Rend. Sem. Mat. Univ. Pol. Torino
Vol. 57, 3 (1999)

F. Salvarani

DIFFUSION LIMITS FOR THE INITIAL-BOUNDARY VALUE PROBLEM OF THE GOLDSTEIN-TAYLOR MODEL

Sommario.
In the paper is studied, in the diffusive scaling, the limiting behaviour of the Goldstein-Taylor model in a box, for a large class of initial and boundary conditions. It is shown that, in the limit, the evolution of the mass density is governed by the heat equation, with initial conditions depending only on the initial data of the hyperbolic system, and conditions on the boundary depending only on the ones of the kinetic model.

1. Introduction

In the kinetic theory of rarefied gases, a challenging problem is given by the study of the transition from the full Boltzmann equation to the Euler or Navier-Stokes equation.

This problem was introduced by Hilbert in the first years of this century, but, until now, many results were obtained only at a formal level [2].

For this reason, in recent years much attention has been devoted to the so called *discrete velocity models* of the Boltzmann equation and, in particular, to the two-velocity ones, which allow to achieve rigorous results.

Two velocity models describe the evolution of the velocity distribution of a gas composed of two kinds of particles moving parallel to the x-axis with constant and equal speeds, either in the positive x-direction with a density $u = u(x, t)$, or in the negative x-direction with a density $v = v(x, t)$.

The most general one, which is in local equilibrium when $u = v$, has the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} & = & k(u, v, x)(v - u) \\ \dfrac{\partial v}{\partial t} - c\dfrac{\partial v}{\partial x} & = & k(u, v, x)(u - v) \qquad x \in \Omega \subseteq \mathbb{R},\ t \geq 0\,, \end{cases}$$

where $k(u, v, x)$ is a nonnegative function which characterizes the interactions between gas particles, and $c > 0$. The most famous model of this kind was introduced by Carleman [1] and it corresponds to the choice $k(u, v, x) = u + v$.

The mathematical theory of these models is well established (see, for example, [9]); recently, in some papers [7], [10], [3], [13], it has been shown that several well known differential equations of mathematical physics (the porous media equation, the Burgers' equation and some kinds of diffusion equations) can be obtained as diffusive limits of Cauchy problems of particular kinetic models.

Moreover, these results have a very useful application, giving the possibility to construct new kinds of numerical schemes for the target equations, as shown in several works (for example, see [4], [5], [8]).

All the previously quoted papers deal with the full initial value problem, or with the initial-boundary value problem with specular or periodic conditions at the boundary. For this reason, in the present paper, we will investigate the hydrodynamical limit (i.e. as $\varepsilon \to 0^+$) of the hyperbolic Goldstein-Taylor model [6], [11]

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + \dfrac{1}{\varepsilon}\dfrac{\partial u_\varepsilon}{\partial x} &= \dfrac{1}{\varepsilon^2}(v_\varepsilon - u_\varepsilon) \\[2ex] \dfrac{\partial v_\varepsilon}{\partial t} - \dfrac{1}{\varepsilon}\dfrac{\partial v_\varepsilon}{\partial x} &= \dfrac{1}{\varepsilon^2}(u_\varepsilon - v_\varepsilon) \qquad \varepsilon > 0 \end{cases} \tag{1}$$

in a bounded domain $\Omega = (-L, L)$, $L \in \mathbb{R}^+$, with initial conditions $u_\varepsilon(x,0) = u_0(x)$, $v_\varepsilon(x,0) = v_0(x) \in L^\infty(\Omega)$ and boundary conditions of type $u_\varepsilon(-L,t) = \varphi^-(t)$, $v_\varepsilon(+L,t) = \varphi^+(t) \in W^{1,\infty}(0,T)$, $T > 0$.

