements can be deduced from their RFS counterpart through a purely logical procedure. Thus RFS contains the whole scientific power of Probability Theory as concerns statistical modelisation.

The fundamental concept of RFS is that of *random number,* i.e. a finite sequence $X = (x_1, ..., x_s) \in \mathbf{R}^s$ with very large size s and such that $fr\{|X| > m\} \approx 0$ for every $m \approx \infty$.. Its *mean value* $M(X)$, *variance* $V(X)$, *deviation* $\sigma(X)$ and *distribution function* are defined as usual by the formulas

$$\begin{aligned} M(X) &= \frac{1}{s}\sum_{j=1}^{s} x_j \\ V(X) &= M((X - M(X))^2) = M(X^2) - (M(X))^2, \\ \sigma(X) &= \sqrt{V(X)}, \\ F_X(t) &= fr\{X \le t\}. \end{aligned}$$

We say that *t*wo random numbers X and Y *have common distribution* if and only if $fr\{X \in I\} \approx fr\{Y \in I\}$ for each interval I. Starting from the concept of random number, we define a *large random sample* as a finite sequence of random numbers $S = (X_1, ..., X_n)$ with n large but not very large. In other words we have a matrix (s_{ij}) with n rows and s columns, where each column is a *sample realization*. Then write $\mu(S)$ for the average of S, $\mu(S) = \frac{\sum X_i}{n}$.

The concept of large random sample can be interpreted as an idealization of the informal discourse which is usual in statistics: if very long independent sampling could be performed repeatedly a large number of times, we would know the phenomenon nearly perfectly.

Let $T = \{t_0, \cdots, t_n\}$ be a set of n real numbers with $t_0 < \cdots < t_n$ and n weakly moderate. A one-dimensional stochastic process indexed by T is a sequence of random numbers $X_{t_0}, \cdots, X_{t_n}$ with the same very large size s. As a random sample, also a stochastic process can be visualized by a nxs matrix, whose columns are the trajectories of the process.

In order to express the low of matter we have introduce the *characteristic function* $\Phi_X(t) = M(\exp(itX))$ of a random number X which satisfies :

(i) if X and Y have common (resp. weakly common) distribution, then $\Phi_X(t) \approx \Phi_Y(t)$ (resp. $\sim$) for every weakly moderate (resp. moderate) t;

(ii) inversion formula:

$$\begin{aligned} fr(a < X \le b) &+ \frac{1}{2}fr\{X = a\} - \frac{1}{2}fr\{X = b\} \\ &\approx \frac{1}{2\pi}\int_{-T}^{+T} \frac{exp(-ita) - exp(-itb)}{it}\Phi_X(t)dt \end{aligned}$$

for every $a < b$, every very large T ;

(iii) for every continuous probability density f with

$$\int_{-\infty}^{+T} f(x)dx \approx \int_{T}^{\infty} f(x)dx \approx 0$$

for all very large T, there is a random number X such that

$$\int_{-\infty}^{x_j} f(x)\,dx = \frac{j}{s+1}.$$

Then

$$\Phi_X(t) \approx \int_{-\infty}^{+\infty} exp(itx)f(x)dx$$

for all weakly moderate t;

(iv) if X is LL^n for some weakly moderate n, then

$$\Phi_X(t+h) = \Phi_X(t) + \sum_{k=1}^{n} \frac{h^k}{k!} M((iX)^k exp(itX)) + h^n \varepsilon$$

where $\varepsilon \approx 0$ for all $h \approx 0$ and weakly moderate t;

(v) if $S = (X_1, ..., X_n)$ is a sample of independent random numbers then

$$\Phi_{X_1+\cdots+X_n}(t) \approx \Phi_{X_1}(t) \cdots \Phi_{X_n}(t).$$

For the proof we refer to [6].

5. Theoretic model inside the RFS context

To formalize the empirical statistical description of §1 we introduce a random number $X_{i,j,k}$ for every cell (i, j, k) where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant p$, $1 \leqslant k \leqslant q$. These random numbers take their values in a discrete finite set E and the numbers n, p, q are large while s is a very large. The cells are supposed to be of small size. Thus the model can be visualized by a multidimensional table $x^a_{i,j,k}$, $1 \leqslant a \leqslant s$.
We have then the following cases:

a linear material is represented by a nxs matrix;

a bidimensional material is represented by a cubic $nxpxs$ matrix;

a tridimensional material is represented by an hypercubic $nxpxqxs$ matrix.

