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Autorizzazione del Tribunale di Torino N. 2962 del 6.VI.1980
Direttore Responsabile: ANDREA BACCIOTTI

QUESTA RIVISTA È STAMPATA CON IL CONTRIBUTO DI:
UNIVERSITÀ DEGLI STUDI DI TORINO
DIPARTIMENTO DI MATEMATICA DEL POLITECNICO DI TORINO
CONSIGLIO NAZIONALE DELLE RICERCHE
FONDAZIONE BANCA CRT

RENDICONTI
DEL SEMINARIO
MATEMATICO

Università e Politecnico di Torino

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Volume 56, N. 3 **1998**

Rend. Sem. Mat. Univ. Pol. Torino
Vol. 56, 3 (1998)

M.Y. Chi (Qi Minyou)

HADAMARD'S FUNDAMENTAL SOLUTION AND MULTIPLE-CHARACTERISTIC PROBLEMS †

Abstract. In this work, Hadamard's construction of fundamental solutions for linear holomorphic PDE is generalized to a rather broad class of linear holomorphic PDE. In the case of simple-characteristics, solutions with polynomial singularity along the characteristic conoid are constructed which are very close to Hadamard's fundamental solution. In the case of multi-characteristics, solutions with exponential singularity are obtained. It is clear that Hadamard's procedure is just a kind of asymptotic method; using this method and singular solutions mentioned above, some new results for the propagation of singularities are also proved which are essentially a generalization of Huygens' Principle.

1. Introduction

Among Hadamard's contributions to the theory of linear PDE, the construction of fundamental solution is very prominent and of basic importance. In a certain sense, this contribution of his summed vast progress in this area to his time, and was very clearly presented in his classical treatise: "Le Problème de Cauchy" [1] and also in his posthumous monograph: "La Théorie des Équations aux Dérivées Partielles" [2], which was published in China (1964) in accordance with his will, hence little known in the West and is now a literature rarity even in China. Hadamard defined the fundamental solution (solution élémentaire) to be solutions with certain singularity and tried to find them in the form of an asymptotic series

$$\Gamma(x, y) = \sum_{h=0}^{\infty} U_h(x, y) k^{p+h} / \Gamma(p + h + 1) \tag{1}$$

where $k = 0$ is the equation for the characteristic conoid, and $k(x, y)$ satisfies an important first relation:

$$A\Big(x, \frac{\partial k}{\partial x}\Big) = 4k\,, \tag{2}$$

where $A(x, \xi)$ is the principal symbol of the 2nd order linear holomorphic partial differential operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u, \tag{3}$$

† This work is supported by Chinese National Natural Sciences Foundation and the Foundation of the State Education Commission of China.

$$a_{ij}(x) = a_{ji}(x), \quad \det(a_{ij}(x)) \neq 0\,.$$

Hadamard's method is actually just the asymptotic method widely used among mathematical physicists. (2) is just the eiconal equation. Hadamard then proceeded and obtained the transport equations for U_h and the convergence of the series (1) was proved by standard majoration methods.
Hadamard's argument is variational: characteristic conoid is composed of bicharacteristics issuing from the vertex, while the latter are geodesics in a certain metric defined through the principal symbol which is quadratic. This approach restricted him to partial differential operators of 2nd order only. But as early as H. Poincaré and E. Cartan, it was known that Hamiltonian variational principles are equivalent to the principle of integral invariants [3]; the latter principle is sympletic in nature and thus can be used in a much broader field. J. Leray is aware of this, in the late 50's and early 60's, he published a series of important papers under the general title "Problème de Cauchy" [4], which contains a generalization of (2) as

$$g(x, k_x) = \frac{k}{m-1}\,. \tag{4}$$

Also, it should be mentioned, the 6th or the last of these series, i.e., L. Gårding, T. Kotake and J. Leray [5] developed systematically the asymptotic approach to the Cauchy problem.

But there is another approach to the theory of fundamental solutions. The year 1930's saw I. G. Petrowsky's work on general PDE which started an algebro-geometrical approach to the theory of linear PDE with constant coefficients. Fourier-Laplace transform is the main tool, and this is the beginning of the period when the theory of distributions "dominated" this field. Fundamental solutions are now defined to be the solutions of the equation $L(u) = \delta$. Ehrenpreis, Hörmander and Malgrange proved the existence of the fundamental solutions for general linear PDE with constant coefficients. For general linear PDE with variable coefficients, Lewy's famous example showed the deep difference between the analytic and C^∞ frames. For the latter, we have now the micro- local analysis, with the help of the PsDO and the FIO, we can construct parametrices, which are approximate fundamental solutions. Thus there arises a problem: can we also construct distributional fundamental solutions for general linear PDE with variable coefficients and what are the relations between these two kinds of fundamental solutions? Another problem is: Hadamard established his theory for normal elliptic and hyperbolic equations, which are all of principal type. Then, what can we say about the multiple-characteristic problems? As will be seen in what follows, Hadamard's theory is actually a Fuchsian theory, while PDE's with multiple-characteristics can be considered as an analogy of the ODE's with irregular singularities. For the latter, we have the arsenal of such things as the asymptotic expansions in a sector, the Borel-Laplace transforms etc. All these come from the exponential growth of the solutions. Thus the Gevrey classes and their dual, the ultra- distributions, offer a natural frame for the multiple-characteristic problems. For a very clear treatment, see [14], also see [15] for an up-to-date survey with comprehensive literature. But can we also consider Hadamard's theory from a distributional point of view? It is the author's aim to give partial answers to these problems. We can prove that Hadamard's fundamental solution is only one from a broad category of solutions with definite singularity, and both approaches are closely related and can be unified. But in the present paper, we must restrict ourselves to first extend Hadamard's approach to a class of linear holomorphic PDE of higher order with simple characteristics, and next construct for a class of linear holomorphic PDE's with multiple- characteristics a solution with exponential singularity following Hadamard's procedure, hence we call it the Hadamard fundamental solution, although its relation with the distributional fundamental solution is not clear yet and will be treated later.

The plan of this paper is as follows. In part I, we consider the simple- characteristic case where (4) is valid. In part II, the simplest case of multiple-characteristic problem is considered where (4) is not valid.

2. Part I. Simple-characteristic Problems

2.1. Notations

Let X be a domain in an analytic complex manifold with complex dimension n, with local coordinates for its element $x = (x_1, \cdots, x_n)$. The complex projective space $\Xi^* = \mathbf{C}^{n+1} \setminus 0/\sim$ ($\sim$ denotes dilation) is just the space of complex affine functions defined on X, and we denote

$$\langle \xi, x\rangle = \xi_0 + \xi \cdot x = \xi_0 + \sum_{i=1}^{n} \xi_i \cdot x_i \,. \tag{5}$$

Let $P(x, \partial_x)$ be a linear holomorphic partial differential operator of order m:

$$P(x, \partial_x) = \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha \,, \tag{6}$$

with principal symbol

$$P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \,. \tag{7}$$

2.2. Characteristic conoid

Bicharacteristics strips for $P(x, \partial_x)$ are defined as orbits of the Hamiltonian system

$$\begin{aligned} &\frac{dx}{dt} = \frac{\partial P_m(x, \xi)}{\partial \xi} \quad x|_{t=0} = y \,, \\ &\frac{d\xi}{dt} = -\frac{\partial P_m(x, \xi)}{\partial x} \quad \xi|_{t=0} = \eta \,, \\ &P_m(y, \eta) = 0 \,, \end{aligned} \tag{8}$$

which give extremal curves for the action integral

$$W = \int \sum \xi_i \, dx_i - H \, dt \,.$$

In analytic dynamics, the Hamiltonian H is the total energy, i.e., the sum of kinectic energy (quadratic in ξ) and potential energy, and in our case, H should be replaced by $P_m(x, \xi)$ which is homogeneous of degree m in ξ now. H. Poincaré and E. Cartan realized that Hamilton's principle of least action is equivalent to the principle of invariance of energy-momentum form. Thus in our case, we should add another equation for ξ_0 to (8)

$$\begin{aligned} &\frac{d\xi_0}{dt} = \sum_i x_i \frac{\partial}{\partial x_i} P_m(x, \xi) - P_m(x, \xi) \,, \\ &\xi_0|_{t=0} = \eta_0 \,, \end{aligned} \tag{9}$$

and for initial values we should assume

$$\langle \eta, y\rangle = \eta_0 + \sum_{i=1}^{n} \eta_i y_i = 0\,,$$
$$P_m(y, \eta) = 0\,.$$

It is readily seen, the system (8), (9) has first integrals

$$P_m(x, \xi),\ \langle \xi, x\rangle + (1-m)t P_m(x, \xi)$$

and also an invariant differential form

$$\omega = \langle d\xi, x\rangle + P_m(x, \xi)\, dt\,. \tag{10}$$

Actually, set $\omega = 0$ and $d\omega = 0$ as differential forms in $dx,\ d\xi$ and dt, we could recover (8) and (9), meaning that this system is the characteristic system for ω and hence ω is invariant form, of (8) and (9) [3] (Chap. 5).

x-components of the system (8), i.e., the bicharacteristic curves in X, through a fixed point y with $P_m(y, \eta) = 0$, $\eta \neq 0$, form a surface with y as a conic point which is called the characteristic conoid with vertex at y. We have

THEOREM 1 (J. LERAY). *The characteristic conoid can be written as $k(x, y) = 0$ where $k(x, y)$ is holomorphic in x and y when $|x - y|$ is small enough, and*

$$P_m(x, k_x) = \frac{k}{m-1} \tag{11}$$

under the assumption

$$\mathrm{Hess}_\eta P_m(y, \eta) \neq 0\,, \tag{12}$$

when $\eta \neq 0$.

Proof. The following proof is reproduced from Leray [4] (paper I) where η is assumed to be complex and (12) holds when $P_m(y, \eta) = 0$. The solution to (8) when $P_m(y, \eta)$ is arbitrary gives a mapping

$$\begin{aligned} x &= y + P_{m\xi}(x, \xi)t + o(t) \\ &= y + P_\eta m(y, \eta)t + o(t) \\ (\xi &= \eta - P_{mx}(x, \xi)t + o(t)) \end{aligned}$$

when $|t|$ is small enough. Thus we may write

$$\begin{aligned} x &= x(t, y, \eta)\,, \\ \xi &= \xi(t, y, \eta)\,, \end{aligned} \tag{13}$$

and prove immediately that

$$\frac{D(x_1(t, y, \eta), \ldots, x_n)}{D(\eta_1, \ldots, \eta_n)} = t\,\mathrm{Hess}_\eta P_m(y, \eta) + o(t)\,,$$

which does not vanish for $t \neq 0$ when $\mathrm{Hess}_\eta P_m(y, \eta) \neq 0$ and $\eta \neq 0$. Thus we may replace (t, x, y) by (t, η, y) and *vice versa*. The invariance of ω (see (10)) gives

$$\langle d\xi, x\rangle = -P_m(x, \xi)\, dt + \langle d\eta, y\rangle$$

from which follows the relation:

$$d\langle\xi, x\rangle = -P_m(x,\xi)\,dt + \sum_{i=1}^n \xi_i\,dx_i - \sum_{i=1}^n \eta_i\,dy_i\,,$$

(note that $\langle y, \eta\rangle = 0$). Taking (t, x, y) as independent variables, we have

$$\begin{aligned} \frac{\partial}{\partial t}\langle\xi, x\rangle &= -P_m(x,\xi)\,, \\ \frac{\partial}{\partial x_i}\langle\xi, x\rangle &= \xi_i\,, \\ \frac{\partial}{\partial y_i}\langle\xi, x\rangle &= -\eta_i\,. \end{aligned} \tag{14}$$

Since $P_m(x,\xi)$ and $\langle\xi, x\rangle + (1-m)t P_m(x,\xi)$ are first integrals,

$$\begin{aligned} P_m(x,\xi) &= P_m(y,\eta)\,, \\ \langle\xi, x\rangle &= (m-1)t P_m(y,\eta)\,. \end{aligned} \tag{15}$$

Substituting (15) into (14), we have

$$\frac{\partial}{\partial t}\langle\xi, x\rangle = \langle\xi, x\rangle/(1-m)t\,.$$

Thus there is a function $k(x, y)$ such that

$$\langle\xi(t,\eta,y), x(t,\eta,y)\rangle = t^{\frac{1}{1-m}} k(x,y)\,. \tag{16}$$

$k(x, y)$ should be holomorphic in x and y for x, y close enough. $x = y$ corresponds to $t = 0$ which is singular for $\langle\xi, x\rangle$ as seen from (16).

From the second equation of (14), we have

$$\xi_i = \frac{\partial}{\partial x_i}\langle\xi, x\rangle = t^{\frac{1}{1-m}} k_x(x,y)\,, \tag{17}$$

substituting again into the first equation of (14), we have

$$-\frac{\partial}{\partial t}\langle\xi, x\rangle = \frac{\langle\xi, x\rangle}{(m-1)t} = \frac{t^{\frac{m}{m-1}}}{m-1} k(x,y)\,,$$

but

$$P_m(x,\xi) = P_m\big(x, t^{\frac{1}{1-m}} k_x\big) = t^{\frac{m}{1-m}} P_m(x, k_x)\,,$$

hence (11).

(11) shows that $k(x, y) = 0$ is characteristic. Now take into consideration that $P_m(y,\eta) = 0$. If $x = x(t,\eta,y)$ is a bicharacteristic curve through $y : x|_{t=0} = y$, then, since $P_m(x,\xi)$ is a first integral,

$$\begin{aligned} k[x(t,\eta,y), y] &= (m-1)P_m[x(t,\eta,y), k_x(x(t,\eta,y),y)] \\ &= (m-1)t^{\frac{m}{m-1}} P_m[x(t,\eta,y), \xi(t,\eta,y)] \\ &= (m-1)t^{\frac{m}{m-1}} P_m(y,\eta) = 0\,. \end{aligned}$$

It means that the whole bicharacteristic curve lies on the surface $k(x, y) = 0$. Thus, it is easily seen, $k(x, y) = 0$ is an equation for the characteristic conoid.

□

In the proof above, the condition $\mathrm{Hess}_\eta P_m(y,\eta) \neq 0$ is very important. It excludes multiple-characteristic problems from consideration. Actually, we have the following

LEMMA 1. *If $a(x,\xi)$ and $b(x,\xi)$ both vanish at a point (x_0,ξ_0) and $\mathrm{grad}_\xi a(x_0,\xi_0)$ is parallel to $\mathrm{grad}_\xi b(x_0,\xi_0)$, then $\mathrm{Hess}_\xi[a(x_0,\xi_0)b(x_0,\xi_0)] = 0$, here $x \in \mathbb{C}^n$, $0 \neq \xi \in \mathbb{C}^n$.*

Proof. Denote differentiation in ξ by sub-indices. Since

$$\begin{aligned}[a(x,\xi)b(x,\xi)]_{ij} &= a(x,\xi)b_{ij}(x,\xi) + a_i(x,\xi)b_j(x,\xi)\\ &\quad + a_j(x,\xi)b_i(x,\xi) + a_{ij}(x,\xi)b(x,\xi)\,,\end{aligned}$$

$\mathrm{Hess}_\xi[a(x,\xi)b(x,\xi)]$ can be written as a sum of 2^n determinants and we may arrange them such that every row is of either the following forms:

$$(a_ib_1, a_ib_2, \ldots, a_ib_n) = a_i(b_1, b_2, \ldots, b_n)$$

or

$$(a_1b_i, a_2b_i, \ldots, a_nb_i) = b_i(a_1, a_2, \cdots, a_n)$$

Since these vectors are parallel, each determinant contain 2 linearly dependent rows making this determinant vanishing. Thus the lemma is proved.

□

Because of this lemma, arguments in this part are usually invalid for partial differential operator of the form

$$P_m(x,\partial_x) = [a(x,\partial_x)]^2$$

which is the simplest double-characteristic case, and would be treated in part II of this paper.

2.3. Characteristic projection

Now we discuss further properties of $k(x,y)$.

It is easy to see that the solutions of (8), (9) enjoy the following homogeneity properties:

$$\begin{aligned}\xi_i(\theta^{1-m}t, \theta\eta, y) &= \theta\xi_i(t,\eta,y)\,,\\ x_i(\theta^{1-m}t, \theta\eta, y) &= x_i(t,\eta,y)\,.\end{aligned} \tag{18}$$

Hence ξ is a characteristic projection in Leray's sense [4] (IV). Set

$$\tau = t^{\frac{1}{m-1}},\ \theta = \tau, \tag{19}$$

we have

$$\begin{aligned}\tau\xi_i(t,\eta,y) &= \xi_i(1,\tau\eta,y)\,,\\ x_i(t,\eta,y) &= x_i(1,\tau\eta,y)\,.\end{aligned}$$

Thus, denote $\tau\eta = \zeta$, $\tau\xi = \pi$, we see $x_i(t,\eta,y)$ are actually functions $x_i(1,\zeta,y)$ of (ζ,y), $\pi_i(t,\eta,y)$ are $\xi_i(1,\zeta,y)$, both are holomorphic in (ζ,y). Hence we write them hereafter simply as $x(\zeta,y)$, $\pi(\zeta,y)$. (16) and (17) now give

$$\begin{aligned}k(x,y) &= \langle\pi(\zeta,y), x(\zeta,y)\rangle\,,\\ k_{x_i}(x,y) &= \pi_i(\zeta,y)\,,\end{aligned} \tag{20}$$

and $k(x, y)$ is also holomorohic in (ζ, y).

The solutions of (8) now give

$$\begin{aligned} x_i &= y_i + P_{m\zeta_i}(x, \zeta) + \dots, \\ \pi_i &= \zeta_i - \tau P_{mx_i}(x, \zeta) + \dots, \end{aligned} \tag{21}$$

"..." are terms of higher order. Hence, from the implicit function theorem, we have from the second equation of (21),

$$\zeta_i = \pi_i + o(\tau).$$

Substitute it into the first equation of (21) gives

$$x_i = y_i + P_{m\zeta_i}(y, \pi) + o(\tau^m).$$

Differentiate both side with respect to x_j, we have

$$\begin{aligned} \delta_{ij} &= \sum_{l=1}^{n} P_{mil}(y, \pi)\frac{\partial \pi_l}{\partial x_j} + \cdots \\ &= \tau^{m-2}\sum_{l=1}^{n} P_{mil}(y, \xi)\frac{\partial \pi_l}{\partial x_j} + \cdots \\ &= \sum_{l=1}^{n} P_{mil}(y, k_x)\frac{\partial^2 k}{\partial x_i\, \partial x_j} + \cdots \end{aligned}$$

Thus,

$$\sum_{i,j=1}^{n} P_{mij}(x, k_x)\frac{\partial^2 k}{\partial x_i\, \partial x_j} = n + F(\zeta, y). \tag{22}$$

where $F(\zeta, y)$ is holomorphic in (ζ, y) and $F(0, y) = 0$. Henceforth, sub-indices always denote differentiation in fiber variables.

2.4. Construction of Hadamard fundamental solution

Now we will look for a solution to $Pu = 0$ where P is defined in (6) in the form

$$u(x, y) = \sum_{h=0}^{\infty} U_h k^{p+h}/\,\Gamma(p + h + 1), \tag{23}$$

p and U_h are to be decided. Whereas (11) plays the role of eiconal equation, we are to find the transport equations. We proceed first to calculate $P_m(x, \partial_x)U_0 k^p/\,\Gamma(p + 1)$. By generalized Leibniz's formula, we have

$$\begin{aligned} P_m(x, \partial_x)U_0 k^p/\,\Gamma(p + 1) &= U_0 P_m(x, \partial_x)k^p/\,\Gamma(p + 1) \\ &\quad + \sum_{i=1}^{n} \frac{\partial U_0}{\partial x_j} P_{mj}(x, \partial_x)k^p/\,\Gamma(p + 1) + \cdots, \end{aligned} \tag{24}$$

where the dots stand for terms containing a factor of the form $P^{(\alpha)}_{m\xi}(x, \partial_x)k^p/\Gamma(p+1)$, $|\alpha| \geq 2$. For the first term, a careful calculation gives

$$\begin{aligned}(25)\qquad P_m(x, \partial_x)k^p/\Gamma(p+1) &= P_m(x, k_x)k^{p-m}/\Gamma(p-m+1)\\ &\quad + \frac{1}{2}\sum_{i,j=1}^{n} P_{mij}(x, k_x)\frac{\partial^2 k}{\partial x_i\, \partial x_j}\cdot k^{p-m+1}/\Gamma(p-m+2) + \cdots .\end{aligned}$$

Verification of this result, particularly the coefficient $\frac{1}{2}$ in front of the second term is tedious. But a general procedure would be given in Part II, showing that (25) is correct.