The macroscopic variables for this model are the mass density $\rho_\varepsilon = u_\varepsilon + v_\varepsilon$, and the flux

$$j_\varepsilon(x,t) = \frac{u_\varepsilon(x,t) - v_\varepsilon(x,t)}{\varepsilon}.$$

It is interesting to remark that, since u_ε and v_ε can be expressed in terms of ρ_ε and j_ε, system (1) is equivalent to the following macroscopic equations for the mass density and the flux

$$\begin{cases} \dfrac{\partial \rho_\varepsilon}{\partial t} + \dfrac{\partial j_\varepsilon}{\partial x} &= 0 \\[2ex] \varepsilon^2 \dfrac{\partial j_\varepsilon}{\partial t} + \dfrac{\partial \rho_\varepsilon}{\partial x} &= -2 j_\varepsilon \qquad (x,t) \in \Omega \times (0,T), \end{cases} \tag{2}$$

where the boundary conditions for the macroscopic variables are partially unknown. We will show that the density $\rho_\varepsilon = u_\varepsilon + v_\varepsilon$ converges weakly in L^2, as $\varepsilon \to 0^+$, to $\rho = u + v$ where u and v are, respectively, the limits of u_ε and v_ε. Moreover ρ is governed by the heat equation

$$\frac{\partial \rho}{\partial t} - \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2} = 0 \tag{3}$$

satisfying the initial and boundary conditions:

$$\rho(x,0) = u_0(x) + v_0(x)$$

and

$$\rho(-L,t) = 2\varphi^-(t)$$
$$\rho(+L,t) = 2\varphi^+(t).$$

The paper is organized as follows: in the next section we prove many preliminary results on the hyperbolic model; in part III, we study the limiting behaviour of the macroscopic density on the boundary. Section IV is devoted to the study of uniform bounds for the flux and, finally, in part V, we investigate the hydrodynamical limit.

It is necessary to point out that many of the forthcoming results are deeply connected to the linearity of the problem, and they are not easily extendible to nonlinear situations. This will be the object of our future research.

2. A maximum principle

In this section we prove a maximum principle for system (1). Local existence of solutions of such kind of hyperbolic systems is well known (see, for example, [12] and the references within). Therefore, a maximum principle implies that the solution is global and unique. Since the problem is linear, we will consider some different sub-problems, which are easier to study.

PROBLEM 5.1. *We study the system*

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + \dfrac{1}{\varepsilon}\dfrac{\partial u_1}{\partial x} = \frac{1}{\varepsilon^2}(v_1 - u_1) \\[2ex] \dfrac{\partial v_1}{\partial t} - \dfrac{1}{\varepsilon}\dfrac{\partial v_1}{\partial x} = \frac{1}{\varepsilon^2}(u_1 - v_1), \end{cases} \tag{4}$$

with the initial and boundary conditions:

$$\begin{aligned} u_1(x,0) &= u_0(x) \in L^\infty(\Omega) \\ v_1(x,0) &= v_0(x) \in L^\infty(\Omega) \\ u_1(-L,t) &= 0 \\ v_1(+L,t) &= 0. \end{aligned}$$

PROBLEM 5.2. *We study the system*

$$\begin{cases} \dfrac{1}{\varepsilon}\dfrac{\partial u_2}{\partial x} = \frac{1}{\varepsilon^2}(v_2 - u_2) \\[2ex] \dfrac{1}{\varepsilon}\dfrac{\partial v_2}{\partial x} = \frac{1}{\varepsilon^2}(v_2 - u_2), \end{cases} \tag{5}$$

with boundary conditions:

$$\begin{aligned} u_2(-L,t) &= \varphi^-(t) \in W^{1,\infty}(0,T) \\ v_2(+L,t) &= \varphi^+(t) \in W^{1,\infty}(0,T). \end{aligned}$$

PROBLEM 5.3. *We study the system*

$$\begin{cases} \dfrac{\partial u_3}{\partial t} + \dfrac{1}{\varepsilon}\dfrac{\partial u_3}{\partial x} = \frac{1}{\varepsilon^2}(v_3 - u_3) + f_\varepsilon(x,t) \\[2ex] \dfrac{\partial v_3}{\partial t} - \dfrac{1}{\varepsilon}\dfrac{\partial v_3}{\partial x} = \frac{1}{\varepsilon^2}(u_3 - v_3) + g_\varepsilon(x,t), \end{cases} \tag{6}$$

where $f_\varepsilon(x,t)$ and $g_\varepsilon(x,t)$ are suitable functions that will be specified later, with the following initial and boundary conditions:

$$\begin{aligned} u_3(x,0) &= 0 \\ v_3(x,0) &= 0 \\ u_3(-L,t) &= 0 \\ v_3(+L,t) &= 0. \end{aligned}$$

The functions

$$\begin{aligned} u_\varepsilon(x,t) &= u_1(x,t) + u_2(x,t) + u_3(x,t), \\ v_\varepsilon(x,t) &= v_1(x,t) + v_2(x,t) + v_3(x,t) \end{aligned}$$

satisfy, by linearity, the differential system (1), with the correct initial-boundary conditions, provided that

$$f_\varepsilon(x,t) = -\frac{\partial u_2}{\partial t}$$

and

$$g_\varepsilon(x,t) = -\frac{\partial v_2}{\partial t}.$$

In addition, if a maximum principle holds separately for Problem 5.1, 5.2, and 5.3, then the original problem admits itself a maximum principle.