The statistical matter behavior law can be expressed by means of the following conditional frequencies:

$$fr(X_{i,j,k} = a \mid X_{i_1,j_1,k_1} = a_1, X_{i_2,j_2,k_2} = a_2, \cdots , X_{i_r,j_r,k_r} = a_r).$$

The model suggests the following rough classification of behaviors:

A) Local behaviors: among them we distinguish between the independent case

$\forall a \in E$, $\forall a_1, \cdots , a_r \in E$

$$\left| \begin{array}{c} fr\left(X_{i,j,k} = a \mid X_{i_1,j_1,k_1} = a_1, X_{i_2,j_2,k_2} = a_2, \cdots , X_{i_r,j_r,k_r} = a_r\right) - \\ fr\left(X_{i,j,k} = a\right) \end{array} \right| \approx 0$$

and the weakly dependence case expressed by the conditions

$\forall a \in E$, $\forall a_1, \cdots , a_r \in E$

$$\left| \begin{array}{c} fr\left(X_{i,j,k} = a \mid X_{i_1,j_1,k_1} = a_1, X_{i_2,j_2,k_2} = a_2, \cdots , X_{i_r,j_r,k_r} = a_r\right) - \\ fr\left(X_{i,j,k} = a\right) \end{array} \right| \sim 0$$

B) Non local behavior, where we distinguish between the short range dependence expressed by the two conditions

$\forall a \in E$, $\forall a_1, \cdots , a_r \in E$

$$\left| fr\left(X_{i,j,k} = a\right) - fr\left(X_{i,j,k} = a \mid X_{i_r,j_r,k_r} = a_r\right) \right| not \approx 0$$

$$\left| \begin{array}{c} fr\left(X_{i,j,k} = a \mid X_{i_1,j_1,k_1} = a_1, X_{i_2,j_2,k_2} = a_2, \cdots , X_{i_r,j_r,k_r} = a_r\right) - \\ fr\left(X_{i,j,k} = a \mid X_{i_r,j_r,k_r} = a_r\right) \end{array} \right| \approx 0$$

and the weak short range dependence case expressed by:

$$\left| fr\left(X_{i,j,k} = a\right) - fr\left(X_{i,j,k} = a \mid X_{i_r,j_r,k_r} = a_r\right) \right| not \approx 0$$

$$\left| \begin{array}{c} fr\left(X_{i,j,k} = a \mid X_{i_1,j_1,k_1} = a_1, X_{i_2,j_2,k_2} = a_2, \cdots, X_{i_r,j_r,k_r} = a_r\right) - \\ fr\left(X_{i,j,k} = a \mid X_{i_r,j_r,k_r} = a_r\right) \end{array} \right| \sim 0$$

the cells (i_r, j_r, k_r) range into the neighborhood of the (i, j, k)cell;

C) the long range dependence case expressed by the conditions:

$\forall a \in E$, $\forall a_1, \cdots, a_r \in E$

$\left| fr\left(X_{i,j,k} = a\right) - fr\left(X_{i,j,k} = a \mid X_{i_r,j_r,k_r} = a_r\right)\right| \, not \approx 0$

and

$\left| fr\left(X_{i,j,k} = a\right) - fr\left(X_{i,j,k} = a \mid X_{i_r,j_r,k_r} = a_r\right)\right| \nsim 0$

where the cells $(i_r, j_r, k_r)$are not necessary in the neighborhood of the (i, j, k)cell.

This classification may be refined if one relates the dependences with the distances. The inversion formula of the characteristic function may be useful to treat the information in order to eliminate the white noise and to put in evidence the intrinsic characteristic distances of the concerned matter.

6. Conclusion

This model gives a formal tool to characterize the behavior of matter in terms of local and non-local interactions.

A theoretical model inside the probabilistic context would replace the random numbers by the random variable and frequency by the probabilities. Such a modelization hides the intuitive interpretation of the model since the probability is a mathematical concept, which has no direct statistical interpretation. However, in the statistical context of RFS, which works within the mathematical framework of ZFL_2 the description of the behavior is at the same time intuitive and formal.

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STRAIN SOLITARY WAVES IN AN ELASTIC ROD WITH MICROSTRUCTURE

Abstract. The nonlinear longitudinal strain solitary waves are studied inside cylindrical elastic rod with microstructure. The problem is solved using the pseudo-continuum Cosserat model and the Le Roux continuum model. A procedure is developed for derivation of a governing equation for longitudinal nonlinear strain waves. Exact solution of the equation has the form of a travelling bell-shaped solitary wave. The influence of microstructure on the solitary wave propagation is studied. Possible experimental determination of the parameters of the microstructure is discussed.