For the second term in (24), noting that

$$P_{mj}(x, \partial_x)k^p/\Gamma(p+1) = P_{mj}(x, k_x)k^{p-m+1}/\Gamma(p-m+2) + \cdots ;$$

and $k_x = t^{\frac{1}{m-1}}\xi$, $P_{mj}(x, \xi) = \frac{dx_j}{dt}$, we have

$$\sum_{j=1}^{n}\frac{\partial U_0}{\partial x_j}P_{mj}(x, \partial_x)k^p/\Gamma(p+1) = t\sum\frac{\partial U_0}{\partial x_j}\frac{dx_j}{dt}k^{p-m+1}/\Gamma(p-m+2) + O(k^{p-m+2}) .$$

Thus, using (22) and (11)

$$\begin{aligned}P_m(x, \partial_x)U_0k^p/\Gamma(p+1) &= \Big[\frac{p-m+1}{m-1} + \frac{n}{2}\\ &\quad + \frac{1}{2}F(\zeta, y)\Big]U_0k^{p-m+1}/\Gamma(p-m+2)\\ &\quad + t\frac{dU_0}{dt}k^{p-m+1}/\Gamma(p-m+2) + O(k^{p-m+2})\\ &= \Big[t\frac{dU_0}{dt} + \Big(\frac{p-m+1}{m-1} + \frac{n}{2}\\ &\quad + \frac{1}{2}F(\zeta, y)\Big)U_0\Big]k^{p-m+1}/\Gamma(p-m+2)\\ &\quad + O(k^{p-m+2}) .\end{aligned}$$

Since $t\frac{d}{dt} = \frac{1}{m-1}\tau\frac{d}{d\tau}$, $F(\zeta, y) = O(\tau)$, when we are to look for a bounded U_0, p must be so chosen such that

$$(26)\qquad \frac{n}{2} + \frac{p-m+1}{m-1} = 0, \text{ i.e., } p = (1-m)\left(\frac{n}{2} - 1\right) .$$

After p is decided, we come next to degree $m-1$ (in ∂_x) part of $P(x, \partial_x)$, i.e. $P_{m-1}(x, \partial_x)$. By similar calculation as above, we have

$$\begin{aligned}P_{m-1}(x, \partial_x)U_0k^p/\Gamma(p+1) &= U_0P_{m-1}(x, \partial_x)k^p/\Gamma(p+1)\\ &\quad + \sum_{|\alpha|>0}\frac{1}{\alpha!}\frac{\partial^\alpha U_0}{\partial x^\alpha}P^{(\alpha)}_{m-l,\xi}(x, \partial_x)k^p/\Gamma(p+1)\\ &= U_0P_{m-1}(x, k_x)k^{p-m+1}/\Gamma(p-m+2)\\ &\quad + O(k^{p-m+2})\\ &= tP_{m-1}(x, \xi)U_0k^{p-m+1}/\Gamma(p-m+2)\\ &\quad + O(k^{p-m+2}) .\end{aligned}$$

Summing up, we have proved

THEOREM 2. *We have the following expansion*

$$
\begin{aligned}
P(x,\partial_x)U_0k^p/\Gamma(p+1) = {}& \frac{k^{p-m+1}}{\Gamma(p-m+2)}\Big[\frac{1}{m-1}\tau\frac{dU_0}{d\tau} \\
&+\Big(\frac{n}{2}+\frac{p-m+1}{m-1}+O(\tau)\Big)U_0\Big] \\
&+\frac{k^{p-m+2}}{\Gamma(p-m+3)}L_2(U_0)+\cdots \\
&+\frac{k^p}{\Gamma(p+1)}L_m(U_0)\,,
\end{aligned} \tag{27}
$$

where $L_2,\dots,L_m$ are linear holomorphic partial differential operators of order $2,3,\dots,m$ respectively. Thus for the existence of a bounded U_0, we must take $p=(1-m)\left(\frac{n}{2}-1\right)$ and we have the transport equation for U_0:

$$
\tau\frac{dU_0}{d\tau}+\Big[(m-1)\Big(\frac{n}{2}+\frac{p-m+1}{m-1}\Big)+\tau A(\tau)\Big]U_0=0\,. \tag{28}
$$

We may take

$$
U_0=\exp\left(\int_0^\tau A(\tau)\,d\tau\right). \tag{29}
$$

The transport equations for U_h are similar to (28) only with $p+h$ to replace h and a linear form of $U_{h-1},\dots,U_{h-m+1}$ and the derivatives of U_{h-j} up to order $j+1$ as the right hand side:

$$
\begin{aligned}
&\tau\frac{dU_h}{d\tau}+\Big[p+h-m+1+\frac{n}{2}(m-1)+\tau A(\tau)\Big]U_h \\
&\quad = L(U_{h-1},\dots,U_{h-m+1})\,.
\end{aligned} \tag{30}
$$

Setting $p=(1-m)\left(\frac{n}{2}-1\right)$ as in (26), we have the transport equation for U_h

$$
\tau\frac{dU_h}{d\tau}+[h+\tau A(\tau)]\,U_h=L(U_{h-1},\dots,U_{h-m+1})\,. \tag{31}
$$

Its unique bounded solution is

$$
U_h=\frac{U_0}{\tau^h}\int_0^\tau\frac{\tau^{h-1}}{U_0}L(U_{h-1},\dots,U_{h-m+1})\,d\tau\,. \tag{32}
$$

The argument above fails when p is a negative integer when we should replace (23) by

$$
\begin{aligned}
u(x,y) = {}& U_0k^p+\cdots+U_{-p-1}k^{-p-1}+U_{-p}\log k+\cdots \\
&+U_{-p+h}k^{(h)}+\cdots+\sum_{h=0}^{\infty}V_hk^h/h!\,,
\end{aligned} \tag{33}
$$

where $k^{(h)}$ is a h-th primitive of k, i.e.,

$$
k^{(h)}=\int_0^k\cdots\int_0^k\log k\,dk=\frac{k^h}{h!}\Big(\log k-1-\cdots-\frac{1}{h}\Big)\,,
$$
$$
(h-\text{fold integration})\,.
$$

The transport equations for $U_0, \dots, U_{-p-1}$ are the same as (30), in the deduction of which we should note that

$$kk^{(h)} = (h+1)k^{(h+1)} + k^{h+1}/(h+1)!\,,$$

while for V_h we have

$$\tau \frac{dV_h}{d\tau} + \Big[h + \frac{n}{2}(m-1) + \tau A(\tau)\Big] V_h = U_{-p+h} + L(V_{h-1}, \dots, V_{h-m+1})\,. \tag{34}$$

2.5. Convergence of the formal solution

For the convergence of (23) (that for (33) are quite similar, hence omitted) we use the majorant method as J. Hadamard did in [1].

Let $\sigma = \frac{\tau}{\alpha} + \eta_1 + \dots + \eta_n$, $0 < \alpha < 1$. Since U_0 is holomorphic in (τ, η) we can find $0 < r < 1$ and constant $A_0 > 0$ such that

$$U_0 \ll A_0 \Big/ \Big(1 - \frac{\sigma}{r}\Big)\,. \tag{35}$$

Assume that for $U_1, \dots, U_{h-1}$ it is already known that

$$U_j \ll MA_j \Big/ \Big(1 - \frac{\sigma}{r}\Big)^{2j+1}\,,$$

then for U_h we should first estimate $L(U_{h-1}, \dots, U_{h-m+1})$. Since L contains derivatives of U_{h-j} up to orders $j+1$, we have

$$\begin{aligned} U_{h-j}^{(j+1)} &\ll MA_{h-j}(2h-2j+1)\cdots(2h-2j+j+1) \Big/ \Big(1-\frac{\sigma}{r}\Big)^{2h-2j+j+2} \\ &\ll M_1 2h(2h+1)(p+h)_{j-1} A_{h-j} \Big/ \Big(1-\frac{\sigma}{r}\Big)^{2h+2}\,. \end{aligned}$$

Thus

$$\begin{aligned} L(U_{h-1}, \dots, U_{h-m+1}) \ll \frac{K2h(2h+1)}{\big(1-\frac{\sigma}{r}\big)^{2h+2}} \big[&A_{h-1} + (p+h-1)A_{h-2} + \cdots \\ &+ (p+h-1)_{(m-1)} A_{h-m+1}\big]\,. \end{aligned}$$

Hence

$$\begin{aligned} U_h \ll \frac{K2h(2h+1)}{4(h+1)} \big[&A_{h-1} + (p+h-1)A_{h-2} + \cdots \\ &+ (p+h-1)_{(m-1)} A_{h-m+1}\big] \Big/ \Big(1-\frac{\sigma}{r}\Big)^{2h}\,, \\ (\lambda)_j = \lambda(\lambda-1)&\cdots(\lambda-j+1)\,. \end{aligned}$$

Setting

$$\begin{aligned} A_h &= \Lambda K(p+h+1)\big[A_{h-1} + (p+h-1)A_{h-2} + \cdots \\ &\quad + (p+h-1)_{(m-1)} A_{h-m+1}\big]\,, \\ \Lambda &= \sup_h \frac{2h(2h+1)}{4(h+1)(p+h+1)}\,. \end{aligned}$$

We have

$$U_h \ll A_h \Big/ \Big(1-\frac{\sigma}{r}\Big)^{2h}\,.$$

Since the power series

$$\sum A_h k^{p+h} / \Gamma(p+h+1)$$

converges for $|k|$ small enough, we know that the series (23) converges for $|k|$ small enough. Summing up we have the main result of Part I.

THEOREM 3. *Let*

$$\begin{aligned} P(x, \partial_x) &= \sum_{|\alpha| \le m} a_\alpha(x) \Big(\frac{\partial}{\partial x}\Big)^\alpha \\ &= P_m(x, \partial_x) + P_{m-1}(x, \partial_x) + \cdots \\ &= \sum_{|\alpha| = m} a_\alpha(x) \Big(\frac{\partial}{\partial x}\Big)^\alpha + \sum_{|\alpha| = m-1} a_\alpha(x) \Big(\frac{\partial}{\partial x}\Big)^\alpha + \cdots, \ m \ge 2 \end{aligned}$$

be a linear holomorphic partial differential operator in a domain X of a complex analytic manifold, $y \in X$ a fixed point, and

$$\mathrm{Hess}_\eta P_m(y, \eta) \ne 0 \textit{ for } P_m(y, \eta) = 0, \ \eta \ne 0 .$$

Then, there exists an Hadamard fundamental solution with an asymptotic expansion

$$u(x, y) = \sum_{h=0}^{\infty} U_h k^{p+h} / \Gamma(p+h+1) ,$$

where $p = (1-m)\left(\frac{n}{2} - 1\right)$. $k(x, y) = 0$ is the equation of characteristic conoid, with k holomorphic for x close enough to y satisfying

$$P_m(x, k_x) = \frac{k}{m-1} .$$

When p is a negative integer, the expansion should be modified as (33).

2.6. Propagation of Singularities

The method above can also be applied to various problems for linear holomorphic partial differential operator, among them, propagation of singularities. J. Leray, Y. Hadamard and C. Wagschal considered in a series of papers [6], [7], [8] the Cauchy problem for

$$P(x, \partial_x) u = v(x) , \tag{36}$$

where

$$v(x) = f(x) / [g(x)]^\lambda , \tag{37}$$

$g(x) = 0$ is a holomorphic hypersurface in X which is not characteristic for $P(x, \partial_x)$. Their main results go roughly as follows: the singularities of the solution $u(x)$ propagate along characteristics issuing from the singularities for the right hand side, i.e., points of the hypersurface $g(x) = 0$. But when $g(x) = 0$ is characteristic, the behavior of the solution is quite different. Actually, we have

THEOREM 4. *If the right hand side of (36) is of the form*

$$P(x, \partial_x)u = f(x)/[k(x, y)]^{\lambda}, \tag{38}$$

where $1 - \lambda \neq 0, -1, -2, \ldots$ *and* $k(x, y) = 0$ *is the characteristic conoid with vertex at* y*. Then there exists a solution of (38) of the form*

$$u(x, y) = \sum_{h=0}^{\infty} U_h(x, y)[k(x, y)]^{p+h}/\Gamma(p + h + 1). \tag{39}$$

where

$$p = m - \lambda - 1. \tag{40}$$

Proof. Using the procedure as above, we obtain the transport equation for U_0 as

$$\tau \frac{dU_0}{d\tau} + \left[\frac{n}{2}(m - 1) - \lambda + \tau A(\tau)\right] U_0 = f(x)\Gamma(1 - \lambda).$$

Denote

$$\mu = \frac{n}{2}(m - 1) - \lambda,$$

We have

$$\frac{d}{d\tau}(\tau^{\mu} U_0) + A(\tau)\tau^{\mu} U_0 = \tau^{\mu - 1} f(x)\Gamma(1 - \lambda),$$

and

$$U_0 = \tau^{-\mu} \int_0^{\tau} \tau^{\mu-1} f(x)\Gamma(1 - \lambda) e^{-\int_0^{\tau} A(s)\, ds}\, d\tau. \tag{41}$$

When $\mu \leq 0$, the integral should be taken in the distributional sense. U_0 is bounded. For the remaining U_h, $h > 0$, we have similar transport equations,

$$\tau \frac{dU_h}{d\tau} + \left[\frac{n}{2}(m - 1) - \lambda - h + \tau A(\tau)\right] U_h = L(U_{h-1}, \ldots, U_{h-m+1}),$$

the unique bounded solution of which is

$$U_h = \frac{U_0}{\tau^h} \int_0^{\tau} \frac{\tau^{h-1}}{U_0} L(U_{h-1}, \ldots, U_{h-m+1})\, d\tau. \tag{42}$$

After (39) is constructed as a formal solution, convergence can be proved as before.

□

REMARK 1. When $1 - \lambda = 0, -1, -2, \ldots$, further modification for (refc1.34) is needed by adding terms containing $\log k$. The details are omitted.

REMARK 2. The implication of this theorem is as follows. If another hyperplane not through y is taken to be the initial hyperplane, its intersection with $k(x, y) = 0$ are new source of singularties. According to results already known on the behavoir of solutions, singularities would propagate along all bicharacteristics issuing from the intersection. But theorem 4 states that they propagate only along $k(x, y) = 0$. This is analogous to the Huygens' principle for the wave equation. According to Huygens' construction, superficially, there would not only be a wave front going forward, but also another wave front going backward. This apparent contradiction was explained by Fresenel in terms of interference [9]. Theorem 4 claims similar results for general partial differential operators.

REMARK 3. Some new problems are also motivated by the procedure above. For instance, if we introduce new variables $(z_1, \dots, z_n)$ with $z_1 = k(x, y)$. Operator $P(x, \partial_x)$ will have the form

$$z_1\Big(\frac{\partial}{\partial z_1}\Big)^m + Q(z, \partial_z) \tag{43}$$

where $Q(z, \partial_z)$ contains at most $\big(\frac{\partial}{\partial z_1}\big)^{m-1}$. (43) is of Fuchsian type. In this sense Hadamard's theory is also a preliminary Fuchsian theory.
But the transformation $x \to z$ is singular at $x = y$. For the simplest case of $m = 2$, take $y = 0$, $k(x, y) = k(x)$ is holomorphic at $x = 0$. Using Morse lemma, which is also valid in complex holomorphic case (for a proof see e.g. [10]), we will arrive at a holomorphic partial differential operator with $\frac{\partial}{\partial z_1}$ appearing in the principal part in the following form:

$$\Big(z_1\frac{\partial}{\partial z_1} + \sum_{j=2}^{n} z_j\frac{\partial}{\partial z_j}\Big)\frac{\partial}{\partial z_1}\,.$$

There is a critical point in its Hamiltonian field. A preliminary study is given in [11]. For more general case, we must study corresponding uniformization problem.

3. Part II. Multiple-characteristic Problems

3.1. Motivation

In this part, we are to modify the Hadamard's procedure that it would be applicable to multiple-characteristic problems. We are to consider only the simplest case when the principal symbol $P_m(x, \xi)$ is factorizable, more precisely, when

$$P_m(x, \xi) = [a(x, \xi)]^2\,,$$

and to see where lie the main difficulties. For simplicity, we assume $a(x, \xi)$ to be a homogeneous polynoimal in ξ of order m, such that the original operator $P(x, \partial_x)$ is of order $2m$, thus the principal symbol of $P(x, \partial_x)$ is

$$P_{2m}(x, \xi) = [a(x, \xi)]^2\,. \tag{44}$$

Here, as in Part I, we also assume

$$\mathrm{Hess}_\eta a(y, \eta) \neq 0,\ \text{ when } a(y, \eta) = 0,\ \eta \neq 0\,. \tag{45}$$

We are to use the Hamiltonian system for $a(x, \xi)$ instead of that for $P_{2m}(x, \xi)$, namely, we are to consider

$$\begin{aligned} &\frac{dx}{dt} = a_\xi(x, \xi),\ \ x|_{t=0} = y,\\ &\frac{d\xi}{dt} = -a_x(x, \xi),\ \ \xi|_{t=0} = \eta,\\ &a(y, \eta) = 0. \end{aligned} \tag{46}$$

Let $k(x, y) = 0$ be the characteristic conoid with respect to $a(x, \xi)$, hence

$$a(x, k_x) = \frac{1}{m-1}k\,. \tag{47}$$

Then

$$P_{2m}(x, k_x) = \frac{1}{(m-1)^2} k^2 .$$

If we take $z_1 = k(x, y)$ as a new independent variable, the equation $P(x, \partial_x)u = 0$ becomes

$$z_1^2 \frac{\partial^m u}{\partial z_1^m} + A\Big(x, \frac{\partial}{\partial x}\Big) \frac{\partial^{m-1} u}{\partial z_1^{m-1}} + \cdots = 0 ,$$

or

$$\frac{\partial^m u}{\partial z_1^m} + \frac{1}{z_1^2} A\Big(x, \frac{\partial}{\partial x}\Big) \frac{\partial^{m-1} u}{\partial z_1^{m-1}} + \cdots = 0 .$$

This equation looks like an ordinary differential equation with irregular singularity at $z_1 = 0$. For such equations as

$$\frac{dw}{dt} = t^{-(2+r)} A(t) w, \ \ r = 0, 1, 2, \dots ,$$

with irregular singularity at $t = 0$, the solution can be of the form

$$w = \exp\Big[\sum_{i=1}^{r+1} (a_i t^i)\Big] t^p f(t)$$

(see e.g. [12], Chap. 5, Theorem 2.1). Thus we are motivated to find a solution of

$$P(x, \partial_x)u = 0 \tag{48}$$

in the following form:

$$u = \exp[A(x, y)/k(x, y)] \left(\sum_{h=0}^{\infty} U_h k^{p+h}\right) . \tag{49}$$

where $A(y, y) \neq 0$. In constrast to Part I, now we should need eiconal equations for both A and k. The latter is (47) while that for A is to be sought. Thus, we should have "eiconal system". Transport equations for U_h are also to be constructed.

For brevity in the sequel, we always use sub-indices to denote differentiation with respect to fibre variables such as ξ, η, etc., if applicable.

3.2. Notations

In the following we will use repeatly Euler's formula for homogeneous functions $P(x, \xi)$ of order l:

$$\begin{aligned} P(x, \xi) &= \frac{1}{l} \sum_{i_1=1}^{n} \partial_{i_1} P(x, \xi) \xi_{i_1} \\ &= \frac{1}{l(l-1)} \sum_{i_1, i_2=1}^{n} \partial_{i_1} \partial_{i_2} P(x, \xi) \xi_{i_1} \xi_{i_2} = \dots \\ &= \frac{1}{(l)_r} \sum_{i_1, i_2, \dots, i_r=1}^{n} \partial_{i_1} \partial_{i_2} \dots \partial_{i_r} P(x, \xi) \xi_{i_1} \xi_{i_2} \dots \xi_{i_r} = \dots \\ &= \frac{1}{l!} \sum_{i_1, \dots, i_l=1}^{n} \partial_{i_1} \dots \partial_{i_l} P(x, \xi) \xi_{i_1} \dots \xi_{i_l} . \end{aligned} \tag{50}$$

In these formulas, $i_1, \ldots, i_l$ run dependently from 1 to n. This form of $P(x,\xi)$ is more convenient for our purpose than the usual expression

$$P(x,\xi) = \sum_{|\alpha|=l} \frac{1}{\alpha!} \partial_\xi^\alpha P(x,\xi)\xi^\alpha . \tag{51}$$

Hence, we introduce a new notation

$$D^\beta = \partial_{i_1} \ldots \partial_{i_l} \sim (\partial_{i_1}, \ldots, \partial_{i_l}) \sim (i_1, \ldots, i_l) , \tag{52}$$

here $\beta = (i_1, i_2, \ldots, i_l)$ is an ordered set of l elements from the set $\{1, 2, \ldots, n\}$. Elements of β may coincide among themselves. If ∂^α can be written as a certain D^β, we say D^β is a permutation of ∂^α, denoted as $\beta \in \sigma(\alpha)$, also $|\beta|$ is defined to be $|\alpha|$, actually $|\beta| = \operatorname{card}\beta$. Corresponding to the addition of multi-index α, such as $\alpha = \alpha' + \alpha''$, hence $\partial^\alpha = \partial^{\alpha'}\partial^{\alpha''}$, we have the partition of β into two subsets $\beta = \beta' \cup \beta''$, $\beta' \in \sigma(\alpha')$, $\beta'' \in \sigma(\alpha'')$ and $D^\beta = D^{\beta'}D^{\beta''}$. So we also denote $\beta = \beta' + \beta''$. Similarly, we define $\beta' = \beta \setminus \beta''$ to be deleting from β the subset β'' while keep the original order of elements in β', and also write $\beta' = \beta - \beta''$. Lastly, for multi-index $\alpha = 0$ and $|\alpha| = 0$ means $\alpha = (0, \ldots, 0)$, and $\partial^\alpha = \partial^0 = \mathrm{id}$. For β, $|\beta| = 0$ means $\beta = \emptyset$ and also $D^\emptyset = \mathrm{id}$, hence we also use $\beta = 0$ to denote $\beta = \emptyset$.

DEFINITION 1. *When*

$$\beta = \beta^1 + \cdots + \beta^L = \beta^1 \cup \cdots \cup \beta^L , \tag{53}$$

$\beta^i \neq 0$ *and elements in* β^i *keep their original order in* β, *we say that* $(\beta^1, \ldots, \beta^L)$ *is a partition of* β, *L the partition number. Denote* $\lambda_l = \operatorname{card}\{\beta^i, |\beta^i| = l\}$, *i.e., the number of* l*-element-subsets in (53).* $[\lambda] = (\lambda_1, \ldots, \lambda_{|\beta|})$ *is called the partition type of (53).*

We have an evident but important

LEMMA 2. *For the numbers of subsets of various cardinality, we have*

$$1 \cdot \lambda_1 + 2\lambda_2 + \cdots + |\beta|\lambda_{|\beta|} = |\beta| \tag{54}$$

From (54) we see, $\lambda_1 = |\beta|$ is possible ($\lambda_2 = \cdots = \lambda_{|\beta|} = 0$), it means β is partitioned into the sum of $|\beta|$ singleton subsets. But $\lambda_1 = |\beta| - 1$ is impossible. In fact, the partition type of β is just a non-negative- integer solution of (54) as a Diophantine equation.

LEMMA 3. *The number of partitions (53) (the order of the subsets* β^L *is irrelevant) with the same partition type* $[\lambda]$ *is* $m!/L!(1!)^{\lambda_1} \cdots (m!)^{\lambda_m}$, $|m| = |\beta|$.

Proof. First, construct λ_1 singleton-subsets. There are

$$\binom{m}{1}\binom{m-1}{1}\cdots\binom{m-\lambda_1+1}{1} = \frac{(m)_{\lambda_1}}{1!^{\lambda_1}}$$

different ways. Next for the first doubleton-subsets, we take 2 arbitrary elements from the remaining $m - \lambda_1$ elements in $(m-\lambda_1)_2/2!$ ways. (The chosen elements should keep their original order in β, thus there shouldn't be $(m-\lambda_1)_2$ ways, which is the permutation number and hence contains twice the same set with elements in opposite orders). The second doubleton-subsets can be chosen in $(m-\lambda_1-2)_2/2!$ ways and so on. Summing up, the doubleton-subsets can be chosen in $(m-\lambda_1)_{2\lambda_2}/(2!)^{\lambda_2}$ ways. After the $(m-1)$- element subsets are chosen, there remain $m-[1\cdot\lambda_1+2\cdot\lambda_2+\cdots+(m-1)\cdot\lambda_{m-1}]=m\lambda_m$ elements for m-element subsets, the number of which is λ_m. So they can be chosen in $(m\lambda_m)_{\lambda_m}/(m!)^{\lambda_m}$ ways. Thus we have

$$\frac{(m)_{\lambda_1}(m-\lambda_1)_{2\lambda_2}\cdots(m-[1\cdot\lambda_1+2\lambda_2+\cdots+(m-1)\lambda_{m-1}])_{m_{\lambda_m}}}{(1!)^{\lambda_1}(2!)^{\lambda_2}\cdots(m!)^{\lambda_m}}$$

ways to generate a partition of type $[\lambda]$. But the permutation of L subsets in the partition is irrelevant, hence the number of partitions of β with the same partition type $[\lambda]$ is

$$\frac{m!}{L!(1!)^{\lambda_1}\cdots(m!)^{\lambda_m}}\,.$$

□

3.3. Lemmas on differentiation of composite functions

The computation below relies on differentiation of composite functions. We are to use a modification of the Faa-de Bruno formula [16] (p. 78, Exer. 16). To facilitate the computation, we put the techniques needed as three lemmas.

LEMMA 4. *Let $P(x,\xi)$ be a homogeneous polynomial in ξ of degree l, then*

$$P(x,\xi)=\sum_{|\alpha|=l}\frac{1}{\alpha!}\partial_\xi^\alpha P(x,\xi)\partial_x^\alpha = \frac{1}{l!}\sum_{1\le i\le n}D_\xi^\beta P(x,\xi)D_x^\beta \tag{55}$$

Particularly, for $P(x,\xi)=\partial_x^\alpha$, we have

$$\partial_x^\alpha=\frac{1}{l!}\sum_{\beta\in\sigma(\alpha)}D_x^\beta\,. \tag{56}$$

For D_x^α, $\beta\in\sigma(\alpha)$, we have

$$D_x^\beta=\partial_x^\alpha \tag{57}$$

This lemma is quite trival. (57) shows that D_x^β and ∂_x^α are the same when acting on a function; (56) shows that ∂_x^α is the average of D_x^β's with $\beta\in\sigma(\alpha)$; while (55) makes the Euler's formula for homogeneous function more symmetric.

Assume $\Phi(k)$ to be a smooth function of $k(x)$. By chain-rule, $\partial_x^\alpha\Phi(k)=\{$ linear combination of $\Phi^\gamma(k)\partial_{i_1}k\cdots\partial^2_{j_1j_2}k\cdots\}$.