In order to obtain bounds for Problem 5.1, we multiply the first equation of system (4) by $2pu_1^{2p-1}$ and the second one by $2pv_1^{2p-1}$ ($p \in \mathbb{N}$):

$$\text{(7)} \qquad \begin{cases} \dfrac{\partial u_1^{2p}}{\partial t} + \dfrac{1}{\varepsilon}\dfrac{\partial u_1^{2p}}{\partial x} &= \frac{2}{\varepsilon^2} p u_1^{2p-1}(v_1 - u_1) \\ \dfrac{\partial v_1^{2p}}{\partial t} - \dfrac{1}{\varepsilon}\dfrac{\partial v_1^{2p}}{\partial x} &= \frac{2}{\varepsilon^2} p v_1^{2p-1}(u_1 - v_1). \end{cases}$$

By integration over Ω, adding the resulting equations and using the boundary conditions, we have:

$$\text{(8)} \qquad \begin{aligned} &\frac{d}{dt}\int_\Omega (u_1^{2p} + v_1^{2p})dx + \frac{1}{\varepsilon}\left[u_1^{2p}(L,t) + v_1^{2p}(-L,t)\right] = \\ &= \frac{2p}{\varepsilon^2}\int_\Omega (u_1 - v_1)\left(v_1^{2p-1} - u_1^{2p-1}\right)dx \le 0. \end{aligned}$$

Thus we deduce, at least formally, that

$$\frac{d}{dt}\int_\Omega \left[u_1^{2p} + v_1^{2p}\right]dx \le 0$$

for all $t \ge 0$. Letting p go to $+\infty$, we find that

$$\text{(9)} \qquad \max\{\|u_1(t)\|_\infty, \|v_1(t)\|_\infty\} \le \max\{\|u_0\|_\infty, \|v_0\|_\infty\}.$$

This proves the following lemma.

LEMMA 5.1. *Let* $u_1(x,0) = u_0(x)$, $v_1(x,0) = v_0(x) \in L^\infty(\Omega)$ *and* $u_1(-L,t) = 0$, $v_1(+L,t) = 0$, *for all* $t > 0$. *Then* $u_1(x,t)$, $v_1(x,t) \in L^\infty(\Omega)$ *and*

$$\max\{\|u_1(t)\|_\infty, \|v_1(t)\|_\infty\} \le \max\{\|u_0\|_\infty, \|v_0\|_\infty\}.$$

In order to study the so called "stationary problem", we subtract the two equations of system (5), finding

$$\frac{\partial}{\partial x}(u_2 - v_2) = 0;$$

this means that

$$u_2(x,t) = v_2(x,t) + \alpha(t) \tag{10}$$

almost everywhere in Ω, where $\alpha(t)$ is a function that will be determined later.

Moreover, adding the two equations of system (5), we find that

$$\frac{\partial}{\partial x}(u_2 + v_2) = \frac{2}{\varepsilon}(v_2 - u_2).$$

By using (10) we have

$$\frac{\partial v_2}{\partial x}(x,t) = -\frac{\alpha(t)}{\varepsilon}. \tag{11}$$

If we integrate (11) on the interval (L, x), we obtain

$$v_2(x,t) = \varphi^+(t) - \frac{\alpha(t)}{\varepsilon}(x - L). \tag{12}$$

Using (10) at $x = -L$, we have $v_2(-L,t) = \varphi^-(t) - \alpha(t)$; this result, joined to (12), leads to conclude that

$$\alpha(t) = \varepsilon\frac{\varphi^-(t) - \varphi^+(t)}{\varepsilon + 2L}.$$

Similarly, by integrating on $(-L, x)$, we can prove that

$$u_2(x,t) = \varphi^-(t) - \frac{\alpha(t)}{\varepsilon}(x + L).$$

Thus we have proved the following lemma.