1. Introduction

Sometimes classic elastic theory cannot account for phenomenon caused by the microstructure of a material. A particular case is a dispersion of strain waves in an elastic medium. The influence of microstructure may provide dissipative effects [14, 6, 2], however, here consideration is restricted by non-dissipative case. The theory of microstructure has been developed recently, see [6, 7, 15, 17] and references therein. Most of results belong to the linear theory of elasticity, however, there are findings in the field of the nonlinear theory [6, 7]. Strain waves were studied mainly in the linear approximation [7, 15, 17]. Only a few works are devoted to the nonlinear waves in microstructured non-dissipative media [6, 19, 20, 10, 9]. Waves in elastic *wave-guides* with microstructure were out of considerable investigation. Also the values of the parameters characterizing microstructure, are unknown as a rule, only a few data may be mentioned [20].

It is known that the balance between nonlinearity and dispersion may result in an appearance of bulk localized long bell-shaped strain waves of permanent form (solitary waves or solitons) which may propagate and transfer energy over the long distance along an elastic wave guide. The amplification of them may cause the appearance of plasticity zones or microcracks in a wave guide. This is of importance for an assessment of durability of elastic materials and structures, methods of nondestructive testing, determination of the physical properties of elastic materials, particularly, polymeric solids, and ceramics. Bulk waves provide better suited detection requirements than surface strain waves in setting up a valuable nondestructive test for pipelines.

Recently, the theory has been developed to account for long longitudinal strain solitary waves propagating in a free lateral surface elastic rod [5, 21, 22]. The procedure has been proposed to obtain model equations using boundary conditions on the rod surface [18]. The nonlinearity, caused by both the finite stress values and elastic material properties, and the dispersion resulting from the finite transverse size of the rod, when in balance allow the propagation of

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strain solitary waves v. The equation governing this process is of Boussinesq type, namely, a double dispersive equation

$$v_{tt} - \alpha_1 v_{xx} - \alpha_2 (v^2)_{xx} + \alpha_3 v_{xxtt} - \alpha_4 v_{xxxx} = 0.$$

The coefficients α_i depend upon the elastic parameters of the rod material. Exact solution of the equation has the form of a travelling bell-shaped solitary wave. The amplitude and the velocity of the solitary wave are explicitly connected with the elastic moduli. It allows to propose the estimation of the Murnaghan third order elastic moduli using measurement of the solitary wave parameters [1]. Motivated by analytical theoretical predictions, there has been successful experimental generation of strain solitons in a polystyrene free lateral surface rod using holographic interferometry [3]. The procedure developed in Ref.[18], has been successfully applied for the more complicated modelling of strain waves in a narrowing rod[4] and in a rod interacting with an another external elastic medium [1].

The present paper refers to the study of nonlinear solitary waves inside cylindrical rod with microstructure. The problem is solved using the "pseudocontinuum" Cosserat model and the Le Roux continuum model. A procedure is developed for derivation of the model equation for long longitudinal strain waves inside the rod. The influence of the microstructure on the solitary wave propagation is studied. Possible experimental determination of the parameters of the microstructure is discussed.

2. Modelling of elastic medium with microstructure

Recall some basic ideas following Eringen [7]. Suppose the macroelement of an elastic body contains discrete micromaterial elements. At any time the position of a material point of the αth microelement may be expressed as

$$\mathbf{x}^{(\alpha)} = \mathbf{x} + \xi^{(\alpha)},$$

where $\mathbf{x}$ is the position vector of the center of mass of the macroelement, $\xi^{(\alpha)}$ is the position of a point in the microelement relative to the center of mass. The motion of the center of mass depends upon the initial position $\mathbf{X}$ and time t, $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, while for $\xi^{(\alpha)}$ the axiom of affine motion is assumed,

$$\xi^{(\alpha)} = \chi_K(\mathbf{X}, t)\ \Xi_K^{(\alpha)},$$

where $\Xi^{(\alpha)}$ characterizes initial position of a point relative to the center of mass. Then the square of the arc length is $(ds^{(\alpha)})^2 = d\mathbf{x}^{(\alpha)} d\mathbf{x}^{(\alpha)}$, and the difference between the squares of arc length in the deformed and undeformed body is

$$\begin{aligned} (ds^{(\alpha)})^2 - (dS^{(\alpha)})^2 &= (x_{k,K}\, x_{k,L} - \delta_{KL} + 2x_{k,K}\, \chi_{kM,L}\, \Xi_M \\ &+ \chi_{kM,K}\, \chi_{kN,L}\, \Xi_M\, \Xi_N) dX_K\, dX_L \\ &+ 2\left(x_{k,K}\, \chi_{kL} - \delta_{KL} + \chi_{kL}\, \chi_{kM}\, \Xi_M\right) dX_K\, d\Xi_L \\ &+ \chi_{kK}\, \chi_{kL} d\Xi_K\, d\Xi_L. \end{aligned} \tag{1}$$

where δ_{KL} is the Kronecker delta. Let us introduce vector of macrodisplacements, $\mathbf{U}(\mathbf{X}, t)$ and tensor of microdisplacements, $\Phi(\mathbf{X}, t)$,

$$\begin{aligned} x_{k,K} &= (\delta_{LK} + U_{L,K})\delta_{kL}, \\ \chi_{kK} &= (\delta_{LK} + \Phi_{LK})\delta_{kL} \end{aligned}$$