But to write down explicitly the coefficients is much easier for D_x^β than for ∂_x^α. In fact we have

LEMMA 5. *Let $\Pi(\beta)$ be the set of partitions of β, then*

$$D_x^\beta\Phi(k)=\sum_{\Pi(\beta)}\Phi^{(L)}(k)D^{\beta^1}k\cdots D^{\beta^L}k\,, \tag{58}$$

where $\beta=\beta^1+\cdots+\beta^L$.

Proof. We proceed by induction. For $|\beta| = 1$, e.g., $D_x^\beta = D_x$, (58) becomes

$$D_{x_1}\Phi(k) = \Phi^{(l)}(k)D_{x_1}k$$

which holds evidently. Suppose that the lemma holds for $|\beta| = m - 1$. For $|\beta| = m$, we may assume that $D_x^\beta = D_1 D_x^{\beta'}$, $|\beta'| = m-1$. Then the partitions of β may be one of two kinds. The first is $\beta = \beta^1 \bigcup(\beta^2 \bigcup \cdots \bigcup \beta^L)$, $\beta^1 = \{D_1\}$ and $\beta^2 \bigcup \cdots \bigcup \beta^L = \beta'$ is a partition of β' with partition number $L - 1$. The second kind are those not containing the singleton-subset $\{D_1\}$, hence D_1 is contained in another subset, say D^{β^1} : $D^{\beta^1} = D_1 \cdot D_x^{\beta''}$, $|\beta''| = |\beta'| - 1$. Thus $D_x^\beta = D_1(D_x^{\beta''} D_x^{\beta^2} \cdots D_x^{\beta^L})$ and the factor $D_x^{\beta''} D_x^{\beta^2} \cdots D_x^{\beta^L}$ constitutes a partition of $D_x^{\beta'}$ of partition number also L. $\Pi_1(\beta') = \{\beta^2 \bigcup \cdots \bigcup \beta^L\}$ and $\Pi_2(\beta') = \{\beta'' \bigcup \beta^2 \bigcup \cdots \bigcup \beta^L\}$ exhaust all possible partitions of β'. Hence, by induction hypothesis,

$$\begin{aligned}
D_x^\beta \Phi(k) &= D_1 \cdot [D_x^{\beta'}\Phi(k)] \\
&= D_1 \sum_{\Pi(\beta')} \Phi^{(L')} D^{\beta'} k \cdots D^{\beta^{L'}} k \\
&= \sum_{\Pi_1(\beta')} \Phi^{(L'+1)}(k) D_1 k D^{\beta^2} k \cdots D^{\beta^L} k \\
&\quad + \sum_{\Pi_2(\beta')} \Phi^{(L')}(k)(D_1 \cdot D^{\beta'})k \cdots D^{\beta^L} k \\
&= \sum_{\Pi_1(\beta')} \Phi^{(L)}(k) D_1 k D^{\beta^2} k \cdots D^{\beta^L} k \\
&\quad + \sum_{\Pi_2(\beta')} \Phi^{(L)}(k)(D_1 \cdot D^{\beta'} k) \cdots D^{\beta^L} k \\
&= \sum_{\Pi(\beta)} \Phi^{(L)}(k) D^{\beta^1} k D^{\beta^2} k \cdots D^{\beta^L} k\,.
\end{aligned}$$

□

The generalized Leibniz's formula for D also takes a simpler form, namely, we have

LEMMA 6 (LEIBNIZ'S FORMULA).

(59) $$D^\beta uv = \sum_{\beta = \beta^1 + \beta^2} D^{\beta^1} u \cdot D^{\beta^2} v\,.$$

The binomial coefficients disappear hence make the calculation easier. The proof is omitted for its simplicity.

3.4. Computation of $P_{2m}(x, \xi)E_p$ (General plan)

In order to compute $P(x, \partial_x)u$ where u is expressed as (49), we proceed term by term and start from $P(x, \partial_x)E_p U_0$, where $E_p = \exp[A/k] \cdot k^p$. Since $P = P_{2m} + P_{2m-1} + \cdots + P_0$ where $P_{2m-j}(x, \xi)$ is a homogeneous polynomial in ξ of degree $2m - j$, we first compute

$P_{2m}(x, \partial_x)E_pU_0$ which by Leibniz's formula can be written as

$$P_{2m}(x, \partial_x)E_pU_0 = U_0P_{2m}(x, \partial_x)E_p + \sum_{i=1}^{n} \partial_{\xi_i} P_{2m}(x, \partial_x)E_p \cdot \frac{\partial U_0}{\partial x_i} + \cdots \tag{60}$$

Thus our plan is, first, compute $P_{2m}(x, \partial_x)E_p$, then lower order terms $P_{2m-j}(x, \partial_x)E_p$ and further $\sum_{i=1}^{n} \partial_{\xi_i} P_{2m}(x, \partial_x)E_p \cdot \frac{\partial U_0}{\partial x_i}$ etc. Every term should be expressed as a series in powers of k. It is to be verified, that the lowest order terms are of the order k^{2-4m} and k^{3-4m} for determing $A(x, y)$, p and U_0. We also denote $P_{2m}(x, \xi)$ by $g(x, \xi)$ and note that the results are applicable also to $P_{2m-j}(x, \xi)$ when replacing g by a lower degree polynoimal.

Denote $E_p = \Phi(k_p)$, where $\Phi(\cdot) = \exp(\cdot)$, $k_p = Ak^{-1} + p \log k$. From Lemma 4,

$$g(x, \partial_x)E_p = \frac{1}{(2m)!} \sum_{1\le i\le n} D_\xi^\beta P(x, \xi) \cdot D_x^\beta E_p\,,\ |\beta| = 2m\,,$$

$\beta = (i_1, i_2, \ldots, i_{2m})$ and the i's take arbitrary value from $(1, 2, \ldots, n)$. Using Lemma 5, we have

$$D_x^\beta E_p = \sum_{\Pi(\beta)} \Phi^{(L)}(k_p) D^{\beta^1} k_p \cdots D^{\beta^L} k_p\,,$$

and $\Phi^{(L)}(k_p) = \Phi(k_p)$ since $\Phi(\cdot) = \exp(\cdot)$, hence

$$D_x^\beta E_p = E_p \sum_{\Pi(\beta)} D^{\beta^1} k_p \cdots D^{\beta^L} k_p\,. \tag{61}$$

For $D^{\beta^i} k_p$, for instance, for $D^{\beta^1} k_p$

$$D^{\beta^1} k_p = D^{\beta^1}(Ak^{-1}) + pD^{\beta^1} \log k\,.$$

Apply Leibniz's formula for the first term, we have

$$D^{\beta^1}(Ak^{-1}) = \sum_{\beta^1=\eta^1+\gamma^1} D^{\eta^1} A D^{\gamma^1} k^{-1}\,.$$

Apply Lemma 5 to $D^{\gamma^1} k^{-1}$. γ^1 is partitioned as $\gamma^1 = \delta^{11} + \cdots + \delta^{1l_1}$, then

$$\begin{aligned} D^{\gamma^1} k^{-1} &= \sum_{\Pi(\gamma')} \Phi^{(l_1)}(k) D^{\delta^{11}} k \cdots D^{\delta^{1l_1}} k,\ \Phi(\cdot) = (\cdot)^{-1} \\ &= \sum_{\Pi(\gamma')} (-1)^{l_1} l_1! k^{-(l_1+1)} D^{\delta^{11}} k \cdots D^{\delta^{1l_1}} k\,. \end{aligned}$$

Hence

$$D^{\beta^1}(Ak^{-1}) = \sum_{\beta^1=\eta^1+\gamma^1} \sum_{\Pi(\gamma^1)} D^{\eta^1} A (-1)^{l_1} (l_1!) k^{-(l_1+1)} D^{\delta^{11}} k \cdots D^{\delta^{1l_1}} k\,. \tag{62}$$

Similarly, apply lemma 5 to compute $D^{\beta^1}(\log k)$, we have

$$D^{\beta^1}(\log k) = \sum_{\Pi(\beta^1)} (-1)^{l_1-1}(l_1 - 1)! k^{-l_1} D^{\delta^{11}} k \cdots D^{\delta^{1l_1}} k\,. \tag{63}$$

In order to combine (62) and (63), consider the part of (62) where $\eta^1 = 0$, *i. e.*, $\beta^1 = \gamma^1$, and they can be combined into

$$A \sum_{\Pi(\beta^1)} (-1)^{l_1} l_1! k^{-(l_1+1)} D^{\delta^{11}} k \cdots D^{\delta^{1l_1}} k \,,$$

and the sum of (62) and (63) gives

$$D^{\beta^1}(k_p) = \sum_{\beta^1=\eta^1+\gamma^1} \sum_{\Pi(\gamma^1)} (D^{\eta^1} A)^* (-1)^{l_1} (l_1!) k^{-(l_1+1)} D^{\delta^{11}} k \cdots D^{\delta^{1l_1}} k \,, \tag{64}$$

$(D^{\eta^1} A)^*$ denotes $D^{\eta^1} A$ when $\eta^1 \neq 0$ and $(D^0 A)^* = (A - pk/l_1)$.

Substuting these results into (61), and denote $l = (l_1, \ldots, l_L)$, $|l| = l_1 + \cdots + l_L$, $l! = l_1! \cdots l_L!$, l_i is the partition number of $\eta^i = (\delta^{i_1}, \ldots, \delta^{il_i})$. Re-numbering $D^{\delta^{11}} k, \ldots, D^{\delta^{il_i}} k$, $\ldots$ as $D^{\delta^1} k, \ldots, D^{\delta^N} k$, N is the total partition number. Then by lemma 4,

$$\begin{aligned} g(x, \partial_x) E_p = \frac{E_p}{(2m)!} \sum_{\beta} D_\xi^\beta g(x, \xi) \cdot \sum_{\Pi(\beta)} (D^{\eta^1} A)^* \cdots (D^{\eta^L} A)^* \cdot \\ \cdot (-1)^{|l|} l! k^{-(N+L)} \cdot D^{\delta^1} k \cdots D^{\delta^N} k \,, \ |\beta| = 2m \,. \end{aligned} \tag{65}$$

$\Pi(\beta)$ under the second $\sum$ denotes β twice partitioned $\beta = \beta^1 + \cdots + \beta^L = \sum_{i=1}^L (\eta^i + \sum_{j=1}^{l_i} \delta^{ij})$. If we write $D^\beta = D_{i_1} D_{i_2} \cdots D_{i_{2m}}$, (65) can be further simplified:

$$\begin{aligned} g(x, \partial_x) E_p = \frac{E_p}{(2m)!} \sum_{1 \le i \le n} D_\xi^\beta g(x, \xi) (-1)^{|l|} l! k^{-(N+L)} \cdot (D^{\eta^1} A)^* \cdots \\ \cdot (D^{\eta^L} A)^* \cdot D^{\delta^1} k \cdots D^{\delta^N} k \,. \end{aligned} \tag{66}$$

To organize further computation, we proceed in the descending order of the 2 partition numbers L and N.

In the first partition $\beta = \beta^1 + \cdots + \beta^L$, all $|\beta^i| > 0$ hence

$$L \le |\beta^1| + \cdots + |\beta^L| = |\beta| = 2m \,. \tag{67}$$

In the second partition $\beta^i = \eta^i + \gamma^i$, either $\gamma^i = 0$ and $\beta^i = \eta^i$, either $\gamma^i \neq 0$, $\gamma^i = \delta^{i1} + \cdots + \delta^{il_i}$, all $\delta^{ij} \neq 0$, thus $|\delta^i| > 0$ and

$$N \le |\delta^1| + \cdots + |\delta^N| + |\eta^1| + \cdots + |\eta^L| = \sum_{i=1}^{L} |\beta^i| = 2m \,. \tag{68}$$

We should also take into consideration that

$$g(x, \xi) = P_{2m}(x, \xi) = [a(x, \xi)]^2 \,,$$

hence

$$\begin{aligned} D_i g(x, \xi) &= 2a(x, \xi) a_i(x, \xi) \,, \\ D_i D_j g(x, \xi) &= 2a(x, \xi) a_{ij}(x, \xi) + 2a_i(x, \xi) a_j(x, \xi) \,, \\ D_i D_j D_l g(x, \xi) &= 2a(x, \xi) a_{ijl}(x, \xi) \\ &\quad + 2\left[a_i(x, \xi) a_{jl}(x, \xi) + a_j(x, \xi) a_{il}(x, \xi) + a_l(x, \xi) a_{ij}(x, \xi)\right] . \end{aligned} \tag{69}$$

Since $a(x, k_x) = \frac{k}{m-1}$, we have

$$
\begin{aligned}
g(x, k_x) &= \frac{k^2}{(m-1)^2}\,, \\
D_i g(x, k_x) &= \frac{2k}{m-1} a_i(x, k_i)\,, \\
D_i D_j g(x, k_x) &= \frac{2k}{m-1} a_{ij}(x, k_i) + 2a_i(x, k_x) a_j(x, k_x)\,, \\
D_i D_j D_l g(x, k_x) &= \frac{2k}{m-1} a_{ijl}(x, k_i) + 2\sum a_i(x, k_x) a_{jl}(x, k_x)\,.
\end{aligned}
\tag{70}
$$

Thus in the *coefficients* of (66), there may appear powers of k up to k^2.

3.5. Computation of $P_{2m}(x, \partial_x) E_p$ (cont')

I. First consider those terms in (66) where $L = 2m$. By (67) all $|\beta^i| = 1$, and since $\beta^i = \eta^i + \gamma^i$, then either $|\gamma^i| = 1$ or $|\gamma^i| = 0$. Set $\{1, 2, \dots, 2m\} = I \bigcup J$, $I \bigcap J = \emptyset$, such that for $i \in I$, $|\gamma^i| = 1, |\eta^i| = 0$ hence $|I| = N$; for $i \in J$ $|\gamma^i| = 0, |\eta^i| = 1$. In short, terms in (66) corresponding to $L = 2m$ are

$$
\begin{aligned}
&\frac{E_p}{(2m)!} \sum_{|\beta|=2m} D_\xi^\beta g(x, \xi) \sum_{I,J} (-1)^{|l|} l! k^{-(N+2m)} D^{\delta^1} k \cdots \\
&\quad \cdot D^{\delta^N} k (D^{\eta^1} A)^* \cdots (D^{\eta^L} A)^*\,.
\end{aligned}
$$

(I-A). $N = 2m$. From (68), this is the extreme case when all $\eta^i = 0$, $|\gamma^i| = 1$, thus $\gamma^i = \delta^i$, and the partition number $l_i = 1$, $|l| = 2m$, $l! = 1$, $D^{\delta^1} k \cdots D^{\delta^N} k = k_{i_1} \cdots k_{i_{2m}}$. By Euler's formula, these terms can be summed up to give

$$
\begin{aligned}
E_p g(x, k_x)(A - pk)^{2m} k^{-4m} &= \frac{E_p}{(m-1)^2}(A - pk)^{2m} k^{2-4m} \\
&= E_p \Big[\frac{A^{2m}}{(m-1)^2} k^{2-4m} - \frac{2mpA^{2m-1}}{(m-1)^2} k^{3-4m} \\
&\qquad + O(k^{4-4m}) \Big]\,.
\end{aligned}
\tag{71}
$$

(I-B). $N = 2m - 1$. Delete any element from $\{i_1, i_2, \dots, i_{2m}\}$ and let the remaining correspond to $\delta^1, \dots, \delta^N$. There are $2m$ such chooses and the following is one typical, thus we should multiply the results by $2m$. Let $\{\delta^1, \dots, \delta^N\} = \{i_2, \dots, i_{2m}\}$, then $l_2 = \cdots = l_{2m} = 1$, $l! = 1$, $|l| = 2m - 1$; also $(D^{\eta^1} A)^* = \partial_{i_1} A$, $(D^{\eta^2} A)^* = \cdots = (D^{\eta^L} A)^* = A - pk$. By Euler's formula,

$$
\begin{aligned}
&\frac{1}{(2m)!} \sum_{|\beta|=2m} D_\xi^\beta g(x, \xi) D^{\delta^1} k \cdots D^{\delta^N} k (D^{\eta^1} A)^* (-1)^{2m-1} k^{-(4m-1)}\,. \\
&\cdot (A - pk)^{2m-1} = \frac{1}{2m} \sum_{i=1}^n \partial_i g(x, k_x) \partial_i A (-1)^{2m-1} k^{-(4m-1)} (A - pk)^{2m-1} \\
&\qquad = -\frac{2k}{2m(m-1)} \sum_{i=1}^n a_i(x, k_x) \partial_i A k^{-(4m-1)} (A - pk)^{2m-1}.
\end{aligned}
$$

Since $k_x = \tau\xi$ and from (46) $a_i(x, \xi) = \frac{dx_i}{dt}$, the final results for terms corresponding to $L = 2m$, $N = 2m - 1$ is

$$E_p\left[-\frac{2A^{2m-1}}{m-1}t\frac{dA}{dt}k^{2-4m} + \frac{2A^{2m-2}}{m-1}t\frac{dA}{dt}(2m-1)pk^{3-4m} + O(k^{4-4m})\right]. \tag{72}$$

(I-C). $N = 2m - 2$. Delete for instance $\{i_1, i_2\}$ from $\{i_1, i_2, \ldots, i_{2m}\}$ and let the remaining be $\delta^1, \ldots, \partial^N$ ($\frac{1}{2}(2m)_2$ such chooses), then $l_3 = \cdots = l_{2m} = 1$, $|l| = 2m - 2$, $l! = 1$, $(D^{\eta_1}A)^* = \partial_{i_1}A$, $(D^{\eta_2}A)^* = \partial_{i_2}A$, all other $(D^{\eta}A)^* = A - pk$. Use also Euler's formula and multiply the results by $\frac{1}{2}(2m)_2$, the final result is

$$\begin{aligned}
&\frac{1}{2}E_p\sum_{i_1,i_2=1}^{n} g_{i_1i_2}(x, k_x)\partial_{i_1}A\partial_{i_2}Ak^{2-4m}(A - pk)^{2m-2}\\
&\quad= E_p\Big[\sum_{i_1,i_2=1}^{n}\frac{1}{m-1}a_{i_1,i_2}(x, k_x)\partial_{i_1}A\partial_{i_2}A\cdot A^{2m-2}k^{3-4m}\\
&\qquad+\Big(t\frac{dA}{dt}\Big)^2A^{2m-2}k^{2-4m} - p\Big(t\frac{dA}{dt}\Big)^2(2m-2)A^{2m-3}k^{3-4m}\\
&\qquad+O(k^{4-4m})\Big].
\end{aligned} \tag{73}$$

(I-D). $N = 2m - 3$. A typical example is deleting $\{i_1, i_2, i_3\}$ from $\{i_1, i_2, \ldots, i_{2m}\}$ and let $\{\delta^1, \ldots, \delta^N\} = \{i_4, i_5, \ldots, i_{2m}\}$. A factor $\frac{(2m)_3}{3!}$ should be added to the result and the final result obtained would be

$$\begin{aligned}
&\frac{(2m)_3}{3!}E_p\Big\{\frac{1}{(2m)_3}2\sum_{i_1,i_2,i_3}[a_{i_1}(x, k_x)a_{i_2i_3}(x, k_x) + a_{i_2}(x, k_x)a_{i_2i_3}(x, k_x)\\
&\qquad+a_{i_3}(x, k_x)a_{i_1i_2}(x, k_x)]\,\partial_{i_1}A\,\partial_{i_2}A\,\partial_{i_3}A(-1)^{2m-3}A^{2m-3}k^{3-4m}\\
&\qquad+O(k^{4-4m})\Big\}\\
&\quad= -E_p\Big[\sum_{i_1,i_2}a_{i_1,i_2}(x, k_x)\,\partial_{i_1}A\,\partial_{i_2}A\cdot t\frac{dA}{dt}A^{2m-3}k^{3-4m}\\
&\qquad+O(k^{4-4m})\Big],
\end{aligned} \tag{74}$$

Our computation would stop here. Since for $N \leq 2m - 4$, $k^{-(N+L)} = O(k^{4-4m})$ and hence omitted.

Summing up, when $L = 2m$, we need to consider the cases $N = 2m$, $2m - 1$, $2m - 2$ and $2m - 3$ to obtain (71), (72), (73) and (74), giving an asymptotic expansion of $P_{2m}(x, \partial_x)E_p$ up to $O(k^{4-4m})$ as

$$
\begin{aligned}
&E_p k^{2-4m} A^{2m-2}\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)^2 \\
&\quad - E_p k^{3-4m} A^{2m-3}\Big[(2m-2)p\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)^2 \\
&\quad - \frac{2p}{m-1}\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)A \\
&\quad + \Big(\sum_{i_1,i_2} a_{i_1,i_2}(x,k_x)\,\partial_{i_1} A \partial_{i_2} A\Big)\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)\Big] \\
&\quad + E_p O(k^{4-4m})\,.
\end{aligned} \tag{75}
$$

II. $L = 2m-1$. From (67),

$$|\beta^1| + \cdots + |\beta^{2m-1}| = 2m\,,$$

The only solution is: one of β^i, say $|\beta^{2m-1}| = 2$ and other $|\beta^i| = 1$. A typical choice for the β^i's is $\beta^j = i_j$, $j = 1,\ldots,2m-2$, $\beta^{2m-1} = (i_{2m-1}, i_{2m})$. There are $\frac{1}{2}(2m)_2$ similar choices, so we must add a factor $\frac{1}{2}(2m)_2$ to the results of all particular case.

(II-A). $N = 2m$. This case can be attained only when for $\beta^i = \eta^i + \gamma^i$, set $\eta^i = 0$, and $\gamma^i = \delta^i, \ldots, \gamma^{2m-2} = \delta^{2m-2}$. Hence $l = (l_1,\ldots,l_L) = (1,\ldots,1,2)$, $|l| = 2m$, $l! = 2$. The terms in (66) arising from this pair of particular (L,N) are

$$
\begin{aligned}
&\frac{(2m)_2}{2} E_p \frac{1}{(2m)!} \sum_{1\le i\le n} D_\xi^\beta g(x,\xi)\partial_{i_1} k \cdots \partial_{i_{2m}} k \cdot (-1)^{2m} \cdot 2 \cdot k^{-(4m-1)}\,. \\
&\cdot (A-pk)^{2m} = (2m)_2 E_p g(x,k_x) k^{-(4m-1)} (A-pk)^{2m} \\
&\qquad = E_p\left[\frac{(2m)_2}{(m-1)^2} k^{3-4m} A^{2m} + O(k^{4-4m})\right].
\end{aligned} \tag{76}
$$

(II-B). $N = 2m-1$. Now, we have 2 possibilities for the first partition, one is the same as above, $l = (1,\ldots,1,2)$, $|l| = 2m-1$, $l! = 2$. In this case, roughly speaking, one of D^δ (say ∂_j) should be acted on A, giving $\partial_j A$. Since there are $2m$ choices for D^δ, we obtain corresponding terms in (66) as

$$
\begin{aligned}
&-\frac{(2m)_2}{2} \cdot 2m \cdot E_p \frac{1}{(2m)!} \sum_{1\le i\le n} D_\xi^\beta g(x,\xi)\partial_{i_1} k \cdots \widehat{\partial_j k} \cdots \\
&\qquad \cdot \partial_{i_{2m}} k \cdot \partial_j A \cdot 2 \cdot k^{2-4m} (A-pk)^{2m-1} \\
&\qquad = -(2m)_2 E_p \sum_{j=1}^n \partial_{\xi_j} g(x,k_x) \frac{\partial A}{\partial x_j} k^{2-4m} (A-pk)^{2m-1} \\
&\qquad = -\frac{2(2m)_2}{m-1} E_p\left[t\frac{dA}{dt} k^{3-4m} A^{2m-1} + O(k^{4-4m})\right]
\end{aligned} \tag{77}
$$

("$\bigwedge$" means wanting).