LEMMA 5.2. *The solution of Problem 5.2 is given by the two functions*

$$u_2(x,t) = \varphi^-(t) - \frac{\varphi^-(t) - \varphi^+(t)}{\varepsilon + 2L}(x + L)$$

$$v_2(x,t) = \varphi^+(t) - \frac{\varphi^-(t) - \varphi^+(t)}{\varepsilon + 2L}(x - L).$$

Thanks to the hypotheses on the boundary conditions $\varphi^-(t)$ and $\varphi^+(t)$, $u_2(x,t)$, $v_2(x,t) \in W^{1,\infty}(0,T;C^\infty(\bar{\Omega}))$, uniformly in ε.

Problem 5.3 needs a slightly more complicated proof, which will be given in several steps.

We first notice that f_ε, g_ε belong to $L^\infty(0,T;C^\infty(\bar{\Omega}))$ by Lemma 5.2. Then, we multiply the first equation of system (6) for $2pu_3^{2p-1}$ and the second one by $2pv_3^{2p-1}$; then we integrate on Ω and add the two obtained equations:

$$\frac{d}{dt}\int_\Omega (|u_3|^{2p} + |v_3|^{2p})dx \le 2p\int_\Omega (|f_\varepsilon||u_3|^{2p-1} + |g_\varepsilon||v_3|^{2p-1})dx. \tag{13}$$

Let M be

$$M = \operatorname*{ess\,sup}_{x\in\Omega,\ t\in(0,T)} \{|f_\varepsilon|, |g_\varepsilon|\}.$$

Then, inequality (13) becomes

$$\frac{d}{dt}\int_\Omega (|u_3|^{2p} + |v_3|^{2p})dx \le 2pM\int_\Omega (|u_3|^{2p-1} + |v_3|^{2p-1})dx.$$

Now, by the Hölder-inequality we have:

$$\int_\Omega |u_3|^{2p-1}dx \le (2L)^{1/2p}\left[\int_\Omega |u_3|^{2p}dx\right]^{\frac{2p-1}{2p}}.$$

By the algebraic inequality

$$a^c + b^c \le 4(a+b)^c \qquad a, b \ge 0,\ \frac{1}{2} \le c \le 1,$$

we obtain

$$\frac{d}{dt}\int_\Omega (|u_3|^{2p} + |v_3|^{2p})dx \le 8pM(2L)^{1/2p}\left[\int_\Omega (|u_3|^{2p} + |v_3|^{2p})dx\right]^{\frac{2p-1}{2p}}.$$

Letting

$$y(t) = \int_\Omega [|u_3(x,t)|^{2p} + |v_3(x,t)|^{2p}]dx,$$

we must now solve the ordinary differential inequality:

$$\frac{d}{dt}y(t) \le 8pM(2L)^{1/2p}y(t)^{\frac{2p-1}{2p}}.$$

Its solution is

$$y(t)^{1/2p} \le y(0)^{1/2p} + 4M(2L)^{1/2p}t,$$

i. e., thanks to the initial conditions $u_3(x,0) = v_3(x,0) = 0$

$$\left[\int_\Omega (|u_3|^{2p} + |v_3|^{2p})dx\right]^{1/2p} \leq 4M(2L)^{1/2p}t. \tag{14}$$

Finally, letting $p \to +\infty$, it is possible to show that, for any finite time, the solution of Problem 5.3 is essentially bounded.

In conclusion, considering our global problem, by linearity we have proved the following theorem.

THEOREM 5.1. *Let $u_\varepsilon(x, 0) = u_0(x)$, $v_\varepsilon(x, 0) = v_0(x) \in L^\infty(\Omega)$ and $u_\varepsilon(-L, t) = \varphi^-(t)$, $v_\varepsilon(+L, t) = \varphi^+(t) \in W^{1,\infty}(0, T)$, for all $T > 0$. Then $u_\varepsilon(x, t)$, $v_\varepsilon(x, t) \in L^\infty(\Omega)$ for all $t \in [0, T]$, uniformly in ε.*

3. The solution on the boundary

This section is devoted to the study of the limiting behaviour of $u_\varepsilon(+L, t)$ and $v_\varepsilon(-L, t)$ on the boundary.