Then three tensors characterizing the behavior of microstructured medium follow from (1),

$$\begin{aligned} C_{KL} &= \frac{1}{2}\left(U_{K,L}+U_{L,K}+U_{M,K}U_{M,L}\right), \\ E_{KL} &= \Phi_{KL}+U_{L,K}+U_{M,K}\,\Phi_{ML}, \\ \Gamma_{KLM} &= \Phi_{KL,M}+U_{N,K}\,\Phi_{NL,M}, \end{aligned}$$

where C_{KL} is the Cauchy-Green macrostrain tensor, E_{KL} is the tensor of a reference distortion, Γ_{KLM} is the tensor of microdistortion. Tensor of the second rank E_{KL} accounts for the microelements motion relative to the center of mass of the macroelement, while tensor of the third rank Γ_{KLM} characterizes relative motion of the microelements of one another.

The density of the potential energy Π should be the function of these tensors, $\Pi = \Pi(C_{KL}, E_{KL}, \Gamma_{KLM})$, more precisely upon the invariants of them. The bulk density of the kinetic energy has the form [15]

$$K = \frac{1}{2}\rho_0\left(U^2_{M,t}+J_{KN}\,\Phi_{KM,t}\,\Phi_{NM,t}\right), \tag{2}$$

where ρ_0 is macrodensity of the elastic material, J_{KN} is the inertia tensor. Elastic media with central symmetry posses simpler representation, $J_{KN} = J^*\delta_{KN}$.

One of the main problem is to define integrity basis of three tensors $C_{KL}, E_{KL}, \Gamma_{KLM}$ [23, 8]. Moreover, the basic invariants of the third and higher rank tensors have not been studied. That is why the models were developed based on the additional assumption on a relationship between $\mathbf{U}$ and Φ. One of them is the pseudocontinuum Cosserat model. According to it

$$\Phi_{KL} = -\varepsilon_{KLM}\Phi_M,\ \ \Phi_M = \frac{1}{2}\varepsilon_{MLK}U_{K,L}, \tag{3}$$

where ε_{KLM} is the alternating tensor. The first relationship represents to the classic Cosserat model when only rotations of solid microelements are possible. The last expression in (3) accounts for the pseudocontinuum Cosserat model when micro rotation vector Φ coincides with the macro rotation vector. In this case the density of the potential energy my be either $\Pi = \Pi(C_{KL}, \Gamma_{KLM})$ or $\Pi = \Pi(C_{KL}, \Phi_{K,L})$ [17, 20]. Tensor E_{KL} has the form

$$E_{KL} = \frac{1}{2}\left(U_{K,L}+U_{L,K}+U_{M,K}U_{M,L}-U_{M,K}U_{L,M}\right),$$

and only linear part of E_{KL} coincides with those of C_{KL}. Assume the microstructure is sufficiently weak to be considered in the linear approximation[17, 20], and the Murnaghan model [5, 12, 16] is valid for macro motion. Then the density of the potential energy may be written as

$$\begin{aligned} \Pi &= \frac{\lambda+2\mu}{2}I_1^2 - 2\mu I_2 + \frac{l+2m}{3}I_1^3 - 2mI_1I_2 + nI_3 \\ &+ 2\mu M^2(\Phi_{K,L}\Phi_{K,L}+\eta\Phi_{K,L}\Phi_{L,K}+\beta\Phi_{K,K}\Phi_{L,L}), \end{aligned} \tag{4}$$

where λ and μ are the Lamé coefficients, (l, m, n) are the third order elastic moduli, or the Murnaghan moduli, M, η and β are the microstructure constants, $I_p,\ p = 1, 2, 3$ are the invariants of the tensor $\mathbf{C}$:

$$I_1(\mathbf{C}) = tr\mathbf{C},\ I_2(\mathbf{C}) = [(tr\mathbf{C})^2 - tr\mathbf{C}^2]/2,\ I_3(\mathbf{C}) = \det\mathbf{C}. \tag{5}$$