The second possibility is that $l = (1,1,\ldots,1)$ ($2m-1$ entries), $|l| = 2m-1$, $l! = 1$. The first $2m-2$ D^{β^i} (in $D^{\beta^i} k$) all act on k^{-1} and the last D^{β^L} in $D^{\beta^L} k$ also acts on k^{-1} giving a second order derivature of k. Thus we have

$$\begin{aligned}
&\frac{(2m)_2}{2} E_p \frac{1}{(2m)!} \sum_{1\le i\le n} D_\xi^\beta g(x,\xi) \partial_{i_1} \cdots \partial_{i_{2m-2}} k \partial^2_{i_{2m-1} i_{2m}} k \cdot (-1)^{2m-1} \cdot 1 \cdot \\
&\quad \cdot (A - pk)^{2m-1} k^{2m-4} \\
&\quad = -\frac{E_p}{2} \sum_{i,j} g_{ij}(x,k_x) \partial^2_{ij} k \cdot A^{2m-1} k^{2-4m} \\
&\qquad + \frac{E_p}{2} \cdot p \sum_{i,j} g_{ij}(x,k_x) \partial^2_{ij} k \cdot A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}) .
\end{aligned}$$

Since it holds $g_{ij}(x,k_x) = 2a(x,k_x)a_{ij}(x,k_x) + 2a_i(x,k_x)a_j(x,k_x) = \frac{2k}{m-1} a_{ij}(x,k_x) + 2a_i(x,k_x)a_j(x,k_x)$, we have

$$\begin{aligned}
&-E_p \sum_{i,j} a_i(x,k_x) a_j(x,k_x) \partial^2_{ij} k A^{2m-1} k^{2-4m} \\
&\qquad - E_p \cdot \frac{1}{m-1} \sum_{i,j} a_{ij}(x,k_x) \partial^2_{ij} k A^{2m-1} k^{3-4m} \\
&\qquad + pE_p \sum_{i,j} a_i(x,k_x) a_j(x,k_x) \partial^2_{ij} k A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}) .
\end{aligned}$$

But (22) in part I shows that $\sum_{i,j} a_{ij}(x,k_x)\partial^2_{lj} k = \delta_{il}(1 + O(t))$, and Euler's formula gives $a_j(x,k_x) = \frac{1}{m-1}\sum_{l=1}^n a_{jl}(x,k_x)\partial_l k$, hence the result above can be simplified further to give

$$\begin{aligned}
&-\frac{E_p}{m-1} \sum_{i,j,l} a_i(x,k_x) a_{jl}(x,k_x) \partial^2_{ij} k \partial_l k A^{2m-1} k^{2-4m} \\
&\qquad - \frac{E_p}{m-1} [n + O(t)] A^{2m-1} k^{3-4m} \\
&\qquad + \frac{pE_p}{m-1} \cdot \sum_{i,j,l} a_i(x,k_x) a_{jl}(x,k_x)\, \partial^2_{ij} k\, \partial_l k \cdot A^{2m-2} k^{3-4m} \\
&\qquad + E_p O(k^{4-4m}) \\
&\quad = -\frac{E_p}{m-1} \sum a_l(x,k_x)\, \partial_l k (1 + O(t)) A^{2m-1} k^{2-2m} \\
&\qquad - \frac{E_p}{m-1} [n + O(t)] A^{2m-1} k^{3-4m} \\
&\qquad + \frac{pE_p}{m-1} \sum a_l(x,k_x)\, \partial_l k (1 + O(t)) A^{2m-2} k^{3-4m} + O(k^{4-4m}) \\
&\quad = -\frac{E_p}{m-1} \Big(\frac{m}{m-1} + n \Big) (1 + O(t)) A^{2m-1} k^{3-4m} + E_p O(k^{4-4m}) .
\end{aligned} \tag{78}$$

(II-C). $N = 2m - 2$. There are also 2 possiblities for the partition $\beta = \beta^1 + \cdots + \beta^L$. The first is the same as above, i.e., $\frac{(2m)_2}{2}$ choices of the type $l = (1, 1, \ldots, 2)$ ($2m - 1$ entries), $|l| = 2m$, $l! = 2$. In order that $N = 2m - 2$, among the differentiations D^β (total order $2m$), 2 must be chosen acting on A giving terms containing $\partial_{i_1} A\, \partial_{i_2} A$ (again $\frac{(2m)_2}{2}$ choices for i and j among $\{i_1, i_2, \ldots, i_{2m}\}$). They give

$$
\begin{aligned}
&\frac{(2m)_2}{2} \cdot \frac{(2m)_2}{2} E_p \frac{1}{(2m)!} \sum_{1 \le i \le n} D_\xi^\beta g(x,\xi) \partial_{i_1} k \cdots \widehat{\partial}_{j_1} k \cdots \widehat{\partial}_{j_2} k \cdots \\
&\qquad \partial_{i_{2m}} k \partial_{j_1} A \partial_{j_2} A \cdot (-1)^{2m} \cdot 2 \cdot k^{3-4m} (A - pk)^{2m-2} \\
&\qquad = \frac{(2m)_2}{2} E_p \sum g_{ij}(x, k_x) \partial_i A \partial_j A \cdot A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}) \\
&\qquad = (2m)_2 E_p \left(t \frac{dA}{dt} \right)^2 A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}) \,.
\end{aligned} \tag{79}
$$

The other possiblity is again $l = (1, 1, \ldots, 1)$ ($2m-1$ entries), $|l| = 2m-1$, $l! = 1$. Thus one and only one of $|\beta^i|$ should be 2. There are $\frac{(2m)_2}{2}$ choices for this subset, $\beta^L = (i_{2m-1}, i_{2m})$ is a typical one. Since all entries in l are 1, there would be no further partition for D^{β^L}, either it acts just on A, giving terms containing $\partial^2 A$, all other D^β act on k^{-1} giving $N = 2m-2$ ∂k's; either D^{β^L} acts on k^{-1} giving $\partial^2 k$ and among the remaining D^{β^i}, choose one and only one acting on A ($2m-2$ such choices) and the remaining $2m-3$ act on k^{-1} giving $2m-3$ ∂k's. Thus, there are totally $1 + (2m-3) = 2m-2$ derivatives on k agreeing with $N = 2m-2$. Hence, these terms are

$$
\begin{aligned}
&\frac{(2m)_2}{2} E_p \frac{1}{(2m)!} \sum_{1 \le i \le n} D_\xi^\beta g(x,\xi) \partial_{i_1} k \cdots \partial_{i_{2m-2}} k \partial^2_{2m-1,2m} A \cdot \\
&\qquad (-1)^{2m-1} \cdot 1 \cdot (A - pk)^{2m-2} k^{3-4m} \\
&\qquad + \frac{(2m)_2}{2} (2m-2) E_p \frac{1}{(2m)!} \sum_{1 \le i \le n} D_\xi^\beta g(x,\xi) \partial_{i_1} A \partial_{i_2} k \cdots \partial_{i_{2m-2}} k \cdot \\
&\qquad \cdot \partial^2_{2m-1,2m} k \cdot (-1)^{2m-1} \cdot 1 \cdot (A - pk)^{2m-2} k^{3-4m} \\
&\quad = -E_p \sum_{i,j} a_i(x, k_x) a_j(x, k_x) \partial^2_{ij} A \cdot A^{2m-2} k^{3-4m} \\
&\qquad - \frac{1}{2} E_p \sum_{i,j,l} g_{i,j,l}(x, k_x) \partial_i A \partial^2_{jl} k A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}) \,.
\end{aligned}
$$

But

$$
\begin{aligned}
g_{i,j,l}(x,k_x) &= 2a(x,k_x)a_{ijl}(x,k_x) + 2[a_i(x,k_x)a_{jl}(x,k_x) \\
&\quad + a_j(x,k_x)a_{il}(x,k_x) + a_l(x,k_x)a_{ij}(x,k_x)] \\
&= \frac{2k}{m-1}a_{ijk}(x,k_x) + 2[a_i(x,k_x)a_{jl}(x,k_x) \\
&\quad + a_j(x,k_x)a_{il}(x,k_x) + a_l(x,k_x)a_{ij}(x,k_x)]\,, \\
\sum_{j,l} a_{jl}(x,k_x)\partial^2_{jl}k &= n(1+O(t))\,, \\
\sum_{j} a_j(x,k_x)\partial^2_{jl}k &= \frac{1}{m-1}\sum_{i,l} a_{ij}(x,k_x)\partial_i k\partial^2_{jl}k \\
&= \frac{1}{m-1}\sum_{i}\delta_{il}\partial_i k(1+O(t)) \\
&= \frac{k_l}{m-1}(1+O(t))\,, \\
\sum_{j,l} a_j(x,k_x)a_{il}(x,k_x)\partial^2_{jl}k &= \frac{1}{m-1}\sum_{l} a_{il}(x,k_x)k_l(1+O(t)) \\
&= a_i(x,k_x)(1+O(t))\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&-E_p\sum_{i,j} a_i(x,k_x)a_j(x,k_x)\partial^2_{ij}A\cdot A^{2m-2}k^{3-4m} \\
&\quad - E_p\Big[\sum_{i,j,l} a_i(x,k_x)a_{jl}(x,k_x)\partial_i A\partial^2_{jl}k \\
&\quad + \sum_{i,j,l} a_j(x,k_x)a_{il}(x,k_x)\partial_i A\partial^2_{jl}k \\
&\quad + \sum_{i,j,l} a_l(x,k_x)a_{ij}(x,k_x)\partial_i A\partial^2_{jl}k\Big]A^{2m-2}k^{3-4m} + E_pO(k^{4-4m}) \\
&= -E_p\sum_{i,j} a_i(x,k_x)a_j(x,k_x)\partial^2_{ij}A\cdot A^{2m-2}k^{3-4m} \\
&\quad - E_p\Big[n\Big(t\frac{dA}{dt}\Big)(1+O(t)) \\
&\quad + \frac{2}{m-1}\sum_{i,l} a_{il}(x,k_x)\partial_l k\partial_i A\Big]A^{2m-2}k^{3-4m} + E_pO(k^{4-4m}) \\
&= -E_p\sum_{i,j} a_i(x,k_x)a_j(x,k_x)\partial^2_{ij}A\cdot A^{2m-2}k^{3-4m} \\
&\quad - E_p(n+2)t\frac{dA}{dt}(1+O(t))A^{2m-2}k^{3-4m} + E_pO(k^{4-4m})\,.
\end{aligned} \tag{80}
$$

For $L = 2n-1$, $N \leq 2m-3$, $k^{-(L+N)} = O(k^{4-4m})$, hence omitted.

III. $L = 2m-2$, and $L < 2m-2$. (A) gives now

$$|\beta^1| + \cdots + |\beta^{2m-2}| = 2m\,.$$

For $N = 2m$, all $D^{\beta^i}k$ must partitioned into ∂ acting on k^{-1}, hence there must appear $\partial_{i_1}k \cdots \partial_{i_{2m}}k$, joining them to $D_\xi^\beta g(x,\xi)$ gives $g(x, k_x) = O(k^2)$, hence all these terms are $O(k^{4-4m})$ and omitted. For $N = 2m-1$, there would appear $2m-1$ ∂k's, and Euler's formula associates them to $D_\xi^\beta g(x,\xi)$ to give $\partial_i g(x, k_x) = 2a(x, k_x)a_i(x, k_x) = O(k)$, hence the total order of such terms are again $O(k^{4-4m})$. $N \le 2m-2$ can also be neglected.

$L \le 2m-3$ gives only terms $O(k^{4-4m})$ hence neglected.

Our computation of $g(x, \partial_x)E_p$ terminates here. We tabulate our results as follows:

I. $L = 2m$

(I-A) $L = 2m$, $N = 2m$, (71).

(I-B) $L = 2m$, $N = 2m-1$, (72).

(I-C) $L = 2m$, $N = 2m-2$, (73).

(I-D) $L = 2m$, $N = 2m-3$, (74).

$L = 2m$, $N \le 2m-4$ give terms of order $O(k^{4-4m})$, hence negligible.

Summing up, we have (75) for the case $L = 2m$.

II. $L = 2m-1$

(II-A) $L = 2m-1$, $N = 2m$, (76).

(II-B) $L = 2m-1$, $N = 2m-1$, (77).

After simplifying terms containing the factors $\partial_{ij}^2 k$, we have (78).

(II-C) $L = 2m-1$, $N = 2m-2$, (79).

Simplifying terms containing $\partial_{ij}^2 k$, but reserving $\partial_{ij}^2 A$, we have (80).

$L = 2m-1$, $N \le 2m-3$, negligible, since they are terms of the order $O(k^{4-4m})$.

III. $L \le 2m-2$ gives terms of the order $O(k^{4-4m})$, hence negligible.

These results are summed as the

PROPOSITION 1. *The terms of order k^{2-4m} and k^{3-4m} in the asymptotic expansion of*

$P_{2m}(x, \partial_x)E_p$ *are the following:*

$$
\begin{aligned}
k^{2-4m} :& E_p A^{2m-2}\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)^2 .\\
k^{3-4m} :& - E_p p\Big[2(m-1)A^{2m-3}\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)^2\\
& - \frac{2}{m-1}A^{2m-2}\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)\Big]\\
& + E_p\Big[(2m)_2 A^{2m-2}\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)^2\\
& - 2A^{2m-3}\sum_{i,j} a_{ij}(x, k_x)\partial_i A \partial_j A\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)\\
& - \frac{n(m-1)+m}{(m-1)^2}(1+O(t))A^{2m-1}\\
& - (n+2)t\frac{dA}{dt}\cdot A^{2m-2}(1+O(t))\\
& - \sum_{i,j} a_i(x, k_x)a_j(x, k_x)\partial_{ij}^2 A\cdot A^{2m-2}\Big].
\end{aligned} \tag{81}
$$

3.6. Computation of $P_{2m-j}(x, \partial_x)E_p$

The symbol $P(x, \xi)$ can be written as the sum of homogeneous polynomials in ξ of degree $2m - j$:

$$P(x, \xi) = P_{2m}(x, \xi) + P_{2m-1}(x, \xi) + \cdots . \tag{82}$$

$P_{2m}(x, \partial_x)E_p$ has been discussed in detail, now we proceed to compute

$$
\begin{aligned}
P_{2m-j}(x, \partial_x)E_p U_0 =& P_{2m-j}(x, \partial_x)E_p \cdot U_0\\
& + \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i} P_{2m-j}(x, \partial_x)E_p \cdot \frac{\partial U_0}{\partial x_i} + \cdots ,\ (j \ge 1) .
\end{aligned} \tag{83}
$$

First consider $P_{2m-j}(x, \partial_x)E_p$, it is almost the same as above, only replacing $g(x, \xi)$ by $P_{2m-j}(x, \xi)$, thus for (66) we have now

$$
\begin{aligned}
P_{2m-j}(x, \partial_x)E_p = \frac{E_p}{(2m-j)!} & \sum_{1\le i\le n} D_\xi^\beta P_{2m-j}(x, \xi)(-1)^{|l|} l! k^{-(N+L)} \cdot\\
& (D^{\eta^1}A)^* \cdots (D^{\eta^L}A)^* D^{\delta^1}k \cdots D^{\delta^N}k ,
\end{aligned} \tag{84}
$$

for (67) and (68), we have now

$$L \le |\beta^1| + \cdots + |\beta^L| = 2m - j,\ (j \ge 1) . \tag{85}$$

$$N \le |\delta^1| + \cdots + |\delta^N| + |\beta^1 - \gamma^1| + \cdots + |\beta^L - \gamma^L| = 2m - j,\ (j \ge 1) . \tag{86}$$

Thus, the lowest term in (84) is

$$k^{2j-4m}.\ (j \ge 1)$$

Similarly, $\partial_{\xi_i} P_{2m-j}(x,\xi)$ is a polynoimal in ξ of degree $2m-j-1$, hence

$$\frac{\partial}{\partial \xi_i} P_{2m-j}(x,\partial_x)E_p = O(k^{2+2j-4m}),\ (j \geq 1). \tag{87}$$

Since we need only terms of order k^{2-4m} and k^{3-4m}, so from the contributions of lower order operators $P_{2m-j}(x,\partial_x)E_pU_0$, we need to consider only $P_{2m-1}(x,\partial)E_p \cdot U_0$.

From (84), we have

$$\begin{aligned} P_{2m-1}(x,\partial_x)E_p = \frac{E_p}{(2m-1)!} \sum_{1\leq i\leq n} D_\xi^\beta P_{2m-1}(x,\xi)(-1)^{|l|} l! k^{-(N+L)} \cdot \\ \cdot (D^{\eta^1}A)^* \cdots (D^{\eta^L}A)^* D^{\delta^1}k \cdots D^{\delta^N}k\,. \end{aligned} \tag{88}$$

Consider the case $L = 2m-1$. For $N = 2m-1$, we must have $l = (1,1,\dots,1)$ ($L = 2m-1$ entries), $|l| = 2m-1$, $l! = 1$, $\delta^1 = \cdots = \delta^N = 1$, corresponding terms in (88) are

$$\begin{aligned} &E_p \cdot P_{2m-1}(x,k_x)(-1)^{2m-1}(A-pk)^{2m-1}k^{2-4m} \\ &\quad = E_p[-P_{2m-1}(x,k_x)A^{2m-1}k^{2-4m} \\ &\qquad + (2m-1)pP_{2m-1}(x,k_x)A^{2m-2}k^{3-4m} + O(k^{4-4m})]\,. \end{aligned} \tag{89}$$

Next, let $N = 2m-2$. A typical case is to choose $(\delta^1,\dots,\delta^{2m-2}) = (i_2,\dots,i_{2m-1})$, thus $(D^{\eta^1}A)^* = \partial_{i_1}A$, $(D^{\eta^2}A)^* = \cdots = (D^{\eta^L}A)^* = A - pk$. There are $2m-1$ such choices, so we should multiply the results obtained by $2m-1$ and obtain terms in (88) corresponding to $L = 2m-1$, $N = 2m-2$. They are

$$\begin{aligned} &E_p \sum_{i=1}^n \partial_i P_{2m-1}(x,k_x)\partial_i A k^{3-4m}(A-pk)^{2m-2} \\ &\quad = E_p \sum_{i=1}^n \partial_i P_{2m-1}(x,k_x)\partial_i A \cdot A^{m-2}k^{3-4m} + E_pO(k^{4-4m})\,. \end{aligned} \tag{90}$$

When $N \leq 2m-3$, $k^{-(N+L)} = O(k^{4-4m})$ and is negligible.

Consider the case $L = 2m-2$. From (67), we have

$$2m-2 \leq |\beta^1| + \cdots + |\beta^{2m-2}| = 2m-1\,.$$

There is one and only one $|\beta^i| = 2$, the others: $|\beta^i| = 1$. We may choose $\beta^{2m-2} = (i_{2m-2}, i_{2mm-1})$, $\beta^1 = i_1, \dots, \beta^{2m-3} = i_{2m-3}$. Since there are $\frac{1}{2}(2m-1)_2$ ways of choosing $|\beta^i| = 2$, the results should be multiplied by $\frac{1}{2}(2m-1)_2$.

We need only consider the case $N = 2m-1$. This would require that both differentiations in $D^{\beta^{2m-2}}k$ acting on k and giving $\partial_{i_{2m-2}}k\partial_{i_{2m-1}}k$. Combining the consideration of L and N we obtain

$$\begin{aligned} &\frac{1}{2}(2m-1)_2 E_p \frac{1}{(2m-1)!} \sum_{1\leq i\leq n} D_\xi^\beta P_{2m-1}(x,\xi)\partial_{i_1}k \cdots \partial_{i_{2m-1}}k \cdot \\ &\qquad \cdot k^{3-4m}(-1)^{2m-1}(A-pk)^{2m-1} \\ &\quad = -\frac{(2m-1)_2}{2} E_p P_{2m-1}(x,k_x)A^{2m-1}k^{3-4m} + E_pO(k^{4-4m})\,. \end{aligned} \tag{91}$$

All other terms are neglected. The final result is

$$
\begin{aligned}
P_{2m-1}(x,\partial_x)E_p = E_p\Big[& -P_{2m-1}(x,k_x)A^{2m-1}k^{2-4m} \\
& + (2m-1)pP_{2m-1}(x,k_x)A^{2m-2}k^{3-4m} \\
& + \sum_{i=1}^{n}\partial_i P_{2m-1}(x,k_x)\frac{\partial A}{\partial x_i}\cdot A^{2m-2}k^{3-4m} \\
& + O(k^{4-4m})\Big].
\end{aligned} \tag{92}
$$

3.7. Computation of $\sum_{i=1}^{n}\partial_i P_{2m}(x,\partial_x)E^p\cdot\frac{\partial U_0}{\partial x_i}$

Here we should find the first term in the asymptotic expansion of

$$\partial_i P_{2m}(x,\partial_x)E^p = Q_i(x,\partial_x)E^p\,,$$

$Q_i(x,\xi)$ is a homogeneous polynomial of ξ of order $2m-1$. Hence (92)is appliable, with $Q_i(x,\xi)$ replacing $P_{2m-1}(x,\xi)$. But

$$Q_i(x,\xi) = \partial_i P_{2m}(x,\xi) = \partial_i a^2(x,\xi) = 2a(x,\xi)a_i(x,\xi)\,,$$

hence

$$Q_i(x,k_x) = \frac{2k}{m-1}a_i(x,k_x)\,.$$

Substitute this into (92) with Q_i replacing P_{2m-1}, we have

$$
\begin{aligned}
\sum_{i=1}^{n}\partial_i P_{2m}(x,\xi)E^p\cdot\frac{\partial U_0}{\partial x_0} &= E^p\Big[-\frac{2}{m-1}A^{2m-1}k^{3-4m}\sum_i a_i(x,k_x)\frac{\partial U_0}{\partial x_i} \\
&\quad + 2A^{2m-2}k^{3-4m}t\frac{dA}{dt}\sum_i A_i(x,k_x)\frac{\partial U_0}{\partial x_i} \\
&\quad + O(k^{4-4m})\Big] \\
&= E^p\Big[2\Big(t\frac{dA}{dt}-\frac{A}{m-1}\Big)A^{2m-2}k^{3-4m}t\frac{dU_0}{dt} \\
&\quad + O(k^{4-4m})\Big].
\end{aligned} \tag{93}
$$

After this long computation, combining (81), (92) and (93), we have

PROPOSITION 2. *The first terms in the asymptotic expansion of* $P(x,\partial_x)E_pU_0$ *is*

$$
\begin{aligned}
P(x,\partial_x)E_pU_0 = E^p[&A^{2m-2}\mathcal{E}(A)U_0k^{2-4m} + A^{2m-3}\mathcal{J}(A,U_0)k^{3-4m} \\
&+ \mathcal{R}(A,U_0)]\,.
\end{aligned} \tag{94}
$$

where

$$
\begin{aligned}
\mathcal{E}(A) &= \Big(t\frac{dA}{dt}-\frac{A}{m-1}\Big)^2 - AP_{2m-1}(x,k_x)\,, \\
\mathcal{J}(A,U_0) &= A^{2m-3}[p\mathcal{J}_1(A,U_0)+\mathcal{J}_2(A,U_0)]\,,
\end{aligned} \tag{95}
$$

$$\mathcal{J}_1(A,U_0) = -\Big[2(m-1)\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)^2 - \frac{2}{m-1}A\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big) - (2m-1)AP_{2m-1}(x,k_x)\Big]U_0\,, \tag{96}$$

$$\begin{aligned}\mathcal{J}_2(A,U_0) = {}& 2A\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)t\frac{dU_0}{dt} + \Big[(2m)_2 A\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)^2 \\ & - \frac{n(m-1)+m}{(m-1)^2}(1+O(t))A^2 - (n+2)t\frac{dA}{dt}\cdot A(1+O(t)) \\ & - \sum_{i,j} a_{ij}(x,k_x)\partial_i A\partial_j A\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big) \\ & - A\sum_{i,j} a_{ij}(x,k_x)a_j(x,k_x)\partial_{ij}^2 A \\ & + A\sum_i \partial_i P_{2m-1}(x,k_x)\frac{\partial A}{\partial x_i}\Big]U_0\,,\end{aligned} \tag{97}$$

$\mathcal{R}(A,U_0)$ is the remainder which is of order $O(k^{4-4m})$.