If $1 \leq p < \infty$, equation (8) shows that

$$\frac{d}{dt}\int_\Omega (u_1^{2p} + v_1^{2p})dx + \frac{1}{\varepsilon}\left[u_1^{2p}(L, t) + v_1^{2p}(-L, t)\right] \leq 0$$

and so, by Theorem 5.1

$$\frac{1}{\varepsilon}\int_0^T \left[u_1^{2p}(L, t) + v_1^{2p}(-L, t)\right]dt \leq (\|u_0\|_{2p}^{2p} + \|v_0\|_{2p}^{2p}) = E,$$

where $E \in \mathbb{R}^+$. Then we have

$$\lim_{\varepsilon\to 0} \|u_1(L, t)\|_{L^p(0,T)} = 0,$$
$$\lim_{\varepsilon\to 0} \|v_1(-L, t)\|_{L^p(0,T)} = 0.$$

Furthermore, by Lemma 5.2, we have that

$$\lim_{\varepsilon\to 0} u_2(L, t) = \varphi^+(t)$$
$$\lim_{\varepsilon\to 0} v_2(-L, t) = \varphi^-(t)$$

almost everywhere.

Finally, if we choose $1 \leq p < \infty$, system (6) also implies that

$$\frac{d}{dt}\int_\Omega (|u_3|^{2p} + |v_3|^{2p})dx + \frac{1}{\varepsilon}\left[u_3^{2p}(L, t) + v_3^{2p}(-L, t)\right] \leq 2pM\int_\Omega (|u_3|^{2p-1} + |v_3|^{2p-1})dx.$$

By integration over $(0, T)$ we have

$$\frac{1}{\varepsilon}\int_0^T \left[u_3^{2p}(L,t)+v_3^{2p}(-L,t)\right]dt \le 2pM\int_0^T\int_\Omega (|u_3|^{2p-1}+|v_3|^{2p-1})dxdt+$$
$$+\int_\Omega (|u_3(x,0)|^{2p}+|v_3(x,0)|^{2p}-|u_3(x,T)|^{2p}-|v_3(x,T)|^{2p})dx.$$

By inequality (14), we know that the $L^p(\Omega)$-norms of u_3 and v_3 are bounded by a linear function of the time, and therefore all the integrals on the right-hand side of the above inequality are bounded, provided that $1 \le p < \infty$, for all $T > 0$. Also in this case, we then have

$$\lim_{\varepsilon\to 0} \|u_3(L,t)\|_{L^p(0,T)} = 0,$$
$$\lim_{\varepsilon\to 0} \|v_3(-L,t)\|_{L^p(0,T)} = 0.$$

Therefore, since

$$\rho_\varepsilon = \sum_{i=1}^{3} (u_i + v_i)\,,$$

by using the properties of the norm we have proved the following theorem.

THEOREM 5.2. *Let $\rho_\varepsilon = u_\varepsilon + v_\varepsilon$ the macroscopic density of system (2) and $\varepsilon > 0$. Then, on the boundary,*

$$\lim_{\varepsilon\to 0} \rho_\varepsilon(-L,t) = 2\varphi^-,$$
$$\lim_{\varepsilon\to 0} \rho_\varepsilon(+L,t) = 2\varphi^+$$

strongly in $L^p(0,T)$, provided that $1 \le p < \infty$.

4. Behaviour of the flux

In this section we show that the flux is bounded in $L^2(\Omega \times (0,T))$.

We repeat the splitting of system (1) and study separately the behaviour of the three fluxes: j_1, j_2 and j_3 respectively; by using the classical inequality

$$(a+b+c)^2 \le 3(a^2+b^2+c^2), \qquad a,b,c, \in \mathbb{R} \tag{15}$$

we will then derive a bound for the total flux $j_\varepsilon(x,t)$.

We can indeed multiply the two equations of system (4) by $2u_1$ and $2v_1$ respectively. Then we add and integrate on Ω, obtaining:

$$\frac{d}{dt}\int_\Omega (u_1^2+v_1^2)dx + \frac{1}{\varepsilon}\left[u_1^2(L,t)+v_1^2(-L,t)\right] = -2\int_\Omega \left(\frac{u_1-v_1}{\varepsilon}\right)^2 dx,$$

and so

$$\int_\Omega |j_1|^2 dx \le -\frac{1}{2}\frac{d}{dt}\int_\Omega (u_1^2+v_1^2)dx$$

$$\int_0^T \int_\Omega |j_1|^2 dxdt \le \frac{1}{2}\int_\Omega [u_0^2 + v_0^2 - u_1^2(x,T) - v_1^2(x,T)]dx \le \frac{1}{2}(\|u_0\|_2^2 + \|v_0\|_2^2). \tag{16}$$