Another simplified microstructure model was used by some authors, see [15, 19, 10]. Sometimes it is referred to as the Le Roux continuum [9]. According to it

$$\Phi_{KL} = -U_{L,K}, \;\; \Gamma_{KLM} = -U_{L,KM}.$$

When microstructure is weak and may be considered in the linear approximation the linear part of E_{KL} is zero tensor. It means that there is no difference between deformation of elastic microelement and elastic macrostructure. In this case $\Pi = \Pi(C_{KL}, \Gamma_{KLM})$. Assume again the Murnaghan model for the macro part of the energy density and use the linear Mindlin's model [15] for its micro part one can obtain

$$\begin{aligned} \Pi \;\; = \;\; & \frac{\lambda+2\mu}{2} I_1^2 - 2\mu I_2 + \frac{l+2m}{3} I_1^3 - 2m I_1 I_2 + n I_3 + a_1 \Gamma_{KKM}\Gamma_{MLL} + \\ & a_2 \Gamma_{KLL}\Gamma_{KMM} + a_3 \Gamma_{KKM}\Gamma_{LLM} + a_4 \Gamma^2_{KLM} + a_5 \Gamma_{KLM}\Gamma_{MLK}. \end{aligned} \tag{6}$$

where a_i, $i = 1-5$, are the constant microstructure parameters.

3. Nonlinear waves in a rod with pseudocontinuum Cosserat microstructure

Let us consider the propagation of a longitudinal strain wave in an isotropic cylindrical *compressible* nonlinearly elastic rod. We take cylindrical Lagrangian coordinates (x, r, φ) where x is directed along the axis of the rod, $-\infty < x < \infty$; r is the coordinate along the rod radius, $0 \le r \le R$; φ is a polar angle, $\varphi \,\epsilon [0, 2\pi]$. Neglecting torsions the displacement vector is $\mathbf{U} = (u, w, 0)$. Then nonzero components of the macrostrain tensor $\mathbf{C}$ are

$$\begin{aligned} C_{xx} \;\; &= \;\; u_x + \frac{1}{2}(u_x^2 + w_x^2),\, C_{rr} = w_r + \frac{1}{2}(u_r^2 + w_r^2),\, C_{\varphi\varphi} = \frac{w}{r} + \frac{w^2}{2r^2}, \\ C_{rx} \;\; &= \;\; C_{xr} = \frac{1}{2}\left(u_r + w_x + u_x u_r + w_x w_r\right). \end{aligned} \tag{7}$$

while nonzero components of the rotation tensor Φ_{KL} are

$$\Phi_{\varphi,x} = w_{xx} - u_{rx}, \qquad \Phi_{\varphi,r} = w_{xr} - u_{rr}. \tag{8}$$

The governing equations together with the boundary conditions are obtained using the Hamilton variational principle, i.e., setting to zero the variation of the action functional,

$$\delta S \;=\; \delta \int_{t_0}^{t_1} dt \left[2\pi \int_{-\infty}^{\infty} dx \int_0^R r\, \mathcal{L} dr \right] = 0, \tag{9}$$

where the Lagrangian density per unit volume, $\mathcal{L}$=$K - \Pi$, with K and Π defined by Eqs.(2) (4) correspondingly. The integration in brackets in (9) is carried out at the initial time $t = t_0$. Initially, the rod is supposed to be in its natural, equilibrium state.

The following boundary conditions (b.c.) are imposed:

$$w \;\to\; 0, \qquad \text{at}\; r \;\to\; 0, \tag{10}$$

$$P_{rr} \;=\; 0, \qquad \text{at}\; r \;=\; R, \tag{11}$$

$$P_{rx} \;=\; 0, \qquad \text{at}\; r \;=\; R, \tag{12}$$

where the components P_{rr}, P_{rx} of the modified Piola - Kirchhoff stress tensor **P** are defined from (9) with (4), (2), (7) and (8) being taken into account:

$$\begin{aligned} P_{rr} \quad = \quad & (\lambda+2\mu)\, w_r + \lambda\, \frac{w}{r} + \lambda\, u_x + \frac{\lambda+2\mu+m}{2}\, u_r^2 + \\ & \frac{3\lambda+6\mu+2l+4m}{2}\, w_r^2 + (\lambda+2l)\, w_r \frac{w}{r} + \frac{\lambda+2l}{2} \frac{w^2}{r^2} + \\ & (\lambda+2l)\, u_x w_r + (2l-2m+n)\, u_x \frac{w}{r} + \frac{\lambda+2l}{2}\, u_x^2 + \\ & \frac{\lambda+2\mu+m}{2}\, w_x^2 + (\mu+m)\, u_r w_x + 4\mu M^2\, (u_{rrx} - w_{xxr})\,, \end{aligned} \tag{13}$$