3.8. Eiconal system for A and k

As we mentioned earlier, we should have two equations to determinate k and A. The first is Leray's result

$$a(x,k_x) = \frac{k}{m-1}\,.$$

$\mathcal{E}(A) = 0$ gives the eiconal equation for A:

$$\Big(t\frac{dA}{dt} - \frac{A}{m-1}\Big)^2 = P_{2m-1}(x,k_x)A\,. \tag{98}$$

These two equations constitute the eiconal system.

Let $t = \tau^{m-1}$ and $A = \tau B$, we have

$$\tau^3\Big(\frac{dB}{d\tau}\Big)^2 = (m-1)^2 P_{2m-1}(x,k_x)B\,.$$

Since $k_x = \tau\xi$, using Taylor's formula, we have

$$\Big(\frac{dB}{d\tau}\Big)^2 = (m-1)^2\tau^{2(m-2)}[P_{2m-1}(y,\eta) + O(\tau)]B\,.$$

Integrating this ODE along the Hamiltonian orbit for $a(x,\xi)$, we have

$$2\sqrt{B} = (m-1)\int_0^\tau \tau^{m-2}\sqrt{P_{2m-1}(y,\eta) + O(\tau)}\,d\tau + C\,.$$

Let $C = 0$ and assume that

$$P_{2m-1}(y,\eta) \neq 0\,, \tag{99}$$

We have

$$\begin{aligned} B &= \frac{1}{4}\tau^{2m-2}P_{2m-1}(y,\eta)(1+O(t))\,, \\ A &= \frac{1}{4}\tau^{2m-1}P_{2m-1}(y,\eta)(1+O(t))\,. \end{aligned} \tag{100}$$

We are to obtain transport equation for U_0 from $\mathcal{J}(A, U_0) = 0$. In order to simplify it, substitute (100) into it and we have

$$\begin{aligned} \mathcal{J}_1(A, U_0) &= -\Big[2(m-1)P_{2m-1}(x, k_x)A - \frac{2}{m-1}P^{1/2}_{2m-1}(x, k_x)A^{3/2} \\ &\qquad - (2m-1)P_{2m-1}(x, k_x)A\Big]U_0 \\ &= \frac{m}{4(m-1)}\tau^{4m-2}P^2_{2m-1}(y, \eta)(1 + O(t))U_0 , \end{aligned} \tag{101}$$

$$\begin{aligned} \mathcal{J}_2(A, U_0) = &\frac{1}{4(m-1)}\tau^{4m-2}P^2_{2m-1}(y, \eta)(1 + O(t))\tau\frac{dU_0}{d\tau} \\ &+ \frac{(2m)_2}{16}[\tau^{2m-1}P_{2m-1}(y, \tau)]^3(1 + O(\tau))U_0 \\ &- \frac{1}{16(m-1)^2}[2m^2n - 2mn + 4m^2 - 5m + 2]\cdot \\ &\cdot[\tau^{2m-1}P_{2m-1}(y, \eta)]^2(1 + O(\tau))U_0 \\ &- \frac{1}{4}\tau^{2m-1}P_{2m-1}(y, \eta)(1 + O(\tau))\Big[2\sum_{i,j} a_{ij}(x, k_x)\partial_i A\partial_j A \\ &+ \sum_{i,j} a_i(x, k_x)a_j(x, k_x)\partial^2_{ij}A - \sum_i \partial_i P_{2m-1}(x, k_x)\frac{\partial A}{\partial x_i}\Big]U_0 . \end{aligned} \tag{102}$$

3.9. Transport equation for U_0

$\mathcal{J}(A, U_0) = 0$ is the transport equation for U_0 which is its bounded solution. For this purpose, (102) should be further simplified.

Note that

$$\frac{\partial A}{\partial \eta_i} = \sum_{j=1}^{n} \frac{\partial A}{\partial x_j}\frac{\partial x_j}{\partial \eta_i}$$

and the Hamiltonian system for x gives

$$x_j = y_j + a_j(y, \eta)t + O(t) ,$$

hence

$$\begin{aligned} \frac{\partial x_j}{\partial \eta_i} &= a_{ij}(y, \eta)t + O(t) , \\ \frac{D(x)}{D(\eta)} &= t\mathrm{Hess}_\eta a(y, \eta)(1 + O(t)) . \end{aligned}$$

Denote $[\mathrm{Hess}_\eta a(y, \eta)]^{-1} = (b_{ij}(y, \eta))$. We have

$$\frac{\partial A}{\partial x_j} = t^{-1}\sum_i b_{ij}(y, \eta)(1 + O(t))\frac{\partial A}{\partial \eta_j} .$$

But (100) gives A as a function of η, we have

$$\frac{\partial A}{\partial x_j} = \frac{1}{4}t^{-1}\sum_i b_{ij}(y, \eta)(1 + O(t))P_{2m-1,j}(y, \eta) = O(\tau^m) , \tag{103}$$

thus,

$$\sum_{i,j} a_{ij}(x, k_x)\partial_i A \partial_j A = O(\tau^{m-2+2m}) = \tau^{2m-1} O(\tau^{m-1}) . \tag{104}$$

Next, denote $D = t\frac{d}{dt} = \sum_i a_i(x, k_x)\frac{\partial}{\partial x_i}$, we have

$$\begin{aligned}
\sum_{i,j} a_i(x, k_x)a_j(x, k_x)\partial_{ij} A &= \sum_i a_i(x, k_x)D(\partial_i A) \\
&= \sum_i D(a_i(x, k_x)\partial_i A) - \sum_i \partial_i A \cdot Da_i \\
&= D^2 A - \sum_i \partial_i A D a_i \\
&= D\Big(\frac{A}{m-1} + [P_{2m-1}(x, k_x)A]^{1/2}\Big) - \sum_i \partial_i A D a_i \\
&= \frac{1}{m-1} DA + \frac{PDA + ADP}{2\sqrt{PA}} - \sum \partial_i A D a_i \,, \\
&\quad \text{(here } P = P_{2m-1}(x, k_x)) .
\end{aligned}$$

Since

$$DA = t\frac{dA}{dt} = \frac{A}{m-1} + \sqrt{PA} \,,$$

then, the expression above can be simplified further:

$$\begin{aligned}
&= \frac{\tau^{2m-1}}{4(m-1)^2} P_{2m-1}(y, \eta)(1 + O(\tau)) + \frac{3\tau^{2m-1}}{4(m-1)} P_{2m-1}(y, \eta)(1 + O(\tau)) \\
&\quad + \frac{\tau^{2m-1}}{2} P_{2m-1}(y, \eta)(1 + O(\tau)) + \frac{1}{4} D P_{2m-1}(x, k_x) - \sum \partial_i A D a_i \\
&= \frac{(2m-1)^2}{4(m-1)^2} \tau^{2m-1} P_{2m-1}(y, \eta) - \sum_i \partial_i A D a_i \,.
\end{aligned}$$

Now

$$
\begin{aligned}
\sum_i \partial_i A D a_i &= \sum_{i,j} \frac{\partial A}{\partial x_i} a_j(x,k_x) \frac{\partial}{\partial x_j} a_i(x,k_x) \\
&= \sum_{i,j} \frac{\partial A}{\partial x_i} a_j(x,k_x) \Big(a_{i x_j}(x,k_x) + \sum_l a_{il}(x,k_x) \frac{\partial^2 k}{\partial x_j \partial x_l} \Big) \\
&= \sum_{i,j} \frac{\partial A}{\partial x_i} a_j(x,k_x) (a_{i x_j}(x,k_x) + \delta_{ij}(1+O(\tau)) \\
&= \sum_{i,j} \frac{\partial A}{\partial x_i} a_j(x,k_x) a_{i x_j}(x,k_x) + \sum_i a_i(x,k_x) \frac{\partial A}{\partial x_i}(1+O(\tau)) \\
&= \sum_{i,j} \frac{\partial A}{\partial x_i} a_j(x,k_x) a_{i x_j}(x,k_x) + t \frac{dA}{dt}(1+O(\tau)) \\
&= \tau^{2m-1} O(\tau) + \frac{A}{m-1} + \sqrt{P_{2m-1}(x,k_x) A}(1+O(\tau)) \\
&= \tau^{2m-1} O(\tau) + \Big(\frac{1}{4(m-1)} + \frac{1}{2} \Big) \tau^{2m-1} P_{2m-1}(y,\eta)(1+O(\tau)) \\
&= \frac{2m-1}{4(m-1)} \tau^{2m-1} P_{2m-1}(y,\eta)(1+O(\tau)) \, .
\end{aligned} \tag{105}
$$

For the last term in (102), we have

$$
\sum_i \partial_i P_{2m-1}(x,k_x) \frac{\partial A}{\partial x_i} = O(\tau^{3m-2}) = \tau^{2m-1} O(\tau^{m-1}) \tag{106}
$$

Substituting (104), (105), (106) into (102), we have

$$
\begin{aligned}
\mathcal{J}_2(A,U_0) = {} & \frac{1}{4(m-1)} [\tau^{2m-1} P_{2m-1}(y,\eta)]^2 (1+O(\tau)) \\
& \Big[\tau \frac{dU_0}{d\tau} - \Big(\frac{M}{4(m-1)} + \tau f(\tau,\eta,y) \Big) U_0 \Big] ,
\end{aligned} \tag{107}
$$

here $M = 2m(m-1)(n+1)$, $f(\tau,\eta,y)$ is holomorphic in (τ,η) near $\tau = 0$.

Combining (101) for $\mathcal{J}_1(A,U_0)$ and (107) for $\mathcal{J}_2(A,U_0)$, we obtain finally the transport equation for U_0:

$$
\tau \frac{dU_0}{d\tau} + \Big[mp - \frac{M}{4(m-1)} - \tau f(\tau,\eta,y) \Big] U_0 = 0 \, . \tag{108}
$$

For U_0 to be bounded near τ, we must choose

$$
p = \frac{M}{4m(m-1)} = \frac{n+1}{2} \, , \tag{109}
$$

and

$$
U_0 = \exp \left(\int_0^\tau f(s,\eta,y) \, ds \right) \, . \tag{110}
$$

3.10. Main result

We have been seeking a solution for the equation $P(x, \partial_x)u = 0$ in the form of (49):

$$u(x, y) = \exp[A(x, y)/k(x, y)] \sum_{h=0}^{\infty} U_h h^{p+h} ,$$

what we have attained up to present is only the solution of eiconal system for $k(x, y)$ ((47)) and $A(x, y)$ ((98)) and also the transport equation for U_0 ((108)). Hence, p is defined by (109), U_0 by (110) and $A(x, y)$ by (100).

For U_h, $h \geq 1$, we would obtain similar equation as (108) only with $p + h$ replacing p and the right hand side would be an expression linear in $U_{h-1}, \dots, U_{h-2m}$ and their derivatives, i.e.

$$\tau \frac{dU_h}{d\tau} + [mh - \tau f(\tau, \eta, y)]U_h = L(U_{h-1}, \dots, U_{h-2m+1}) . \tag{111}$$

The only bounded solution of it is

$$U_h = \frac{U_0}{\tau^{mh}} \int_0^{\tau} \frac{s^{mh-1}}{U_0} L(U_{h-1}, \dots, U_{h-2m+1})\, ds . \tag{112}$$

Thus we are brought back to the case of simple-characteristic problems as discussed in Part I and we come to our main result:

THEOREM 5. *Let $P(x, \partial_x)$ be a holomorphic linear partial differential operator of order $2m$ ($m > 1$) in a domain X of an analytic complex mainfold. Let the principal symbol of $P(x, \partial_x)$ be of the form*

$$P_{2m}(x, \xi) = [a(x, \xi)]^2 ,$$

such that for $y \in X$,

$$\mathrm{Hess}_\eta[a(y, \eta)] \neq 0 \textit{ for } a(y, \eta) = 0,\ \eta \neq 0.$$

Further, we also assume that

$$P_{2m-1}(y, \eta) \neq 0 \textit{ and } m \geq 2.$$

Then there exist solutions of $P(x, \partial_x)u = 0$ with the asymptotic expansion

$$u(x, y) = \exp[A(x, y)/k(x, y)] \sum_{h=0}^{\infty} U_h h^{p+h} .$$

Here $k(x, y) = 0$ is an equation of the characteristic conoid, which satisfies

$$a(x, k_x) = \frac{k}{m-1} .$$

$A(x, y)$ is given by (98):

$$A(x, y) = \frac{1}{4}\tau^{2m-1} P_{2m-1}(y, \eta)(1 + O(\tau)) ,$$

*U_0 and U_h, $h = 1, 2, \dots$, are determined by transport equation (108) and (**??**), p is determined by (109). Thus U_0 and U_h are given by (110) and (112). This asymptotic expansion converges for $|k(x, y)|$ small enough.*

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M. Y. CHI (Qi Minyou)
Institute of Mathematics,
Wuhan University
Wuhan, 430072
P. R. China

Lavoro pervenuto in redazione il 1.4.1997.

Rend. Sem. Mat. Univ. Pol. Torino
Vol. 56, 3 (1998)

M.Y. Chi (Qi Minyou)

LEVI CONDITION AND THE HADAMARD FUNDAMENTAL SOLUTION †

Abstract. In a previous paper [1], solutions exponentially singular are constructed for linear holomorphic PDE's with double characteristics which violate the Levi condition. We prove here that Hadamard fundamental solutions with polynomial singularity exist under Levi condition.

1. Introduction

This is a continuation of the author's paper [1] in which Hadamard's construction of fundamental solution is extended to a general linear holomorphic PDE of order m,

$$P(x, \partial_x)u = 0\,, \tag{1}$$

where the principal symbol $P_m(x, \xi)$ satisfies a non-degeneracy condition, which is very important:

$$\mathrm{Hess}_\xi\, P_m(x, \xi) \neq 0\,, \qquad \xi \neq 0\,. \tag{2}$$

As has been pointed out in [1], multiple-characteristic (hereafter, M.C. for short) cases are excluded by this non-degeneracy condition, and in simple-characteristic (S.C. for short) cases, Hadamard fundamental solution [2] with an asymptotic expansion

$$u(x, y) = \sum_{k=0}^{\infty} U_h(x, y) k^{p+h}(x, y)/\,\Gamma(p + h + 1) \tag{3}$$

for (1) can be constructed, where y is a fixed point and $k(x, y) = 0$ is an equation for the characteristic conoid of (1) with y as its vertex.

For M.C. problem, a simplest model is also considered in [1] where

$$\begin{aligned} P(x, \xi) &= P_m(x, \xi) + P_{m-1}(x, \xi) + \cdots \\ &= [a(x, \xi)]^2 + P_{m-1}(x, \xi) + \cdots\,, \end{aligned} \tag{4}$$

$P_{m-j}(x, \xi)$ is a homogeneous polynomial in ξ of degree $m - j$. In this case a non-degeneracy condition is imposed on $a(x, \xi)$ but not on $P_m(x, \xi)$, i.e.

$$\mathrm{Hess}_\xi\, a(x, \xi) \neq 0\,, \qquad \xi \neq 0\,, \tag{5}$$

† This work is supported by the CNNSF of China and the Foundation of the State Education Commission of China. The author would extend his gratitude to Univ. Turin and Univ. Bochum for their hospitality during his visit that this paper was completed there.

then, the characteristic conoid with respect to $a(x, \xi)$ is also constructed, whose equation is $k(x, y) = 0$, which satisfies the next condition (10).

When

$$P_{m-1}(x, k_x) \not\equiv 0 \quad \text{near} \quad k = 0\,, \tag{6}$$

Hadamard fundamental solution can also be constructed but with an asymptotic expansion of exponential growth,

$$u(x, y) = \exp[A(x, y)/k(x, y)] \sum_{h=0}^{\infty} U_h(x, y) k^{p+h}(x, y)/\Gamma(p + h + 1)\,. \tag{7}$$

Now the problem is: What is the essence of the condition (6)? As a motivation, we allow ourselves to be a little vague for the moment. If

$$P_{m-1}(x, k_x) \equiv 0 \quad \text{when} \quad k = 0\,, \tag{8}$$

the Weierstrass preparation theorem would give

$$P_{m-1}(x, k_x) = A(x) k^{\lambda}\,, \tag{9}$$

where $A(x) \not\equiv 0$ near $k = 0$ and $\lambda \in \mathbb{N}$. (Actually, $\lambda = 0$ corresponds to the case $P_{m-1}(x, k_x) \not\equiv 0$ near $k = 0$, hence we assume $\lambda \in \mathbb{N}_+$ hereafter which covers both cases).

The construction above is all based on a very important fact proved by J. Leray [3]:

$$a(x, k_x) = \frac{1}{m_1 - 1} k\,, \tag{10}$$

m_1 is the order of a in ξ, (10) is just the condition

$$P_m(x, k_x) = \frac{k}{m - 1} \tag{11}$$

in S.C. cases. Hence k^{λ} in the right-hand side of (9) is reduced to $(m_1 - 1)^{\lambda}[P(x, k_x)]^{\lambda}$. It is proved in [1] that the dual variable ξ of x in the Hamiltionian formulation is $k_x = \tau\xi$, $\tau = t^{\frac{1}{m-1}}$, hence the left hand side of (9) is homogeneous in ξ of degree $m - 1 = 2m_1 - 1$ while $k^{\lambda} = (m_1 - 1)^{\lambda}[a(x, k_x)]^{\lambda}$ is homogeneous in ξ of degree $\geq \lambda m_1$. Hence the only plausible factorization for $P_{m-1}(x, \xi)$ is $\lambda = 1$ and

$$P_{m-1}(x, \xi) = Q_1(x, \xi) a(x, \xi) \tag{12}$$

where the degree of Q_1 in ξ is $m_1 - 1$.

The discussion above reminds us that the condition (6) implies the violation of the Levi condition. When Levi condition is satisfied, i.e. (9) is satisfied with $\lambda = 1$, the eiconal equation for A becomes

$$t\frac{dA}{dt} - \frac{A}{m - 1} = O(t) A\,, \tag{13}$$

and the transport equation for U_0 becomes

$$U_0 = 0\,. \tag{14}$$

Thus we see, under Levi condition, the PDE (1) in the simplest M.C. case (4) can never possess an Hadamard fundamental solution (7) with exponential growth. The main purpose of this paper is to prove that in this case, (1) again possesses several Hadamard fundamental solutions of polynomial growth like (3).

In this paper, we assume $x = (x_1, \dots, x_n)$ are real variables and $P(x, \partial x)$ has real-valued analytic coefficients. All the results cited below are valid in this case.

In [1], the author mentioned an analogy between (1) under condition (4) and an ODE with irregular singularity $z = 0$:

$$z^2 \frac{\partial^m u}{\partial z^m} + A(z) \frac{\partial^{m-1} u}{\partial z^{m-1}} + \cdots = 0\,. \tag{15}$$

But if

$$A(0) = 0 \tag{16}$$

hence $A(z)$ has a factor z, (15) is in fact a Fuchsian type ODE

$$z^2 \frac{\partial^m u}{\partial z^m} + zA_1(z) \frac{\partial^{m-1} u}{\partial z^{m-1}} + \cdots = 0\,. \tag{17}$$

Condition (16) is just the Levi condition for (15). The main idea of this paper is to show that the role played by Levi condition lies just in reducing irregular singularities into regular singularities.

There are various but essentially equivalent versions of Levi condition (hereafter L condition for short). See Levi [4] and Mizohata [5]. We shall use de Paris' version [6].

Assume that the principal symbol $P_m(x, \xi)$ for (1) can be factorized as

$$P_m(x, \xi) = \prod_{j=1}^{L} H_j^{d_j}(x, \xi)\,, \tag{18}$$

here $H_j(x, \xi)$ is a homogeneous polynomial in ξ of degree m_j, $d_j \in \mathbb{N}_+$. Assume also $H_j(x, \xi)$ and $H_i(x, \xi)$ do not have common zero (x_0, ξ_0), $\xi_0 \neq 0$, when $i \neq j$. Hence micro-locally, i.e. when $x \in \Omega$ (a neighborhood of x_0), $\xi \in \Gamma$ (a conic neighborhood of ξ_0), $P_m(x, \xi)$ can be written as

$$\begin{aligned} P_m(x, \xi) &= Q_0(x, \xi)[h(x, \xi)]^{d_0} && (19)\\ h(x_0, \xi_0) &= 0 \\ Q(x_0, \xi_0) &\neq 0\,. && (20) \end{aligned}$$

Now expand $P(x, \xi)$ in $h(x, \xi)$, i.e. assume that

$$\begin{aligned} P(x, \xi) &= P_m(x, \xi) + P_{m-1}(x, \xi) + \cdots + P_0(x, \xi) \\ &= \sum_{j=0}^{N} Q_j(x, \xi)[h(x, \xi)]^{d_j}\,. \end{aligned} \tag{21}$$

Here $P_{m-j}(x, \xi)$ is a homogeneous polynomial in ξ of degree $m - j$. We also assume

$$\begin{aligned} Q_0(x, \xi)[h(x, \xi)]^{d_0} &= P_m(x, \xi) \\ \deg_\xi Q_j(x, \xi)[h(x, \xi)]^{d_j} &< \deg_\xi Q_{j-1}(x, \xi)[h(x, \xi)]^{d_{j-1}} \end{aligned} \tag{22}$$

$$d_0 > d_1 > \cdots > d_N .$$

De Paris proposed an important condition for the decomposition, i.e.

$$(L) \quad d_j \geq d_0 - j . \tag{23}$$

and he calls the decomposition (17) satisfying (23) a "bonne decomposition". H. Komatsu pointed out in [7] that the "bonne decomposition" condition (23) is just another version of the L condition. He even defined the "irregularity index" to be

$$\sigma = \max\{1; (d_0 - d_j)/j, j = 1, 2, \ldots, N\} . \tag{24}$$

Clearly, $\sigma \geq 1$. Komatsu pointed out in fact, that the case $\sigma > 1$ corresponds to irregular singularity. Now we prove

LEMMA 1. *$\sigma = 1$ iff (21) is a "bonne decomposition".*

Proof. If (21) is a "bonne decomposition", then for all $j = 1, 2, \ldots, N$ we have $d_j \geq d_0 - j$, i.e. $d_0 - d_j \leq j$ hence $\sigma = 1$.