Then, we must consider Problem 5.2: as usual, we use the explicit solution in Lemma 5.2 in order to show that

$$\begin{aligned} j_2 &= \frac{1}{\varepsilon}\left[\varphi^-(t) - \frac{\varphi^-(t) - \varphi^+(t)}{\varepsilon + 2L}(x+L) - \varphi^+(t) + \frac{\varphi^-(t) - \varphi^+(t)}{\varepsilon + 2L}(x-L)\right] \\ &= \frac{1}{\varepsilon + 2L}[\varphi^-(t) - \varphi^+(t)]. \end{aligned} \tag{17}$$

Finally, we multiply the two equations of system (6) for $2u_3$ and $2v_3$ respectively. Then we add and integrate on Ω, obtaining:

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (u_3^2 + v_3^2)dx + \frac{1}{\varepsilon}\left[u_3^2(L,t) + v_3^2(-L,t)\right] = \\ &= -2\int_\Omega \left(\frac{u_3 - v_3}{\varepsilon}\right)^2 dx + 2\int_\Omega (f_\varepsilon u_3 + g_\varepsilon v_3)dx, \end{aligned}$$

that is, by the maximum principle for Problem 5.3 and using the properties of f_ε and g_ε:

$$\int_\Omega |j_3|^2 dx \le -\frac{1}{2}\frac{d}{dt}\int_\Omega (u_3^2 + v_3^2)dx + 2Kt,$$

where K is a positive constant. This means that

$$\int_0^T \int_\Omega |j_3|^2 dxdt \le KT^2. \tag{18}$$

Inequality (15), together with (16), (17) and (18), shows that $j_\varepsilon = j_1 + j_2 + j_3$ is bounded in $L^2(\Omega \times (0,T))$.

THEOREM 5.3. *Let $u_\varepsilon(x,t)$, $v_\varepsilon(x,t)$ be the unique solution of the initial-boundary problem 1. Then, for all $T > 0$ there exists $D \in \mathbb{R}^+$ such that:*

$$\int_0^T \int_\Omega |j_\varepsilon|^2 dxdt \le D,$$

uniformly in ε.

5. The hydrodynamical limit

In this section, we study the limiting behaviour of the solution $(\rho_\varepsilon, j_\varepsilon)$ to system (2) as $\varepsilon \to 0$. In our passage to the limit, we will consider various relatively compact sequences. In these cases, when we say that the sequence converges to a limit, we mean that there exists a subsequence which converges to a limit.

First, since $\rho_\varepsilon = u_\varepsilon + v_\varepsilon$ is bounded in L^∞ (and hence in L^2) by Theorem 5.1, we notice that there exists a subsequence ρ_ε such that $\rho_\varepsilon \rightharpoonup \rho$ in L^2. Moreover, by Theorem 5.3, we have that $j_\varepsilon \rightharpoonup j$ in $L^2_{x,t}$.

Consider now system (2) with the following conditions:

$$\begin{aligned} \rho_\varepsilon(x,0) &= u_0(x) + v_0(x) \\ \rho_\varepsilon(-L,t) &= \rho_*(-L,t,\varepsilon) \\ \rho_\varepsilon(+L,t) &= \rho_*(+L,t,\varepsilon), \end{aligned}$$

where the right-hand sides of the last two conditions are partially unknown, but they approach respectively $2\varphi^-$ and $2\varphi^+$ as $\varepsilon \to 0$ by Theorem 5.2. We can conclude, by substitution, that

$$\frac{\partial \rho_\varepsilon}{\partial t} - \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\partial \rho_\varepsilon}{\partial x} + \varepsilon^2 \frac{\partial j_\varepsilon}{\partial t}\right] = 0, \tag{19}$$

at least at a formal level.