$$\begin{aligned} P_{rx} \quad = \quad & \mu\, (u_r + w_x) + (\lambda+2\mu+m)\, u_r w_r + (2\lambda+2m-n)\, u_r \frac{w}{r} + \\ & (\lambda+2\mu+m)\, u_x u_r + \frac{2m-n}{2}\, w_x \frac{w}{r} + (\mu+m)\, w_x w_r + \\ & (\mu+m)\, u_x w_x \; + 4\mu M^2 [w_{xxx} - u_{xxr} + \frac{1}{r}\, (r(w_{xr} - u_{rr}))_r - \\ & \frac{1}{2} J^* (u_{rtt} - w_{xtt})]. \end{aligned} \tag{14}$$

Exception of torsions provides transformation of the initial 3D problem into a 2D one. Subsequent simplification is caused by the consideration of only long elastic waves with the ratio between the rod radius R and typical wavelength L is $R/L \ll 1$. The typical elastic strain magnitude B is also small, $B \ll 1$. The Hamilton principle (9) yields a set of coupled equations for u and w together with the b.c. (11), (12). To obtain a solution in universal way one usually proceeds to the dimensionless form of the equations and looks for the unknown displacement vector components in the form of power series in the small parameters of the problem (for example R/L), hence, leading to an asymptotic solution of the problem. However, this procedure has some disadvantages. In particular, comparison of the predictions from the dimensionless solution to the experiments suffers from the fact that both B and L, are not well defined. Further, the coefficients of the nonlinear terms usually contain combinations of elastic moduli which may be also small in addition to the smallness of B [21, 22] something not predicted beforehand. Finally, this procedure gives equations of only first order in time, t, while general equations for displacements u and w are of the second order in time. Therefore the solution of the model equation will not satisfy two independent initial conditions on longitudinal strains or displacements [21].

An alternative is to simplify the problem making some assumptions about the behavior of longitudinal and/or shear displacements and/or strains in the elastic wave-guide. Referring to the elastic rod these relationships give explicit dependence of u and w upon the radius, while their variations along the rod axis are described by some unknown function and its derivatives along the axis of the rod. Then the application of Hamilton's principle (9) yields the governing equation in dimensional form for this function. This equation is of the second order of time, hence its solution can satisfy two independent initial conditions. Any combinations of elastic moduli appear in the coefficients of the equation, hence, subsequent scaling may take into account their orders when introducing small parameters.

For an elastic rod, the simplest assumption is the plane cross section hypothesis [13]: the longitudinal deformation process is similar to the beards movement on the thread. Then every cross-section of the rod remains flat, hence, $u = U(x, t)$ does not change along the radius

r. However, this assumption is not enough due to the Poisson effect, i.e., longitudinal and shear deformations are related. That is why Love proposed to use a relationship between w and u: $w = -r\,\nu\,U_x$, with ν the Poisson coefficient[11]. Unfortunately, the plane cross-section hypothesis and Love's hypothesis do not satisfy the boundary conditions that demand vanishing of both the normal and tangential stresses, P_{rr} and P_{rx}, at the lateral surface of the rod with prescribed precision.

Another theory has been proposed in [18] to find the relationships between displacement vector components satisfying b.c. on the lateral surface of the rod (11), (12) as well as the condition for w (10).

Since pure elastic wave are studied, $B \ll 1$, the "linear" and "nonlinear" parts of the relationships may be obtained separately. A power series approximations is used, as generally done for long wave processes. An additional parameter $\gamma = M^2/R^2$ is introduced to characterize the microstructure contribution. Accordingly, the longitudinal and shear displacement in *dimensional* form are:

$$
\begin{aligned}
u &= u_L + u_{NL}, \\
u_L &= u_0(x,t) + r\,u_1(x,t) + r^2 u_2(x,t) + ..., \\
u_{NL} &= u_{NL0}(x,t) + r\,u_{NL1}(x,t) + ...,
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
w &= w_L + w_{NL}, \\
w_L &= w_0(x,t) + r\,w_1(x,t) + r^2 w_2(x,t) + ..., \\
w_{NL} &= w_{NL0}(x,t) + r\,w_{NL1}(x,t) + ...
\end{aligned} \tag{16}
$$