Conversely, if (21) is not a "bonne decomposition", we can find an integer j_0, $1 \leq j_0 \leq N$, such that

$$d_{j_0} < d_0 - j , \tag{25}$$

hence $d_0 - d_j > j$ and $\sigma > 1$. □

In the whole paper below, we must assume a very important non-degeneracy condition

$$\mathrm{Hess}_\xi h(x, \xi) \neq 0 , \qquad (x, \xi) \in \Omega \times \Gamma , \ \xi \neq 0 . \tag{26}$$

The essence of this condition should be studied separately. We should add another important restriction on $\mathrm{Hess}_\xi h(x, \xi)$, namely

$$\mathrm{Hess}_\xi h(x, \xi) \text{ is positive definite} \tag{27}$$

For simplicity, we shall consider only the simplest case of "bonne decomposition", i.e. $N = 2$, and the symbol for the partial differential operator $P(x, \partial_x)$ is

$$\begin{aligned} P(x, \xi) &= P_m(x, \xi) + P_{m-1}(x, \xi) + P_{m-2}(x, \xi) + \cdots \\ &= Q_0(x, \xi)[h(x, \xi)]^2 + Q_1(x, \xi)h(x, \xi) + Q_2(x, \xi) , \end{aligned} \tag{28}$$

where $\deg_\xi Q_j(x, \xi) = \lambda_j$, $j = 0, 1, 2$. $Q_0(x, \xi)$ and $Q_1(x, \xi)$ are homogeneous polynomials in ξ while $Q_2(x, \xi)$ is inhomogeneous in general. Let $\deg_\xi h(x, \xi) = m_1$, $m_1 \geq 2$, we have

$$m = \lambda_0 + 2m_1 , \tag{29}$$

$$m - 1 = \lambda_1 + m_1 , \tag{30}$$

$$m - 2 = \lambda_2 \tag{31}$$

hence

$$\begin{aligned} &\lambda_1 - \lambda_0 = m_1 - 1 , \\ &\lambda_2 - \lambda_0 = m - 2 - (m - 2m_1) = 2(m_1 - 1) \geq 2 , \end{aligned} \tag{32}$$

these relations are useful later.

Now we can state our main result.

THEOREM 1. *Assume that $P(x, \partial_x)$ has a "bonne decomposition" (28) (hence condition L is satisfied). Assume further the non-degeneracy condition (26) and (27). Then the linear holomorphic PDE (1) possesses an Hadamard fundamental solution of polynomial growth as (3).*

2. Asymptotic Expansion of the Hadamard Fundamental Solution

We are to prove our main result in the following order. First, assume that the solution should be sought in the following form:

$$u(x, y) = \sum_{h=0}^{\infty} U_h(x, y)[k(x, y)]^{p+h}/\Gamma(p + h + 1), \tag{33}$$

where y is the conic point of the characteristic conoid $k(x, y) = 0$ which is taken to be 0 without losing generality. Hence, we shall omit y in (33) in the sequel.

We are to decide U_0, U_h and p in the following. For that purpose, we substitute (33) into (1) and collect terms with the same power of k. U_0 should be bounded in a neighborhood of 0, and p is to be decided from this requirement. After U_0 and p are decided, we can decide U_h for $h = 1, 2, \ldots$ requiring also that they are bounded near $x = 0$. The equations defining $U_0, U_1 \ldots$ are the transport equations. In this and the next sections we are to give them. In the last two sections, we study these transport equations to give p and also prove the convergence of the series (33) obtained.

In order to make clear which terms after the action of the operators in (1) would give the same power of k, we should note two points.

The first is, the power of k in any term after substituting into equation (1) are influenced in two ways. The first comes from differentiation of $k^{p+h}/\Gamma(p + h + 1)$ in k, which diminishes the power of k^{p+h} by 1 and at the same time changes also the denominator. For instance

$$\frac{d}{dk}[k^{p+h}/\Gamma(p + h + 1)] = k^{p+h-1}/\Gamma(p + h) \tag{34}$$

without leaving any factor k outside this term. The second way is that when we combine the derivatives of k in x, i.e. $\partial_x^\beta k$, with the coefficients of $P_{m-j}(x, \xi)$ and use Euler's formula for homogeneous functions to simplify the results, as there are factors $h(x, \xi)$ or $h^2(x, \xi)$ in them and

$$\begin{aligned} h(x, k_x) &= \frac{k}{m_1 - 1} \\ [h(x, k_x)]^2 &= \left[\frac{k}{m_1 - 1}\right]^2 \end{aligned} \tag{35}$$

the factor $k^{p+h}/\Gamma(p+h+1)$ would become $k^{p+h+1}/\Gamma(p+h+1)$ and $k^{p+h+2}/\Gamma(p+h+1)$ changing only the numerators but not the denominator. But,

$$\begin{aligned} k^{p+h+1}/\Gamma(p + h + 1) &= (p + h + 1)k^{p+h+1}/\Gamma(p + h + 2), \\ k^{p+h+2}/\Gamma(p + h + 1) &= (p + h + 2)(p + h + 1)k^{p+h+2}/\Gamma(p + h + 3). \end{aligned} \tag{36}$$

We call them the Leray shifts of length 1 and 2 respectively, while in the first way, the Leray shift is 0. When there are Leray shifts, there would appear a factor of polynomial in h. We put this as a lemma.

LEMMA 2. *When the Leray shift of a term is 1 or 2, there would appear a factor of polynomial in h of degree 1 or 2; otherwise there would not appear any factor involving h in the computation.*

The second point is, as was proved in [1],

$$\sum_{j=1}^{n} h_{ij}(x, k_x)k_{jl} = \delta_{il} + O(t)\,,$$

and Morse lemma can be applied. Now we are to use the partition principle to simplify our results, i.e., we use the formula

$$D_x^{\beta} f(k) = \sum_{\Pi(\beta)} \left(\frac{d}{dk}\right)^{L} f(k)\partial^{\beta_1}k \cdots \partial^{\beta_L}k\,.$$

Among the subsets of β (denoted by $\beta_1, \ldots, \beta_L$), there are only L_1 singleton subsets and L_2 doubleton subsets. We also put this fact as a lemma which is useful.

LEMMA 3. *In any partition of β, there would appear only L_1 singleton subsets and L_2 doubleton subsets, etc., and*

$$\begin{gathered} L_1 + L_2 + \cdots + L_{|\beta|} = L\,, \\ L_1 + 2L_2 + \cdots + |\beta| L_{|\beta|} = m\,. \end{gathered} \tag{37}$$

Now we can start the computation. The general plan is as follows. Actually we are to compute terms of the form

$$A(x, \partial_x)U_h(x)k^{p+h}/\,\Gamma(p+h+1)\,. \tag{38}$$

From Leray [3], it can be proved that in any compact set in the domain $x\ :\ k(x) > 0 \cap x\ :\ \|x\| < \rho$ with ρ sufficiently small, there exist constants c and C such that

$$0 < c \leq |\partial^{\alpha} k| / \|x\|^{\frac{m}{m-1} - |\alpha|} \leq C\,.$$

In the following, we always assume this to be true.

For simplicity we denote $k^{(p+h)} = k^{p+h}/\,\Gamma(p+h+1)$. By applying Leibniz's formula,

$$\begin{aligned} A(x, \partial_x)U_h k^{p+h} =& U_h A(x, \partial_x)k^{(p+h)} \\ &+ \sum_{i=1}^{n} \frac{\partial U_h}{\partial x_i}\frac{\partial A}{\partial \xi_i}(x, \partial_x)k^{(p+h)} \\ &+ \frac{1}{2!}\sum_{i,j=1}^{n} \frac{\partial^2 U_h}{\partial x_i \partial x_j}\frac{\partial^2 A}{\partial \xi_i \partial \xi_j}(x, \partial_x)k^{(p+h)} \\ &+ \cdots \\ &+ \frac{1}{m!}\sum_{|\beta|=m} \frac{\partial^{\beta} U_0}{\partial x^{\beta}}\frac{\partial^{|\beta|} A}{\partial \xi^{\beta}}(x, \partial_x)k^{(p+h)}\,. \end{aligned} \tag{39}$$

The general plan is then divided into 4 steps:

I. Let $A(x,\partial_x) = P_m(x,\partial_x) = Q_0(x,\partial_x)[h(x,\partial_x)]^2$ and $h = 0$ (this h here is the sub-index of U_h which should not be confused with the function $h(x,\xi)$) and compute the 4 terms in (39).

II. Do the same for $A(x,\partial_x) = P_{m-1}(x,\partial_x) = Q_1(x,\partial_x)h(x,\partial_x)$, the sub-index $h = 0$ too.

III. Do the same for $A(x,\partial_x) = P_{m-2}(x,\partial_x) + \cdots = Q_2(x,\partial_x)$ which is inhomogeneous in ∂_x, $h = 0$.

IV. Repeat these steps above for further sub-indices $h > 0$.

The first step is carried out in this section and the remaining steps in the next section.

I. Computation of $P_m(x,\partial_x)U_0k^{(p)}$

As was done in [1], we write

$$
\begin{aligned}
P_m(x,\partial_x)U_0k^{(p)} &= \frac{1}{m!}\sum_{|\beta|=m} P_\beta(x)D^\beta(U_0k^{(p)})\\
&= \frac{1}{m!}\sum_{|\beta|=m} P_\beta(x)\sum_{\beta_1+\beta_2=\beta} D^{\beta_1}U_0D^{\beta_2}k^{(p)}\\
&= U_0P_m(x,\partial_x)k^{(p)} + \sum_{i=1}^{n}\frac{\partial U_0}{\partial x_i}\frac{\partial P_m}{\partial \xi_i}(x,\partial_x)k^{(p)}\\
&\quad + \frac{1}{2!}\sum_{i,j}\frac{\partial^2 U_0}{\partial x_i\partial x_j}\frac{\partial^2 P_m}{\partial\xi_i\partial\xi_j}(x,\partial_x)k^{(p)}\\
&\quad + \sum_{|\beta|\geq 3}\frac{1}{|\beta|!}D_x^\beta U_0 D_\xi^\beta P_m(x,\partial_x)k^{(p)}\\
&= T_1 + T_2 + T_3 + T_4\,.
\end{aligned}
\tag{40}
$$

A. $T_1 = U_0P_m(x,\partial_x)k^{(p)}$

Following [1], we have

$$
\begin{aligned}
P_m(x,\partial_x)k^{(p)} &= \frac{1}{m!}\sum_{|\beta|=m} P_\beta(x)D^\beta k(p)\\
&= \frac{1}{m!}\sum_{|\beta|=m}\sum_{L=1}^{m} P_\beta(x)k^{(p-L)}\partial^{\beta_1}k\cdots\partial^{\beta_L}k\,.
\end{aligned}
\tag{41}
$$

Here, $(\beta_1,\ldots,\beta_L)$ is a partition of β with partition number L and

$$
|\beta_1| + \cdots + |\beta_L| = m\,, \qquad |\beta_i| \geq 1\,. \tag{42}
$$

Consider separately those terms in (41) corresponding to $L = m$, $m-1$ and $m-2$.

For $L = m$, from (37) $L_1 = m$ and $L_2 = 0$ hence $|\beta_1| = \cdots = |\beta_L| = 1$ and

$$
\begin{aligned}
\{\text{terms in (41) for } L = m\} &= \frac{1}{m!}\sum_{|\beta|=m} P_\beta(x)\partial^{\beta_1}k\cdots\partial^{\beta_m}kk^{(p-m)}\\
&= P_m(x,k_x)k^{(p-m)}\\
&= \frac{Q_0(x,k_x)}{(m_1-1)^2}(p-m+1)(p-m+2)k^{(p-m+2)}
\end{aligned}
\tag{43}
$$

with Leray shift 2. Since it is assumed $Q_0(x,\xi)\neq 0$ in a conic neighborhood of (x_0,ξ_0) where $h(x_0,\xi_0)=0$ we may assume that $Q_0(x,k_x)\neq 0$.

For $L=m-1$, from (37), $L_1=m-2$ and $L_2=1$. We may select any two elements from $1,2,\dots,m$ to make this doubleton set. A typical selection is $\beta_1=i_1,\dots,\beta_{m-2}=i_{m-2},\beta_{m-1}=(i_{m-1},i_m)=(i,j)$ and there are $\frac{(m)_2}{2!}$ such selections with identical result. Hence

$$\begin{aligned}
&\{\text{terms in (41) for } L=m-1\}\\
&\quad=\frac{(m)_2}{2!}\frac{1}{m!}\sum_{|\beta|=m}P_\beta(x)k^{(p-m+1)}\partial^{\beta_1}k\cdots\partial^{\beta_{m-1}}k\\
&\quad=\frac{1}{2!(m-2)!}\sum_{|\beta|=m}P_{i_1\cdots i_m}(x)k_{i_1}\cdots k_{i_{m-2}}\frac{\partial^2 k}{\partial x_i\partial x_j}k^{(p-m+1)}\\
&\quad=\frac{1}{2!}\sum_{i,j}\partial^2_{i,j}P_m(x,k_x)\frac{\partial^2 k}{\partial x_i\partial x_j}k^{(p-m+1)}\,,
\end{aligned}\tag{44}$$

where as in [1], sub-indices always denote differentiation in fiber variables when applicable.

Now consider the Leray shift. We have

$$\begin{aligned}
\partial^2_{ij}P_m(x,\xi)&=\partial^2_{ij}Q_0(x,\xi)h^2(x,\xi)\\
&\quad+2h(x,\xi)\left[\frac{\partial Q_0}{\partial\xi_i}\frac{\partial h}{\partial\xi_j}+\frac{\partial Q_0}{\partial\xi_j}\frac{\partial h}{\partial\xi_i}+Q_0\frac{\partial^2 h}{\partial\xi_i\partial\xi_j}\right](x,\xi)\\
&\quad+2Q_0(x,\xi)h_i(x,\xi)h_j(x,\xi)\,.
\end{aligned}$$

So

$$\begin{aligned}
\partial^2_{ij}P_m(x,k_x)&=\frac{k^2}{(m_1-1)^2}\partial^2_{ij}Q_0(x,\xi)\\
&\quad+\frac{k}{m_1-1}[Q_{0i}h_j+Q_{0j}h_i+Q_0h_{ij}](x,k_x)\\
&\quad+2Q_0(x,k_x)t^2\frac{dx_i}{dt}\frac{dx_j}{dt}\,.
\end{aligned}\tag{45}$$

Substituting (45) into (44), the first term gives Leray shift 2 and the second term gives Leray shift 1. But we should be more careful in that we have seen in [1] that

$$\sum_{i,j=1}^n h_{ij}(x,k_x)\frac{\partial^2 k}{\partial x_i\partial x_j}=n+O(t)\,.$$

Further

$$\begin{aligned}
h_i(x,k_x)&=th_i(x,\xi)=t\frac{dx_i}{dt}\,,\\
Q_{0i}(x,k_x)&=Q_0(x,k_x)O(1)\,,
\end{aligned}$$

since we have $Q_0(x,k_x)\neq 0$. Substituting the second term into (44) gives $\dfrac{2k}{m_1-1}\,[n+O(t)]\cdot$ $\cdot k^{(p-m+1)}$.

For the third term, since from the Hamiltonian equation for x,

$$x_i = t\frac{dx_i}{dt} + o(t) = h_i(x, k_x) + o(t)\,.$$

The result of substitution into (44) can be proved to be

$$\sum_{i,j} Q_0(x, k_x)[k_{ij}x_ix_j + O(x^3)] = Q_0(x, k_x)k[1 + O(t)]\,.$$

Hence, the third term also gives Leray shift 1 and we have

$$\begin{aligned}
&\{\text{terms in (41) for } L = m-1\}\\
&\quad= \frac{1}{2(m_1-1)^2}\sum_{i,j} Q_{0ij}(x, k_x)k_{ij}(x)k^{p-m+3}/\,\Gamma(p-m+2)\\
&\quad\quad+\frac{Q_0(x,k_x)}{m_1-1}[(n+m_1-1)+O(t)]k^{p-m+2}/\,\Gamma(p-m+2)\\
&\quad= \frac{1}{2(m_1-1)^2}\left[\sum_{i,j} Q_{0ij}(x, k_x)k_{ij}(x)\right](p-m+2)(p-m+3)k^{(p-m+3)}\\
&\quad\quad+\frac{Q_0(x,k_x)}{m_1-1}[(n+m_1-1)+O(t)](p-m+2)k^{(p-m+2)}
\end{aligned}\tag{46}$$

with Leray shift 2 and 1 respectively.

Finally, we come to those **terms in (41) for** $L = m-2$. From (37) we have $L_1 = m-4$ and $L_2 = 2$. Let the doubleton sets be $\{i_1, i_2\}$, $\{i_3, i_4\}$, there are $\frac{(m)_2}{2!}\frac{(m-2)_2}{2!} = \frac{(m)_4}{4}$ different ways of selecting the doubleton sets, and Euler's formula gives

$$\{\text{terms in (41) for } L = m-2\} = \frac{1}{4}\sum P_{mi_1i_2i_3i_4}(x, k_x)k_{i_1i_2}k_{i_3i_4}k^{(p-m+2)}\,.\tag{47}$$

Consider now the Leray shift related to (47). Here $P_m(x,\xi) = iQ_0(x,\xi)[h(x,\xi)]^2$ is differentiated 4 times and then ξ is substituted by k_x. If $[h(x,\xi)]^2$ is not touched, Leray shift 2 would appear. If only 1 factor $h(x,\xi)$ is not touched, Leray shift 1 would appear. If there remains a factor of the form $h_i(x,\xi)$, then $h_i(x,k_x) = th_i(x,\xi) = t\frac{dx_i}{dt} = O(t)$ would also elevate the order of the term. Thus the lowest order term in (47) is

$$\frac{1}{2}Q_0(x,k_x)\sum(h_{i_1i_2}h_{i_3i_4} + h_{i_1i_3}h_{i_2i_4} + h_{i_1i_4}h_{i_2i_3})k_{i_1i_2}k_{i_3i_4}k^{(p-m+2)}\,.$$

But

$$\begin{aligned}
\sum h_{i_1i_2}h_{i_3i_4}k_{i_1i_2}k_{i_3i_4} &= \left(\sum_{i_1,i_2} h_{i_1i_2}k_{i_1i_2}\right)\left(\sum_{i_3,i_4} h_{i_3i_4}k_{i_3i_4}\right)\\
&= [n+O(t)][n+O(t)] = n^2 + O(t)\,,\\
\sum h_{i_1i_3}h_{i_2i_4}k_{i_1i_2}k_{i_3i_4} &= \sum_{i_2i_3i_4} h_{i_2i_4}k_{i_3i_4}\left(\sum_{i_1} h_{i_1i_3}k_{i_1i_2}\right)\\
&= \sum_{i_2i_3i_4} h_{i_2i_4}k_{i_3i_4}(\delta_{i_2i_3} + O(t))\\
&= n + O(t)\,.
\end{aligned}$$

Hence the lowest order term in (47) is

$$\frac{1}{2}Q_0(x,k_x)(n^2+2n+O(t))k^{(p-m+2)}\,.$$

Substitute this result into (43), we have

$$\begin{aligned}\{\text{terms in (41) for } L=m-2\} &= \frac{1}{2}Q_0(x,k_x)(n^2+2n+O(t))k^{(p-m+2)}\\ &\quad + O(1)k^{(p-m+3)} + O(1)k^{(p-m+4)}\,.\end{aligned}\tag{48}$$

The last 2 terms arise from Leray shift 1 and 2, hence may contain p in their coefficients up to at most its square.

For $L \le m-3$. Since $k^{(p-L)} = O(1)k^{(p-m+3)}$, we have

$$\{\text{terms in (41) for } L \le m-3\} = O(1)k^{(p-m+3)}\,.\tag{49}$$

$O(1)$ may contain p up to at most square.

Summing up (43), (46), (48), and (49) we have

LEMMA 4. *The expansion in powers of k for $U_0P_m(x,\partial_x)k^{(p)}$ is*

$$\begin{aligned}U_0(x)P_m(x,\partial_x)k^{(p)} &= U_0(x)Q_0(x,k_x)\Bigg[\frac{(p-m+1)(p-m+2)}{(m_1-1)^2}\\ &\quad + \frac{(p-m+2)(n(m_1-1)^2+m_1)}{(m_1-1)^2}\\ &\quad + \frac{n^2+2n}{2} + O(1)\Bigg]k^{(p-m+2)}\\ &\quad + U_0\sum_{i=3}^{m-1}P_m^i(x)k^{(p-m+i)}\end{aligned}\tag{50}$$

where the unspecified terms $O(1)$ and $P_m^i(x)$ may contain powers of p up to at most 2.

B. $T_2 = \displaystyle\sum_{i=1}^{n}\frac{\partial U_0}{\partial x_i}\frac{\partial p_m}{\partial \xi_i}(x,\partial_x)k^{(p)}$

This is much like the computation of T_1. We should only replace U_0 by $\frac{\partial U_0}{\partial x_i}$ and $P_m(x,\partial_x)$ by $\tilde{P}^{(i)}_{m-1}(x,\partial_x) = \frac{\partial P_m}{\partial \xi_i}(x,\partial_x)$ which is of order $m-1$ and

$$\tilde{P}^{(i)}_{m-1}(x,\partial_x) = Q_{0i}(x,\xi)[h(x,\xi)]^2 + 2Q_0(x,\xi)h(x,\xi)h_i(x,\xi)\,.$$

Thus, T_2 is splitted into two parts, i.e., $(Q_{0i}h^2)(x,\partial_x)$ and $2(Q_0hh_i)(x,\partial_x)$ acting on $k^{(p)}$ respectively. The computation in the first part is almost the same as that of T_1, only with $Q_{0i}(x,\xi)$ replacing $Q_0(x,\xi)$ and diminish the order m by 1.

Thus, denote

$$Q_{0i}(x,\partial_x)[h(x,\partial_x)]^2 = \frac{1}{(m-1)!}\sum_{|\beta|=m-1}A^{(i)}_\beta(x)D_x^\beta\,,$$

we also have

$$\sum_{i=1}^{n} Q_{0i}(x,\partial_x)[h(x,\partial_x)]^2 k^{(p)} \tag{51}$$
$$= \frac{1}{(m-1)!}\sum_{|\beta|=m-1} A_\beta^{(i)}(x) D_x^\beta k^{(p)}$$
$$= \frac{1}{(m-1)!}\sum_{|\beta|=m-1}\sum_{\Pi(\beta)} A_\beta^{(i)}(x)\partial^{\beta_1}k\cdots\partial^{\beta_L}k k^{(p-L)},$$

with

$$|\beta_1|+\cdots+|\beta_L| = m-1\,,\quad |\beta_i|\geq 1\,,\quad 1\leq L\leq m-1\,.$$

Computation as above gives

$$\begin{aligned}&\{\text{terms in (51) for } L = m-1\}\\ &\quad = Q_{0i}(x,k_x)h^2(x,k_x)k^{(p-m+1)}\\ &\quad = \frac{Q_{0i}(x,k_x)}{(m_1-1)^2}(p-m+2)(p-m+3)k^{(p-m+3)}\end{aligned} \tag{52}$$

$$\begin{aligned}&\{\text{terms in (51) for } L = m-2\}\\ &= \frac{1}{2}\sum_{j,l=1}^{n}\partial_{jl}^2[Q_{0i}(x,k_x)h^2(x,k_x)]\partial_{jl}^2 kk^{(p-m+2)}\\ &= \frac{1}{2}\frac{(p-m+3)(p-m+4)}{(m_1-1)^2}\left[\sum_{j,l=1}^{n}Q_{0ijl}(x,k_x)\partial_{jl}^2 k\right]k^{(p-m+4)}\\ &\quad +\frac{-(p-m+3)}{m_1-1}\sum_{j,l=1}^{n}[Q_{0i}h_{jl}+Q_{0ij}h_l+Q_{0il}h_j](x,k_x)\partial_{jl}^2 kk^{(p-m+3)}\\ &\quad +Q_{0i}(x,k_x)\sum_{j,l=1}^{n}h_j(x,k_x)h_l(x,k_x)\partial_{jl}^2 kk^{(p-m+2)}\\ &= O(1)k^{(p-m+3)}\,.\end{aligned} \tag{53}$$

The last term gives a Leray shift of 1 as in the proof of (46) and $O(1)$ is a polynomial in p of degree 2.