Let $\phi(x,t)$ be a test function of class C^∞ that vanishes outside the rectangle $(-L, L) \times (0, T)$. Multiplying equation (19) by ϕ and then integrating in $(-L, L) \times (0, T)$, we obtain the weak formulation:

$$\int_0^T \int_{-L}^L \frac{\partial \rho_\varepsilon}{\partial t}\phi dxdt - \frac{1}{2}\int_0^T \int_{-L}^L \frac{\partial}{\partial x}\left[\frac{\partial \rho_\varepsilon}{\partial x} + \varepsilon^2 \frac{\partial j_\varepsilon}{\partial t}\right]\phi dxdt = 0.$$

This equation coincides with the weak formulation of the heat equation, provided that

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int_0^T \int_{-L}^L \frac{\partial}{\partial x}\left(\frac{\partial j_\varepsilon}{\partial t}\right)\phi dxdt = 0,$$

and the initial-boundary conditions approach the correct ones as $\varepsilon \to 0$.

Indeed, we have that

$$\varepsilon^2 \int_0^T \int_{-L}^L \frac{\partial}{\partial x}\left(\frac{\partial j_\varepsilon}{\partial t}\right)\phi dxdt = \varepsilon^2 \int_0^T \left[\frac{\partial j_\varepsilon}{\partial t}\phi\right]_{-L}^L dt - \varepsilon^2 \int_0^T \int_{-L}^L \frac{\partial j_\varepsilon}{\partial t}\frac{\partial \phi}{\partial x} dxdt,$$

where the first term on the right-hand side vanishes by the conditions on the support of ϕ. Therefore, it remains only the second term; we now prove that it approaches zero as $\varepsilon \to 0$.

Since

$$\varepsilon^2 \int_0^T \int_{-L}^L \frac{\partial j_\varepsilon}{\partial t}\frac{\partial \phi}{\partial x} dxdt = \varepsilon^2 \int_{-L}^L \left[j_\varepsilon \frac{\partial \phi}{\partial x}\right]_0^T dx - \varepsilon^2 \int_0^T \int_{-L}^L j_\varepsilon \frac{\partial^2 \phi}{\partial x \partial t} dxdt,$$

we may consider the two terms on the right-hand side separately. We have that

$$\varepsilon^2 \int_0^T \int_{-L}^L \left| j_\varepsilon \frac{\partial^2 \phi}{\partial x \partial t}\right| dxdt \le \varepsilon^2 \left[\int_0^T \int_{-L}^L j_\varepsilon^2 dxdt\right]^{\frac{1}{2}} \left[\int_0^T \int_{-L}^L \left(\frac{\partial^2 \phi}{\partial x \partial t}\right)^2 dxdt\right]^{\frac{1}{2}} \to 0$$

because of Theorem 5.3 and the smoothness of ϕ; furthermore, we obtain

$$\varepsilon^2 \int_{-L}^{L} \left[j_\varepsilon \frac{\partial \phi}{\partial x} \right]_0^T dx = \varepsilon \int_{-L}^{L} \left[(u_\varepsilon - v_\varepsilon) \frac{\partial \phi}{\partial x} \right]_0^T dx \to 0$$

because of the maximum principle (see Theorem 5.1) and the smoothness of ϕ.

Since j_ε is bounded in L^2, we deduce that $\partial j_\varepsilon / \partial x$ belongs to H^{-1}, and so we can derive both members of the second equation in system (2) with respect to x. Therefore ρ, which satisfies the boundary conditions $\rho(-L, t) = 2\varphi^-$, $\rho(L, t) = 2\varphi^+$ in $L^p(0, T)$ for $p \in [1, \infty)$ and the initial condition $\rho(x, 0) = u_0 + v_0$, solves by subsequences the heat equation

$$\frac{\partial \rho}{\partial t} - \frac{1}{2} \frac{\partial^2 \rho}{\partial x^2} = 0 \tag{20}$$

in a weak sense, in the rectangle $\Omega \times (0, T)$.

Since we have assumed the initial values $u_0, v_0 \in L^\infty(\Omega)$, also the initial density $\rho_0(x) = \rho(x, t = 0) \in L^\infty(\Omega)$. On the other hand, the heat equation (3), which is compatible with the initial-boundary value problem for system (1), admits a unique global solution in $\mathcal{D}'$. The uniqueness result guarantees the existence of a unique limit for the whole family.