Substituting the linear parts u_L and w_L (15), (16) into the b.c. (10) and in the linear parts of b.c. (11), (12), and equating to zero terms at equal powers of r one obtains u_k and w_k. Using these results the nonlinear parts u_{NL} , w_{NL} are similarly obtained from the full b.c. We get

$$
u = U(x,t) + \frac{\nu r^2}{2}\frac{1+4\gamma}{1-4\gamma}U_{xx}, \tag{17}
$$

$$
\begin{aligned}
w &= -\nu r U_x - \frac{\nu}{2(3-2\nu)(1-4\gamma)}\left[\nu + 4\gamma(2+\nu)\right] r^3 U_{xxx} - \\
&\left[\frac{\nu(1+\nu)}{2} + \frac{(1-2\nu)(1+\nu)}{E}\Big(l(1-2\nu)^2 + 2m(1+\nu) - n\nu\Big)\right] r U_x^2,
\end{aligned} \tag{18}
$$

where ν is the Poisson ratio, E is the Young modulus. Other terms from the series (15), (16) for $i > 3$ may be found in the same way, however, they are omitted here because of no influence on the final model equation for the strain waves. Substituting (17), (18) into (9), and using Hamilton's principle we obtain that longitudinal strains, $v = U_x$, obey a double dispersive nonlinear equation:

$$
v_{tt} - \alpha_1\,v_{xx} - \alpha_2(v^2)_{xx} + \alpha_3\,v_{xxtt} - \alpha_4\,v_{xxxx} = 0, \tag{19}
$$

where $\alpha_1 = c_*^2 = E/\rho_0$, $\alpha_2 = \beta/(2\rho_0)$, $\beta = (\,3E + 2l(1-2\nu)^3 + 4m(1+\nu)^2(1-2\nu) + 6n\nu^2)$, $\alpha_3 = \nu(1-\nu)R^2/2$,

$$
\alpha_4 = \frac{\nu E R^2}{2\rho_0}\frac{1+4\gamma}{1-4\gamma}.
$$

Hence the microstructure affects only dispersion in Eq.(19). The solitary wave solution of Eq.(19) is

$$v = \frac{6\nu E R^2 k^2}{\beta}\left(\frac{1+4\gamma}{1-4\gamma} - \frac{(1-\nu)V^2}{c_*^2}\right)\cosh^{-2}(k\,(x-Vt)), \tag{20}$$

where V is a free parameter while the wave number k is defined by

$$k^2 = \frac{2\rho_0(V^2 - c_*^2)}{\nu E R^2\left(\frac{1+4\gamma}{1-4\gamma} - \frac{(1-\nu)V^2}{c_*^2}\right)}. \tag{21}$$

Therefore the contribution of the microstructure results in the widening of the permitted solitary wave velocities,

$$1 < \frac{V^2}{c_*^2} < \frac{1}{1-\nu}\frac{1+4\gamma}{1-4\gamma}.$$

Also the characteristic width of the solitary wave proportional to $1/k$ becomes larger relative to the wave width in pure elastic case, $\gamma = 0$. We consider γ to be rather small due to the experimental data from Ref. [20]. Then the type of the solitary wave (compression/tensile) is defined by the sign of the nonlinearity parameter β like in case without microstructure.

4. Nonlinear waves in a rod with Le Roux continuum microstructure

The procedure of obtaining the governing equations is similar to those used in previous section. The nonzero components of the tensor Γ_{KLM} are

$$\begin{aligned}
\Gamma_{xxx} &= -u_{xx}, \Gamma_{xxr} = \Gamma_{rxx} = -u_{xr}, \Gamma_{xrx} = -w_{xr},\\
\Gamma_{xrr} &= \Gamma_{rrx} = -w_{xr}, \Gamma_{rxr} = -u_{rr}, \Gamma_{rrr} = -w_{rr}.
\end{aligned}$$

The b.c. (11), (12) are satisfied for the strain tensor components

$$\begin{aligned}
P_{rr} = {} & (\lambda+2\mu)\,w_r + \lambda\,\frac{w}{r} + \lambda\,u_x + \frac{\lambda+2\mu+m}{2}\,u_r^2 + \frac{3\lambda+6\mu+2l+4m}{2}\,w_r^2 + \\
& (\lambda+2l)\,w_r\frac{w}{r} + \frac{\lambda+2l}{2}\frac{w^2}{r^2} + (\lambda+2l)\,u_x w_r + (2l-2m+n)\,u_x\frac{w}{r} + \\
& \frac{\lambda+2l}{2}\,u_x^2 + \frac{\lambda+2\mu+m}{2}\,w_x^2 + (\mu+m)\,u_r w_x + 2J^*(2u_{xtt} + w_{rtt}) - \\
& 2a_1 u_{xxx} - 2(a_1+2a_2)w_{xxr} - 2(a_1+a_2)\frac{1}{r}\,(r(w_{rr}))_r - a_1\frac{1}{r}\,(r(u_{xr}))_r\,,
\end{aligned} \tag{22}$$

$$\begin{aligned}
P_{rx} = {} & \mu\,(u_r + w_x) + (\lambda+2\mu+m)\,u_r w_r + (2\lambda+2m-n)\,u_r\frac{w}{r} + \\
& (\lambda+2\mu+m)\,u_x u_r + \frac{2m-n}{2}\,w_x\frac{w}{r} + (\mu+m)\,w_x w_r + \\
& (\mu+m)\,u_x w_x + 2J^* u_{rtt} - a_1 w_{xrr} - 2(a_1+2a_2)u_{xxr} - 2a_2\frac{1}{r}\,(r(u_{rr}))_r\,.
\end{aligned} \tag{23}$$