For the second part, denote

$$2Q_0(x,\partial_x)h(x,\partial_x)h_i(x,\partial_x)k^{(p)} = \tilde{A}_i(x,\partial_x)k^{(p)}\,,$$

where $\tilde{A}_i(x,\partial_x)$ is an order $m-1$ opertor, hence we have

$$\begin{aligned}&\tilde{A}_i(x,\partial_x)k^{(p)} = \frac{1}{(m-1)!}\sum_{|\beta|=m-1}\sum_{\Pi(\beta)}A_\beta^{(i)}(x)\partial^{\beta_1}k\cdots\partial^{\beta_L}kk^{(p-L)}\,,\\ &|\beta_1|+\cdots+|\beta_L| = m-1\,,\quad |\beta_i|\geq 1\,,\quad 1\leq L\leq m-1\,.\end{aligned} \tag{54}$$

Hence

$$
\begin{aligned}
\{\text{terms in (53) for } L = m-1\} &= \tilde{A}_i(x, k_x)k^{(p-m+1)} \\
&= \frac{2Q_0(x, k_x)}{m_1 - 1} k h_i(x, k_x) k^{(p-m+1)} \\
&= 2\frac{(p-m+2)}{m_1 - 1} Q_0(x, k_x) t \frac{dx_i}{dt} k^{(p-m+2)} .
\end{aligned} \tag{55}
$$

$$
\{\text{terms in (54) for } L = m-2\} = \frac{1}{2}\sum_{j,l} \partial^2_{jl}\tilde{A}_i(x, k_x)\partial^2_{jl} k k^{(p-m+2)} . \tag{56}
$$

Now

$$
\begin{aligned}
\partial^2_{jl}\tilde{A}_i(x, k_x) = \frac{2k}{m_1 - 1}&[Q_{0jl}h_i + Q_{0l}h_{ij} + Q_{0j}h_{il} + Q_0 h_{ijl}](x, k_x) \\
&+ 2[Q_{0j}h_i h_l + Q_{0l}h_i h_j + Q_0 h_i h_{jl} + Q_0 h_j h_{il} + Q_0 h_l h_{ij}](x, k_x)k .
\end{aligned}
$$

We also have by Euler's formula,

$$
h_l(x, k_x) = \frac{1}{m_1 - 1}\sum_{\lambda} h_{l\lambda}(x, k_x)k_\lambda ,
$$

and

$$
\begin{aligned}
\sum_l h_{l\lambda}(x, k_x)\partial^2_{jl} k &= \delta_{j\lambda} + O(t) , \\
\sum_l h_{il}(x, k_x)\partial^2_{jl} k &= \delta_{ij} + O(t) , \\
\sum_{j,l} h_{jl}(x, k_x)\partial^2_{jl} k &= n + O(t) .
\end{aligned}
$$

Using these formulas in (56), we obtain

$$
\begin{aligned}
&\{\text{terms in (54) for } L = m-2\}\\
&= \frac{k}{m_1-1}\left[\sum_{j,l}(Q_{0jl}h_i)(x,k_x)\partial_{jl}^2 k + \sum_{j,l}(Q_{0l}h_{ij})(x,k_x)\partial jl^2 k\right.\\
&\quad\left. + \sum_{j,l}(Q_{0j}h_{il})(x,k_x)\partial_{jl}^2 k + \sum_{j,l}(Q_0 h_{ijl})(x,k_x)\partial_{jl}^2 k\right] k^{(p-m+2)}\\
&\quad + \left[2\sum_{j,l}(Q_{0j}h_i h_j)(x,k_x)\partial_{jl}^2 k + 2\sum_{j,l} Q_0 h_j h_{il}(x,k_x)\partial_{jl}^2 k\right.\\
&\quad\left. + \sum_{j,l}(Q_0 h_i h_{jl})(x,k_x)\partial_{jl}^2 k\right] k^{(p-m+2)}\\
&= \frac{(p-m+3)}{m_1-1}\left[\sum_{j,l} Q_{0jl}(x,k_x)\partial_{jl}^2 k h_i(x,k_x) + 2Q_{0i}(x,k_x)\right. \qquad (57)\\
&\quad\left. + \sum_{j,l}(Q_0 h_{ijl})(x,k_x)\partial_{jl}^2 k + O(t)\right] k^{(p-m+3)}\\
&\quad + \frac{2}{m_1-1}\left[\sum (Q_{0j}h_i)(x,k_x)k_j + (m_1-1)(n+2)(Q_0 h_i)(x,k_x)\right.\\
&\quad\left. + O(t)h_i(x,k_x)\right] k^{(p-m+2)}\\
&= \frac{p-m+3}{m_1-1}\left[\sum_{j,l} Q_{0jl}(x,k_x)\partial_{jl}^2 k t\frac{dx_i}{dt} + 2Q_{0i}(x,k_x)\right.\\
&\quad\left. + \sum_{j,l}(Q_0 h_{ijl})(x,k_x)\partial_{jl}^2 k\right] k^{(p-m+3)}\\
&\quad + \frac{2}{m_1-1} t\frac{dx_i}{dt}[\lambda_0 Q_0(x,k_x) + (m_1-1)(n+2)Q_0(x,k_x)\\
&\quad + O(t)]k^{(p-m+2)}
\end{aligned}
$$

where l_0 is the degree of $Q_0(x,\xi)$ with respect to ξ and $O(t)$ may be a degree 2 polynomial in p.

$$\{\text{terms in (54) for } L \leq m-3\} = O(1)k^{(p-m+3)} \qquad (58)$$

with a degree 2 polynomial in p as $O(1)$.

Summing up (49), (57) and (58), we have

LEMMA 5. *Result of computation of T_2 in (40) is*

$$T_2 = \frac{Q_0(x,k_x)}{m_1-1}[2\lambda_0 + (m_1-1)(n+2) + O(t)]t\frac{dU_0}{dt}k^{(p-m+2)} + \sum_{i=1}^{m-3} Q^{(i)}\left(p, x, \frac{\partial U_0}{\partial x}\right)k^{(p-m+2+i)} \tag{59}$$

where $Q^{(i)}$ is linear in $\frac{\partial U_0}{\partial x}$, and quadratic in p, $O(t)$ is also quadratic in p.

C. Now we come to the computation of T_3

$$T_3 = \frac{1}{2}\sum_{i,j}\frac{\partial^2 U_0}{\partial x_i \partial x_j}\frac{\partial^2 P_m}{\partial \xi_i \partial \xi_j}(x, \partial_x)k^{(p)}\,.$$

First,

$$\frac{\partial^2 P_m(x,\xi)}{\partial \xi_i \partial \xi_j} = (Q_{0ij}h^2)(x,\xi) + 2h[(Q_{0i}h_j)(x,\xi) + (Q_{0j}h_i)(x,\xi) + (Q_0 h_{ij})(x,\xi)] + 2(Q_0 h_i h_j)(x,\xi)\,.$$

Denote the first term of the right hand side above which is of degree $m-2$ in ξ by

$$\tilde{A}_{m-2}(x,\partial_x) = Q_{0ij}(x,\partial_x)h^2(x,\partial_x) = \frac{1}{(m-2)!}\sum_{|\beta|=m-2}\tilde{A}_\beta(x)\partial_x^\beta\,,$$

we have

$$\tilde{A}_{m-2}(x,\partial_x)k^{(p)} = \frac{1}{(m-2)!}\sum_{|\beta|=m-2}\sum_{\Pi(\beta)}\tilde{A}_\beta(x)\partial^{\beta_1}k\cdots\partial^{\beta_L}kk^{(p-L)}\,,$$

here $1 \le L \le m-2$. For $L = m-2$, we obtain

$$\tilde{A}_{m-2}(x,k_x)k^{(p-m+2)} = \frac{(p-m+3)(p-m+4)}{(m_1-1)^2}Q_{0ij}(x,k_x)k^{(p-m+4)}\,.$$

For $L \le m-3$, we obtain terms of order $O(t)k^{(p-m+3)}$ with $O(t)$ a polynomial of p of degree 2.

The second term in the expression of $\frac{\partial^2 P_m(x,\xi)}{\partial \xi_i \partial \xi_j}$ is an operator also of degree $m-2$

$$\tilde{A}_{m-2}(x,\partial_x) = 2h(x,\partial_x)[(Q_{0i}h_j) + (Q_{0j}h_i) + (Q_0 h_{ij})](x,\partial_x)\,.$$

We also have

$$\tilde{A}_{m-2}(x,\partial_x)k^{(p)} = O(1)k^{(p-m+3)}\,.$$

What is most remarkable is the contribution of the term $2Q_0(x,\partial_x)h_i(x,\partial_x)h_j(x,\partial_x)$ to T_3, which can be written as

$$2Q_0(x,k_x)h_i(x,k_x)h_j(x,k_x)k^{(p-m+2)} + O(1)k^{(p-m+3)}\,.$$

Since

$$h_i(x,k_x) = th_i(x,\xi) = t\frac{dx_i}{dt}\,,$$

we can sum up the results above to give

LEMMA 6. *T_3 in (40) can be expanded as*

$$\frac{1}{2}\sum_{i,j}\frac{\partial^2 U_0}{\partial x_i \partial x_j}\frac{\partial^2 P_m}{\partial \xi_i \partial \xi_j}(x,\partial_x)k^{(p)} = Q_0(x,k_x)t^2\frac{dx_i}{dt}\frac{dx_j}{dt}\frac{\partial^2 Q_0}{\partial x_i \partial x_j}k^{(p-m+2)} + O(1)k^{(p-m+3)} \tag{60}$$

where the unspecified $O(1)$ can be quadratic in p.

D. The last term T_4 in (40) is

$$\sum_{|\beta|\geq 3}\partial_x^\beta U_0 \partial_\xi^\beta P_m(x,\partial_x)k^{(p)} = \sum_{|\beta|\geq 3}\sum_{i=0}^{m-|\beta|} T_{\beta,i}(p,x,\partial^\beta U_0)k^{(p-m+|\beta|+i)}\,, \tag{61}$$

where $T_{\beta,i}$ are quadratic in p.

The final result of step I is

PROPOSITION 1. *The result of the first term in de Paris' decomposition $Q_0(x,\partial_x)\cdot[h(x,\partial_x)]^2$ acting on $U_0k^{(p)}$ is*

$$\begin{aligned} Q_0(x,k_x)\Big\{t^2\sum_{i,j}\frac{\partial^2 U_0}{\partial x_i\partial x_j}\frac{dx_i}{dt}\frac{dx_j}{dt} &+ \frac{1}{m_1-1}[p+\lambda_0+(m_1-1)n+O(t)]t\frac{dU_0}{dt} \\ &+ [\Phi(p)+O(t)]U_0\Big\}k^{(p-m+2)} \\ &+ \sum_{i=3}^{m} P_m^{(i)}(p,x,U_0,\ldots,\partial^i U_0)k^{(p-m+i)}\,, \end{aligned} \tag{62}$$

with

$$\Phi(p) = \frac{(p-m+1)(p-m+2)}{(m_1-1)^2} + \frac{(p-m+2)(n(m_1-1)^2+m_1)}{(m_1-1)^2} + \frac{n^2+2n}{2}\,, \tag{63}$$

$P_m^{(i)}$ and unspecified $O(1)$ can be square of p.

3. Further Computation for the Asymptotic Expansion (33)

Step I is accomplished as above, now we go on for the steps II, III and IV in this section. Since the principle is the same as before, we proceed more briefly.

II is the result of the action of the second term in de Paris' decomposition $P_{m-1}(x,\partial_x) = Q_1(x,\partial_x)h(x,\partial_x)$ on $U_0k^{(p)}$. The orders λ_1 of $Q_1(x,\partial_x)$ and m_1 of $h(x,\partial_x)$ satisfy the relation

$$m-1 = \lambda_1 + m_1\,, \qquad \lambda_1 - \lambda_0 = m_1 - 1 \geq 1\,.$$

We proceed as in section 1, thus

$$
\begin{aligned}
P_{m-1}(x,\partial_x)U_0k^{(p)} &= U_0P_{m-1}(x,\partial_x)k^{(p)} \\
&\quad+\sum_{i=1}^{n}\frac{\partial U_0}{\partial x_i}\frac{\partial P_{m-1}}{\partial \xi_i}(x,\partial_x)k^{(p)} \\
&\quad+\frac{1}{2!}\sum i,j\,\frac{\partial^2 U_0}{\partial x_i\partial x_j}\frac{\partial^2 P_{m-1}}{\partial \xi_i\partial \xi_j}(x,\partial_x)k^{(p)} \\
&\quad+\sum_{|\beta|=3}^{m-1}\frac{1}{|\beta|!}\partial_x^{\beta}U_0\partial_\xi^{\beta}P_{m-1}(x,k_x)k^{(p)} \\
&= R_1+R_2+R_3+R_4\,.
\end{aligned}
\tag{64}
$$

A. First we compute R_1.

$$
P_{m-1}(x,\partial_x)k^{(p)}=\frac{1}{(m-1)!}\sum|\beta|=m-1\sum_{\Pi(\beta)}A_\beta(x)\partial^{\beta_1}k\cdots\partial^{\beta_L}kk^{(p-L)}\,.
\tag{65}
$$

The terms for $L=m-1$ are

$$
\begin{aligned}
P_{m-1}(x,k_x)k^{(p-m+1)} &= \frac{p-m+2}{m_1-1}Q_1(x,k_x)k^{(p-m+2)} \\
&= \frac{p-m+2}{m_1-1}Q_0(x,k_x)\frac{Q_1(x,k_x)}{Q_0(x,k_x)}k^{(p-m+2)} \\
&= \frac{p-m+2}{m_1-1}Q_0(x,k_x)O(\tau^{\lambda_1-\lambda_0})k^{(p-m+2)} \\
&= Q_0(x,k_x)O(t)k^{(p-m+2)}\,,
\end{aligned}
\tag{66}
$$

with coefficients possibly contain p^2.

The terms for $L=m-2$ are

$$
\frac{1}{2}\sum_{i,j}\frac{\partial^2 P_{m-1}}{\partial\xi_i\partial\xi_j}(x,k_x)\partial_{ij}^2kk^{(p-m+2)}\,.
\tag{67}
$$

But

$$
\begin{aligned}
\frac{\partial^2 P_{m-1}}{\partial\xi_i\partial\xi_j}(x,k_x) &= Q_{1ij}(x,k_x)h(x,k_x) \\
&\quad+(Q_{1i}h_j)(x,k_x)+(Q_{1j}h_i)(x,k_x)+(Q_1h_{ij})(x,k_x) \\
&= \frac{Q_{1ij}(x,k_x)}{m_1-1}k+\frac{Q_{1i}(x,k_x)}{m_1-1}\sum_{l=1}^{n}h_{jl}(x,k_x)k_l \\
&\quad+\frac{Q_{1j}(x,k_x)}{m_1-1}\sum_{l=1}^{n}h_{il}(x,k_x)k_l+(Q_1h_{ij})(x,k_x)\,.
\end{aligned}
$$

Substituting into (67) gives

$$
\begin{aligned}
&\frac{1}{2}\sum_{i,j}\frac{\partial^2 P_{m-1}}{\partial\xi_i\partial\xi_j}(x,k_x)\partial^2_{ij}kk^{(p-m+2)}\\
&\quad=\frac{(p-m+3)}{2(m_1-1)}\sum_{i,j}Q_{1ij}(x,k_x)k_{ij}k^{(p-m+3)}\\
&\quad\quad+\sum_i\left[\frac{Q_{1i}(x,k_x)}{m_1-1}k_i+\frac{1}{2}Q_1(x,k_x)(n+O(t))+O(\tau^{\lambda_1-1}t)\right]k^{p-m+2)}\\
&\quad=\frac{p-m+3}{2(m_1-1)}\left(\sum_{i,j}Q_{1ij}(x,k_x)k_{ij}\right)k^{(p-m+3)}\\
&\quad\quad+Q_1(x,k_x)\left[\frac{\lambda_1}{m_1-1}+\frac{n}{2}\right]+O(\tau^{\lambda_1-1}t)k^{(p-m+2)}\\
&\quad=\frac{p-m+3}{2(m_1-1)}\left(\sum_{i,j}Q_{1ij}(x,k_x)k_{ij}\right)k^{(p-m+3)}+Q_0(x,k_x)O(t)k^{(p-m+2)}\,.
\end{aligned}\tag{68}
$$

Terms for $L\le m-3$ are

$$
\sum_{|\beta|=3}^{m-1}P_{m-1,\beta}(p,x,U_0)k^{(p-m+|\beta|)}
$$

The final result for R_1 is

$$
Q_0(x,k_x)[O(t)U_0k^{(p-m+2)}+\sum_{|\beta|=3}^{m-1}P_{m-1,\beta}(p,x,U_0)k^{(p-m+|\beta|)}\tag{69}
$$

B. Next, compute R_2. Notice that

$$
\frac{\partial P_{m-1}(x,\xi)}{\partial\xi_i}=Q_{1i}(x,\xi)h(x,\xi)+Q_1(x,\xi)h_i(x,\xi)=\tilde P_{m-2}(x,\xi)
$$

which is of degree $m-2$ in ξ, we see

$$
\begin{aligned}
\frac{\partial P_{m-1}(x,\partial_x)}{\partial\xi_i}k^{(p)}&=\frac{1}{(m-2)!}\sum_{|\beta|=m-2}\sum_{\Pi(\beta)}D^\beta_\xi\tilde P_{m-2}(x,\xi)\partial^{\beta_1}k\cdots\partial^{\beta_L}kk^{(p-L)}\\
&\qquad 1\le L\le m-2\,.
\end{aligned}\tag{70}
$$

Hence

$$
\begin{aligned}
\{\text{terms in (70) for } L=m-2\}&=\tilde P_{m-2}(x,k_x)k^{(p-m+2)}\\
&=\frac{(p-m+3)}{m_1-1}Q_{1i}(x,k_x)k^{(p-m+3)}\\
&\quad+Q_1(x,k_x)h_i(x,k_x)k^{(p-m+2)}\\
&=Q_0(x,k_x)\frac{Q_1(x,k_x)}{Q_0(x,k_x)}t\frac{dx_i}{dt}k^{(p-m+2)}\\
&\quad+\frac{p-m+3}{m_1-1}Q_{1i}(x,k_x)k^{(p-m+3)}\,.
\end{aligned}
$$

Corresponding terms in R_2 are

$$Q_0(x, k_x)O(t)t\frac{dU_0}{dt}k^{(p-m+2)} + \frac{p-m+3}{m_1-1}\sum_{i=1}^{n} Q_{1i}(x, k_x)\frac{\partial U_0}{\partial x_i}k^{(p-m+3)}, \tag{71}$$

here use is made of the fact $\lambda_1 - \lambda_0 = m_1 - 1$.

Terms for $L \leq m-3$ can be summed up as

$$\sum_{|\beta|=3}^{m-1} \tilde{Q}_{m-1,\beta}(p, x, \partial Q_0)k^{(p-m+|\beta|)}.$$

Hence, we have

$$\begin{aligned} R_2 = {} & Q_0(x, k_x)O(t)t\frac{dU_0}{dt}k^{(p-m+2)} \\ & + \sum_{|\beta|=3}^{m-1} Q_{m-1,\beta}(p, x, \partial U_0)k^{(p-m+|\beta|)}. \end{aligned} \tag{72}$$

Here $Q_{m-1,\beta}(p, x, \partial U_0)$ is a differential operator of order 1 in U_0 with coefficients polynomials in p to degree 2.

C. R_3 **and** R_4 can be combined as

$$R_3 + R_4 = \sum_{|\beta|=4}^{m-1} R_{m-1,\beta}(p, x, U_0, \ldots, \partial^\beta U_0)k^{(p-m+|\beta|)}. \tag{73}$$

Thus we have proved the

PROPOSITION 2. *The second term $Q_1(x, \partial_x)h(x, \partial_x) = P_{m-1}(x, \partial_x)$ of de Paris' decomposition acting on $U_0k^{(p)}$ would give*

$$\begin{aligned} P_{m-1}(x, \partial_x)U_0h^{(p)} = {} & Q_0(x, k_x)\left[O(t)U_0 + O(t)t\frac{dU_0}{dt}\right]k^{(p-m+2)} \\ & + \sum_{i=3}^{m-1} P_{m-1,i}(p, x, U_0, \ldots, \partial^i U_0)k^{(p-m+i)} \end{aligned} \tag{74}$$

$P_{m-1,i}$ is a polynomial in p of degree 2.

III. Now we can complete step III, i.e., to compute the result of the third term $P_{m-2}(x, \partial_x)$ which is an $(m-2)th$ order inhomogeneous PDO acting on $U_0k^{(p)}$. Computation as above gives

PROPOSITION 3.

$$\begin{aligned} P_{m-2}(x, \partial_x)U_0k^{(p)} = {} & Q_0(x, k_x)O(t^2)U_0k^{(p-m+2)} \\ & + \sum_{i=3}^{m-2} P_{m-2,i}(p, x, U_0, \ldots, \partial^i U_0)k^{(p-m+i)}. \end{aligned} \tag{75}$$

Summing up Prop. 1 (62) Prop. 2 (74) and Prop. 3 (75), we have

PROPOSITION 4.

$$
\begin{aligned}
P(x,\partial_x)U_0k^{(p)} = Q_0(x,k_x)\Big\{&t^2\sum_{i,j}\frac{\partial^2U_0}{\partial x_i\partial x_j}\frac{dx_i}{dt}\frac{dx_j}{dt}\\
&+\frac{1}{m_1-1}[p+\lambda_0+(m_1-1)n+O(t)]t\frac{dU_0}{dt}\\
&+[\Phi(p)+O(t)]U_0\Big\}k^{(p-m+2)}\\
&+\sum A_i^{(0)}(p,x,U_0,\ldots,\partial^iU_0)k^{(p-m+i)},
\end{aligned} \tag{76}
$$

where $A_i^{(0)}$ is a PDO in U_0 of order i with coefficients holomorphic in x and polynomial in p of degree 2 and

$$
\Phi(p)=\frac{(p-m+1)(p-m+2)}{(m_1-1)^2}+\frac{(p-m+2)(n(m_1-1)^2+m_1)}{(m_1-1)^2}+\frac{n^2+2n}{2}. \tag{77}
$$

IV. We shall continue to consider the action of $P(x,\partial_x)$ on further terms $U_hk^{(p+h)}$ in (33). Results should be the same as (76) with only p and U_0 replaced by $p+h$ and U_h respectively:

$$
\begin{aligned}
P(x,\partial_x)U_hk^{(p+h)} = U_0(x,k_x)\Big\{&t^2\sum_{i,j}\frac{\partial^2U_h}{\partial x_i\partial x_j}\frac{dx_i}{dt}\frac{dx_j}{dt}\\
&+\frac{1}{m_1-1}[p+h+\lambda_0+(m_1-1)(n-2)+O(t)]t\frac{dU_h}{dt}\\
&+[\Phi(p+h)+O(t)]U_h\Big\}k^{(p+h-m+2)}\\
&+\sum_{i=3}^{m}A_i^{(h)}(p+h,x,U_h,\ldots,\partial^iU_h)k^{(p+h-m+i)}.
\end{aligned} \tag{78}
$$

Summing (76) and (78) (for various h) and collect terms of the same powers of k, we have the basic result in formal series

$$
\begin{aligned}
P(x,\partial_x)\left[\sum_{h=0}^{\infty}U_hk^{(p+h)}\right] = Q_0(x,k_x)\sum_{h=0}^{\infty}\{&T_h(p+h,x,t\partial_t,\partial_x)U_h\\
&+B_h(p+h,x,U_{h-1},\ldots,U_{h-m})\}k^{(p+h-m+2)},
\end{aligned} \tag{79}
$$

where

$$
\begin{aligned}
T_hU_h \equiv\ &t^2\sum_{ij}\frac{\partial^2U_h}{\partial x_i\partial x_j}\frac{dx_i}{dt}\frac{dx_j}{dt}\\
&+\frac{1}{m_1-1}[p+h+\lambda_0+(m_1-1)n+O(t)]t\frac{dU_h}{dt}\\
&+[\Phi(p+h)+O(t)]U_h,
\end{aligned} \tag{80}
$$

B_h are PDO in $U_{h-1},\ldots,U_{h-m}$ of order i with respect to U_{h-i}, and U_{h-i} should be supplied by 0 for $i>h$, the coefficients of B_h are holomorphic in x and polynomial in p of degree at most 2.