Therefore, the main results of this paper may be summarized as follows:

THEOREM 5.4. *Let $(\rho_\varepsilon, j_\varepsilon)$ be a sequence of solutions to the initial-boundary value problem for system (2), where the initial values $u_0, v_0 \in L^\infty(\Omega)$, and the boundary conditions $u_\varepsilon(-L, t) = \varphi^-(t)$, $v_\varepsilon(+L, t) = \varphi^+(t) \in W^{1,\infty}(0, T)$. Then, there exists $\rho \in L^\infty$ such that $\rho_\varepsilon(x, t)$ converges to $\rho(x, t)$ in L^2. Moreover $\varepsilon j_\varepsilon$ converges to zero strongly in $L^2(\Omega \times (0, T))$. The limit density $\rho(x, t)$ is the (unique) weak solution to the initial-boundary value problem for the heat equation (3), in $\mathcal{D}'(\Omega \times (0, T))$, with initial datum $\rho_0 = u_0 + v_0$, and boundary conditions*

$$\rho(-L, t) = u(-L, t) + v(-L, t) = 2\varphi^-,$$
$$\rho(L, t) = u(L, t) + v(L, t) = 2\varphi^+$$

in $L^p(0, T)$, $1 \le p < \infty$.

6. Conclusions

The paper shows that the heat equation with initial-boundary conditions can be obtained as the hydrodynamical limit (in a weak sense) for the Goldstein-Taylor model of the Boltzmann equation, provided that initial data belong to L^∞ and boundary conditions belong to $W^{1,\infty}$. There are still two open problems. It seems, indeed, that it is possible a n-dimensional generalization, which will be considered in a later work.

Even more interesting might be the nonlinear case, which needs new techniques, not connected to the properties of the linearity of the problem.

7. Acknowledgements

The author is deeply grateful to Professor G. Toscani for helpful discussions and constant encouragement. Moreover he thanks Professors U. Gianazza, R. Monaco and E. Vitali for their friendly support.

This work has been partially supported by the European TMR project "asymptotic methods in kinetic theory", contract ERB FMRX CT97 0157.

Riferimenti bibliografici

[1] CARLEMAN T., *Problèmes mathématiques dans la théorie cinétique des gaz*, Almqvist-Wiksells, Uppsala 1957.

[2] CERCIGNANI C., *The Boltzmann equation and its applications*, Springer-Verlag, New York 1988.

[3] GABETTA E. AND PERTHAME G., *Scaling limits for the Ruijgrok-Wu model of the Boltzmann equation*, Pubbl. IAN-CNR Pavia **1101** (1998), 1–24.

[4] GABETTA E., PARESCHI L. AND TOSCANI G., *Relaxation schemes for nonlinear kinetic equations*, SIAM J. Numer. Anal. **34** (1997), 2168–2194.

[5] JIN S., PARESCHI L. AND TOSCANI G., *Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations*, SIAM J. Numer. Anal. **35** (1998), 2405–2439.

[6] GOLDSTEIN S., *On diffusion by discontinuous movements, and on the telegraph equation*, Quart. J. Mech. Appl. Math. **4** (1951), 129–156.

[7] LIONS P.L. AND TOSCANI G., *Diffusive limits for finite velocities Boltzmann kinetic models*, Rev. Mat. Iberoamericana **13** (1997), 473–513.

[8] NALDI G. AND PARESCHI L., *Numerical schemes for kinetic equations in diffusive regimes*, Appl. Math. Lett. **11** (1998), 29–35.

[9] PŁATKOWSKI T. AND ILLNER R., *Discrete velocity models of the Boltzmann equation: a survey on the mathematical aspects of the theory*, SIAM Review **30** (1988), 213–255.

[10] PULVIRENTI A. AND TOSCANI G., *On a diffusion problem for a Carleman-like equation*, to appear on Math. Models and Meth. in Appl. Sci. (1998).

[11] TAYLOR G.I., *Diffusion by continuous movements*, Proc. London Math. Soc. **20** (1922), 196–212.

[12] TOSCANI G. AND VAN DER MEE C.V.M., *An abstract approach to nonlinear Boltzmann-type equations*, Ann. Univ. Ferrara - Sez. VII - **XXXIV** (1988), 75–100.

[13] VICARIO P., *Diffusive relaxation to scalar viscous conservation laws*, preprint (1997).

AMS Subject Classification: 76P05, 82C40, 82C70.

Francesco SALVARANI
Dipartimento di Matematica
Politecnico di Torino
corso Duca degli Abruzzi, 24
10129 Torino, ITALIA
e-mail: `salvaran@calvino.polito.it`

Lavoro pervenuto in redazione il 05.03.1999 e, in forma definitiva, il 23.11.1999.