Then the approximations for the components of the displacement vector have the form

$$u = U(x,t) + \frac{\nu r^2}{2}\frac{1}{1-N}U_{xx}, \tag{24}$$

$$w = -\nu r U_x - \frac{4J^*(2-\nu)(1+\nu)(1-2\nu)}{E(3-2\nu)R^2} r^3 U_{xtt} -$$
$$\frac{\nu^2 - (1-2\nu)(1-N)(G(1-\nu) - 2\nu N)}{2(3-2\nu)(1-N)} r^3 U_{xxx} -$$
$$\left[\frac{\nu(1+\nu)}{2} + \frac{(1-2\nu)(1+\nu)}{E}\Big(l(1-2\nu)^2 + 2m(1+\nu) - n\nu\Big)\right] r U_x^2, \quad (25)$$

where $G = 2a_1/\mu R^2$, $N = 2a_2/\mu R^2$. Like in previous section the governing equation for longitudinal strain $v = U_x$ is the double dispersive equation (19) whose coefficients are defined now as

$$\alpha_1 = c_*^2, \alpha_2 = \frac{\beta}{2\rho_0}, \alpha_3 = \frac{\nu R^2}{2(1-N)} - \frac{\nu^2 R^2}{2} + 2J^*\nu(2-\nu), \alpha_4 = \frac{\nu c_*^2 R^2}{2(1-N)}.$$

Solitary wave solution has the form

$$v = \frac{6\nu E R^2 k^2}{\beta}\left(\frac{1}{1-N} - \left[\frac{1}{1-N} - \nu + \frac{4J^*(2-\nu)}{R^2}\right]\frac{V^2}{c_*^2}\right)\cosh^{-2}(k\,(x - Vt)), \quad (26)$$

where V is a free parameter, and the wave number k is defined by

$$k^2 = \frac{2(1-N)\rho_0(V^2 - c_*^2)}{\nu E R^2\left[c_*^2 - V^2\left(1 - \nu(1-N) + 4J^*(1-N)(2-\nu)/R^2\right)\right]}. \quad (27)$$

Physically reasonable case corresponds to rather small N, $N < 1$. Then the influence of the microstructure yields an alteration of the permitted solitary wave velocities interval,

$$1 < \frac{V^2}{c_*^2} < \frac{1}{1 - \nu(1-N) + 4J^*(1-N)(2-\nu)/R^2}.$$

The widening or narrowing of the interval depends upon the relationship between N and the parameter of microinertia J^*. Like in previous section the type of the solitary wave is governed by the sign of the nonlinearity parameter β. At the same time the characteristic width of the solitary wave proportional to $1/k$ turns out smaller than the wave width in a pure macroelastic case, $N = 0$, $J^* = 0$.

5. Discussion

It is found that the double dispersive equation (19) accounts for longitudinal strain wave propagation inside the rod even in presence of the microstructure, and only dispersion term coefficients alter in comparison with the pure macroelastic case. The procedure proposed in [18] is profitably applied for the derivation of the governing equation in dimensional form for both the Cosserat and the Le Roux models. The assumption of the linear contribution of the microstructure is correct since its nonlinear contribution, being weaker, may provide alterations only in the neglected higher order nonlinear and dispersion terms in the governing equation. Hence we don't need in an additional nonlinear terms in the density of the potential energy Π thus avoiding the additional unknown parameters (like Murnaghan's third order moduli) describing the nonlinear contribution of the microstructure.

The alterations of the amplitude and the wave width, caused by the microstructure, have been found in both case under study. The important result is in the opposite changing of the wave width which gives a possibility to distinguish the Cosserat and the Le Roux models in possible experiments.

The dispersion caused by the microstructure may be observed experimentally, and numerical data on microstructure parameters my be obtained[20]. In experiments on solitary waves propagation [3] the amplitude and the velocity of the wave may be measured. Therefore expressions (20), (21) provide possible estimation of the parameter M in the pseudocontinuum Cosserat model. In case of the Le Roux continuum there is an extra parameter J^*, see (26), (27), and parameters N and J^* cannot be estimated separately.

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