The subsequent task is to determine p and U_h and prove the convergence of series (33).

4. Study of the Transport Equations

The transport equation for U_0, i.e. equation $T_h U_h = 0$ (see (79) for $h = 0$),

$$t^2 \sum_{i,j} \frac{\partial^2 U_0}{\partial x_i \partial x_j} \frac{dx_i}{dt} \frac{dx_j}{dt} + \frac{1}{m_1 - 1}[p + \lambda_0 + (m_1 - 1)n + O(t)] t \frac{\partial U_0}{\partial t} + [\Phi(p) + O(t)] U_0 = 0 , \tag{81}$$

where

$$\Phi(p) = \frac{(p - m + 1)(p - m + 2)}{(m_1 - 1)^2} + \frac{(p - m + 2)(n(m_1 - 1)^2 + m_1)}{(m_1 - 1)^2} + \frac{n^2 + 2n}{2} , \tag{82}$$

is a Fuchsian type PDE as considered by M.Y. Chi [8], [9] and M.S. Baouendi and Goulaouic [10]. In fact, we have

$$\begin{aligned} t^2 \sum_{i,j} \frac{\partial^2 U_0}{\partial x_i \partial x_j} \frac{dx_i}{dt} \frac{dx_j}{dt} &= \sum_j t \frac{dx_j}{dt} \left(\sum_i t \frac{\partial}{\partial x_i} \left(\frac{\partial U_0}{\partial x_j} \right) \frac{dx_i}{dt} \right) \\ &= \sum_j t \frac{dx_j}{dt} \left[t \frac{d}{dt} \left(\frac{\partial U_0}{\partial x_j} \right) \right] \\ &= t \frac{d}{dt} \left(\sum_j t \frac{dx_j}{dt} \frac{\partial U_0}{\partial x_j} \right) - t^2 \sum_j \frac{\partial U_0}{\partial x_j} \frac{d^2 x_j}{dt^2} . \end{aligned}$$

Substituting into (81) gives the transport equation for U_0 in the form

$$\left(t \frac{\partial}{\partial t} \right)^2 U_0 + \frac{1}{m_1 - 1}[p + \lambda_0 + (m_1 - 1)n + O(t)] \left(t \frac{\partial}{\partial t} \right) U_0 + [\Phi(p) + O(t)] U_0 - t^2 \sum_j \frac{\partial U_0}{\partial x_j} \frac{d^2 x_j}{dt^2} = 0 . \tag{83}$$

It is easy to see now that the transport equation becomes a PDE when the curvature of the bicharacteristic plays a role. If the latter are straight lines, i.e., $\frac{d^2 x_j}{dt^2} = 0$, which is the case when $h(x, \xi)$ is a constant coefficient polynomial in ξ, the transport equation (83) is still an ODE.

To see that (83) is of the Fuchsian type studied in [8], [9] and [10], we should make a transformation of variables.

Since

$$\mathrm{Hess}_\xi h(x, \xi) \neq 0 , \quad \xi \neq 0 ,$$

and Euler's formula gives

$$(m_1 - 1) h_{\xi_i} = \sum_j h_{ij} \xi_j ,$$

we see $\mathrm{grad}_\xi h(x, \xi) \neq 0$ for $\xi \neq 0$, hence we may assume that

$$\begin{aligned} &h_{\xi_1}(x, \xi) \neq 0 , && \xi \neq 0 . \\ &(\text{or } h_1(x, k_x) = t h_1(x, \xi) \neq 0 , && t \neq 0) . \end{aligned}$$

Here and after, the condition $\xi \neq 0$ is satisfied when t is small enough since

$$\xi = \eta - t h_x(0, \eta) + o(t)\,,$$
$$g(0, \eta) = 0\,, \quad \eta \neq 0\,,$$

hence $h_{\xi_1}(x, \xi) \neq 0$ is always satisfied. Now, for any function f

$$t\frac{\partial f}{\partial t} = \sum_i t\frac{\partial f}{\partial x_i}\frac{dx_i}{dt} = \sum_i \frac{\partial f}{\partial x_i} h_i(x, k_x)\,,$$
$$\frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial x_j}\,, \quad j = 2, \ldots, n,$$

the Jacobian for the right-hand side is

$$h_1(x, k_x) = t h_1(x, \xi)\,,$$

hence $\frac{\partial f}{\partial x_i}$ can be expressed linearly in $\left(t\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$ with holomorphic coefficients. Thus the last term of (83) can be written as

$$-t\sum_j \left(t\frac{\partial U_0}{\partial x_j}\right)\frac{d^2 x_j}{dt^2} = t^2 A\left(t, x, \frac{\partial}{\partial x}, t\frac{\partial}{\partial t}\right) U_0\,,$$

$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$. Further, use (t, x) to denote $(t, x_2, \ldots, x_n)$, (83) can be reduced to the form

$$\left(t\frac{\partial}{\partial t}\right)^2 U_0 + \frac{1}{m_1 - 1}[p + \lambda_0 + (m_1 - 1)n + O(t)]t\frac{\partial U_0}{\partial t} + [\Phi(p) + O(t)]U_0 + tA\left(t, x, \frac{\partial}{\partial x}, t\frac{\partial}{\partial t}\right) U_0 = 0 \tag{84}$$

which is certainly of Fuchsian type.

When studying Fuchsian type PDE, one should first consider its “indicial equation”, which is

$$\lambda^2 + \frac{1}{m_1 - 1}[p + \lambda_0 + (m_1 - 1)n]\lambda + \Phi(p) = 0 \tag{85}$$

at present. In order to obtain a solution U_0 bounded near the vertex of the characteristic conoid $x = 0$, i.e., $t = 0$, the indicial equation (85) should have a root $\lambda = 0$. Hence we should take p to be a root of $\Phi(p) = 0$, or

$$\frac{(p - m + 1)(p - m + 2)}{(m_1 - 1)^2} + \frac{(n(m_1 - 1)^2 + m_1)(p - m + 2)}{(m_1 - 1)^2} + \frac{n^2 + 2n}{2} = 0\,,$$

which has 2 roots p_1 and p_2 assumed to have non-integral difference. Set $p = p_1$ for instance, (85) has 2 roots:

$$\lambda_1^{(1)} = 0\,, \quad \lambda_1^{(2)} = -\frac{p + \lambda_0 + (m_1 - 1)n}{m_1 - 1}\,. \tag{86}$$

Corresponding to $\lambda_1^{(1)} = 0$, we can obtain a solution U_0 for (83) which is holomorphic in a domain Ω and bounded at $t = 0$.

REMARK 1. The indicial equation (85) has another root $\lambda^{(2)}$ which would give another solution

$$U_0 = t^{\lambda^{(2)}} V_0(t, x) .$$

Substituting into (83) gives an equation for V_0:

$$\left(t\frac{\partial}{\partial t}\right)^2 V_0 + \frac{1}{m_1 - 1}[p + \lambda + (m_1 - 1)n]\left(t\frac{\partial}{\partial t}\right) V_0 + t^2 \bar{A}(t, x, \partial_x, t\partial_t) V_0 = 0 ,$$

and V_0 is a holomorphic solution of this equation. Thus, for every root p of $\Phi(p) = 0$, there correspond 2 solutions (33) for (1). Since $k = 0$ at $t = 0$, the order of these solutions are different at $t = 0$.

Totally, we should obtain four solutions of (1) with asymptotic expansion (33) of different order at $t = 0$. Of course this conclusion is justified only when the convergence is proved. This proof would be given later, but since the proofs are the same for these four solutions, we shall give only one of them, namely, the one for $p = p_1$ and $\lambda^{(1)} = 0$.

Similarly, for U_h ($h > 0$), we have

$$\begin{aligned} &\left(t\frac{\partial}{\partial t}\right)^2 U_h + \frac{1}{m_1 - 1}[p + h + \lambda_0 + (m_1 - 1)n]\left(t\frac{\partial}{\partial t}\right) U_h \\ &\quad + \Phi(p + h)U_h + t^2 A\left(t, x, \frac{\partial}{\partial x}, t\frac{\partial}{\partial t}\right) U_h \\ &\quad = \sum_{i=1}^{m} B_{hi}\left(p + h, t, x, \frac{\partial}{\partial x}, t\frac{\partial}{\partial t}\right) U_{h-i} . \end{aligned} \tag{87}$$

We are to solve them inductively in h. When $U_0, U_1, \ldots, U_{h-1}$ are founded to be holomorpfic in Ω and bounded at $t = 0$, (87) becomes inhomogeneous Fuchsian type PDE with holomorphic right-hand side.

Since $\Phi(p + h) \neq 0$ for $h > 0$, we can find U_h in the form

$$U_h(t, x) = V_h^{(0)}(x) + V_h^{(1)}(x)t + \cdots + V_h^{(N-1)}(x)t^{N-1} + V_h(t, x) , \tag{88}$$

where $V_h^{(0)}(x), \ldots, V_h^{(N-1)}(x)$ are known and $V_h = O(t^N)$ is a holomorphic function to be decided.

As mentioned in [9], $V_h(t, x)$ should be a solution for the integro-differential equation

$$\begin{aligned} V_h(t, x) = &-\int_0^1\int_0^1 \sigma_1^{-\lambda_1 - 1}\sigma_2^{-\lambda_2 - 1}\left[tA\left(t, x, \frac{\partial}{\partial x}, t\frac{\partial}{\partial t}\right) V_h\right]_{t\to\sigma_1\sigma_2 t} d\sigma_1 d\sigma_2 \\ &+ \int_0^1\int_0^1 \sigma_1^{-\lambda_1 - 1}\sigma_2^{-\lambda_2 - 1}[t^N g(t, x)]_{t\to\sigma_1\sigma_2 t} d\sigma_1 d\sigma_2 , \end{aligned} \tag{89}$$

where λ_1, λ_2 are the roots of the equation

$$\lambda^2 + \frac{1}{m_1 - 1}[p + h + \lambda_0 + (m_1 - 1)n]\lambda + \Phi(p + h) = 0 ;$$

and $g(t, x)$ is known to be holomorphic in Ω. In [9], it is proved that there exists a unique solution $V_h(t, x)$ holomorphic in Ω which is of order t^N, where N is large enough, rendering all the integrals in (89) convergent.

Summing up, we have constructed a formal asymptotic solution (33) for the equation (1), with all coefficients U_h holomorphic in the same domain Ω, what remains is to prove its convergence.

5. Convergence of the Formal Solution

We are to prove convergence of the formal asymptotic solution (33) by majoration method. Before that we should note that the transport equation $T_0 U_0 = 0$ is a homogeneous Fuchsian type PDE with $\lambda = 0$ as one of its indicial roots. Hence, when seeking $U_0(t, x)$ as a holomorphic solution of it, one single initial condition is required and we may assign arbitrary value to $U_0(0, x)$. As for U_h, $h > 0$, $T_h U_h = -B_h(p + h, x, U_{h-1}, \ldots, U_{h-m})$ is non-homogeneous, and since $\Phi(p + h) \neq 0$ for $h > 0$, $\lambda = 0$ is no longer an indicial root, hence in deciding $U_h(t, x)$ to be holomorphic in Ω and bounded at $t = 0$, no initial condition is requird. Before we come to the convergence proof, we should check these equations more closely.

By introducing $t = \epsilon\tau$, we can add a factor ϵ to A, hence we replace $A\left(t, x, \frac{\partial}{\partial x}, t\frac{\partial}{\partial t}\right)$ by $\epsilon A\left(t, x, \frac{\partial}{\partial x}, t\frac{d}{dt}\right)$ afterwards. Set $z = t + x_2 + \cdots + x_n$, we can find a majorant operator for A:

$$\frac{\epsilon M_1}{1 - z/r}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_2} + \cdots + \frac{\partial}{\partial x_n}\right) = \frac{\epsilon n M_1}{1 - z/r}\frac{\partial}{\partial z}, \tag{90}$$

here the factor t of $t\frac{\partial}{\partial t}$ is absorbed into the coefficients and r is a suitable positive constant.

For coefficients of B_h:

$$B_h(p + h, t, x, U_{h-1}, \ldots, U_{h-m}) = \sum_{i=1}^{m} B_{hi}\left(p + h, t, x, \frac{\partial}{\partial x}, t\frac{\partial}{\partial t}\right) U_{h-i},$$

since the coefficients may contain atmost square of $p + h$, and B_{hi} are PDO's of order i, we also can choose corresponding majorant operators of B_{hi} as

$$\frac{h^2 M_2}{1 - z/r}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_2} + \cdots + \frac{\partial}{\partial x_n}\right)^i = \frac{h^2 n^i M_2}{1 - z/r}\left(\frac{\partial}{\partial z}\right)^i.$$

Hence for the transport equation $T_h U_h = \sum_{i=1}^{m} B_{hi} U_{h-i}$, we choose

$$\begin{aligned} &\left(t\frac{\partial}{\partial t}\right)^2 W_h + \frac{1}{m_1 - 1}[p + h + \lambda_0 + (m_1 - n)]\left(t\frac{\partial}{\partial t}\right) W_h \\ &\quad + \Phi(p + h) W_h + \frac{\epsilon_n M_1 t^2}{1 - z/r}\left(\frac{\partial}{\partial z}\right) W_h \\ &\quad = \frac{h^2 M_2}{1 - z/r}\sum_{i=1}^{m} n^i \left(\frac{d}{dz}\right)^i W_{h-i} \end{aligned} \tag{91}$$

as majorant equation.

Now, choose $W_0, W_1, \ldots, W_{h-1}$ to be majorants of $U_0, U_1, \ldots$ and U_{h-1} respectively, we are now to find a suitable majorant for U_h. Since all U_h are holomorphic in the same domain Ω, we can choose r which appears in the majorant for the coefficients such that U_h has a majorant of the form

$$\frac{A_h}{(1-z/r)^{h+1}} \tag{92}$$

Note that we are to use function of the type $f(t) = A(1-t/r)^{-k}$, k a natural number, as majorant, for such functions $f(t) = \sum_{n=0}^{\infty} a_n t^n$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1/r$, so, the ratio of corresponding coefficients of $f'(t)$ and $tf'(t)$, which is $\frac{(n+1)a_{n+1}}{na_n}$, tends to $1/r$. Thus, for such functions, we can replace $tf'(t)$ by $\alpha f'(t)$, where $\alpha \sim 1$. Thus if we can find majorant $W_h(z)$ for $U_h(t,x)$ in the class $W_h(z) = A(1-z/r)^{-k}$, the left-handside can be replaced by

$$\begin{aligned}
&\alpha\frac{\partial^2 W_h}{\partial t^2} + \frac{1}{m_1-1}[p+h+\lambda_0+(m_1-n)]\alpha\frac{\partial W_h}{\partial t}\\
&\qquad + \Phi(p+h)W_h + \frac{\epsilon_n M_1 t^2}{1-z/r}\frac{\partial W_h}{\partial z}\\
&\quad = \alpha\frac{\partial^2 W_h}{\partial z^2} + \frac{1}{m_1-1}[p+h+\lambda_0+(m_1-n)]\alpha\frac{\partial W_h}{\partial z}\\
&\qquad + \Phi(p+h)W_h + \frac{\epsilon_n M_1 t^2}{1-z/r}\frac{\partial W_h}{\partial z}\\
&\quad \geq Ah\frac{\partial W_h}{\partial z}\,.
\end{aligned}$$

We obtain this last inequality, since all terms of the last expression are power series with non-negative coefficients, hence

$$Ah\frac{dW_h}{dz} \leq \frac{h^2 M_2}{1-z/r}\sum_{i=1}^{m} n^i \left(\frac{d}{dz}\right)^i W_{h-i}\,. \tag{93}$$

Also we can replace the last inequality by an equation

$$Ah\frac{dW_h}{dz} = \frac{h^2 M_2}{1-z/r}\sum_{i=1}^{m} n^i \left(\frac{d}{dz}\right)^i W_{h-i}\,. \tag{94}$$

We can prove inductively that we can choose A_k in (92) to be Ck^kL^k and

$$W_k = \frac{Ck^kL^k}{(1-z/r)^{k+1}}\,, \quad L = \left(\frac{2mn^m}{A}\right)M_2 \tag{95}$$

In fact, (94) is valid for $k=0$, if it is valid for k up to $h-1$, let $W_h = A_h/(1-z/r)^{h+1}$, we have

$$\begin{aligned}
Ah(h-1)A_h &\leq Ch^2 M_2 \sum_{i=1}^{m} n^i (h-i)^{h-i}(h-i+1)\cdots h\left(\frac{2mn^m}{A}M_2\right)^{h-1}\\
&\leq Ch^{h+2}mn^m\left(\frac{2mn^m}{A}M_2\right)^{h-1} M_2\,.
\end{aligned}$$

So,

$$A_h \leq C h^h \left(\frac{2mn^m M_2}{A} \right)^h ,$$

hence (95) is proved.

After all, we obtain a majorant series for (33):

$$C \sum_{h=0}^{\infty} \frac{h^h}{\Gamma(p+h+1)} \left(\frac{2mn^m M_2}{A} \right)^h \frac{k^{p+h}}{(1-z/r)^{h+1}} ,$$

using Stirling's formula, we see that the series above converges for $|k|$ small enough.

Now, since the series (33) converges, our main result the Theorem 1 is proved.

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M. Y. CHI (Qi Minyou)
Institute of Mathematics,
Wuhan University
Wuhan, 430072
P. R. China

Lavoro pervenuto in redazione il 1.4.1997.

Rend. Sem. Mat. Univ. Pol. Torino
Vol. 56, 3 (1998)

G. E. Karadzhov

RIESZ SUMMABILITY OF MULTIPLE HERMITE SERIES†

Abstract.

Let $E_\lambda^\alpha f(y) = \sum (1 - \mu_k/\lambda)^\alpha f_k \phi_k(y)$, $\mu_k < \lambda$ be the Riesz means of order α of the multiple Hermite series of the function f. The following main results are established: (i) The convergence $E_\lambda^\alpha f \to f$ as $\lambda \to \infty$ in the space $L_m^1 = L_m^1(\mathbb{R}^n)$, $n \geq 2$, with a norm $\int (1 + |x|)^m |f(x)| dx$ under sharp conditions on the parameters α and m. (ii) The localization principle for the functions in the spaces L_m^p under sharp conditions on the parameters α, m, p. (iii) The convergence $E_\lambda^\alpha f(y) \to f(y)$ on the Lebesgue set of the function f from the spaces L_m^p.

1. Introduction and statement of the results

Consider the multiple Hermite series

$$f(y) \sim \sum f_k \phi_k(y)\,, \quad f_k = \int f(y)\phi_k(y)\,dy\,,$$

and the corresponding spherical partial sum

$$E_\lambda f(y) = \int e(\lambda, x, y) f(x)\,dx\,,$$

where

$$e(\lambda, x, y) = \sum \phi_k(x)\phi_k(y)\,, \quad \mu_k < \lambda$$

and $\mu_k = 2|k| + n$, $\phi_k(x) = H_k(x)\exp(-x^2/2)$ are the eigenvalues and orthonormalized eigenfunctions (Hermite functions) of the operator $-\Delta + x^2$ in $L^2(\mathbb{R}^n)$, $n \geq 2$. Here and later on the integral $\int$ means the integral over $\mathbb{R}^n$. Let

$$E_\lambda^\alpha f(y) = \sum (1 - \mu_k/\lambda)^\alpha f_k \phi_k(y)\,, \quad \mu_k < \lambda\,,$$

be the Riesz means of order $\alpha > 0$ of the multiple Hermite series. Then

$$E_\lambda^\alpha f(y) = \int I^\alpha e(\lambda, x, y) f(x)\,dx\,, \tag{1}$$

where

$$I^\alpha e(\lambda, x, y) = \int_0^\lambda (1 - \mu/\lambda)^\alpha\,de(\mu, x, y)$$

is the Riesz kernel of order α.

†Research partially supported by the Ministry of Education, Science and Tecnology under Contract MM410/94 and CNR

We are interested in the problem of the convergency $E_\lambda^\alpha f \to f$ as $\lambda \to \infty$. The case of compact manifolds without boundary and general elliptic operators is treated in several publications [2], [3], [4], [8], [9], [14], [17], where this convergence is in L^p norm or almost everywhere. The corresponding problems in $\mathbb{R}^n$ seem to be more difficult because of the arising both local and global obstacles. Some results for Schrödinger operators are obtained in [7], [10], [18], [20] and the classical Laguerre and Hermite series are considered, for example, in [1], [5], [12], [13], [15], [16], [21], [22].

Here we investigate further the multiple Hermite series. Our goal is to clarify the following three problems:

(i) The convergence in the space $L_m^1 = L_m^1(\mathbb{R}^n)$ with a norm $\|f\|_{1,m} = \int (1+|x|)^m |f(x)|\, dx$ for optimal conditions on the parameters α and m. The case $m = 0$ is treated also in [1], [22].

(ii) The localization principle for the functions in the space $L_m^p = L_m^p(\mathbb{R}^n)$ with a norm $\|f\|_{p,m} = (\int (1+|x|)^{mp} |f(x)|^p\, dx)^{1/p}$ for optimal conditions on the parameters α, m.

(iii) The convergence on the Lebesgue set of the function f from the spaces L_m^p. The case of L^p spaces for $p \geq 2$ is considered also in [22].

Let $\|E_\lambda^\alpha\|_{m,p}$ be the corresponding operator norm in the space L_m^p.

THEOREM 1. *If* $\alpha > (n-1)/2$ *and* $-m_1 \leq m \leq 2\alpha + 1 - n$, *where* $m_1 = \min(3\alpha + 1 - n/2, 2\alpha + 2/3)$, *then* $\|E_\lambda^\alpha\|_{m,1} \leq c$.

Here and later on all estimates are uniform with respect to the parameter $\lambda > 1$ and c is a positive constant.

COROLLARY 1. *Under the conditions of Theorem 1,* $E_\lambda^\alpha f \to f$ *in the space* L_m^1 *or in the separable subspace* $C_m \subset L_m^\infty$, *consisting of all continuous functions* f *on* $\mathbb{R}^n$ *such that* $|x|^m f(x) \to 0$ *as* $|x| \to \infty$.

REMARK 1. The restrictions on the parameters α and m in Theorem 1 are optimal. Namely, there exists a sequence $\lambda \to \infty$ such that:

$$\|E_\lambda^\alpha\|_{m,1} \geq c\lambda^{m/2+(n-1)/2-\alpha}, \tag{2}$$

$$\|E_\lambda^\alpha\|_{m,1} \geq c(\lambda^{-m/2-\alpha-1/3} + \lambda^{-m/3-\alpha-1/3+n/6}), \tag{3}$$

$$\|E_\lambda^\alpha\|_{m,1} \geq \lambda^{(n-1)/4-\alpha/2}, \tag{4}$$

$$\|E_\lambda^\alpha\|_{m,1} \geq c \log \lambda \text{ if } \alpha = (n-1)/2. \tag{5}$$

Let

$$m_o = m_o(\alpha, p) = 2\alpha + 1 - n + \min(n/p, n/p + 1 - 4/3p)\,. \tag{6}$$

THEOREM 2 (LOCALIZATION). *Let* $\alpha \geq (n-1)/2$, $\delta > 0$. *Then as* $\lambda \to \infty$

$$R_\lambda^\alpha f(y) \overset{def}{=} E_\lambda^\alpha f(y) - \int_{|x-y|<\delta} f(x) I^\alpha e(\lambda, x, y)\, dx \to 0, \text{ locally uniformly}, \tag{7}$$

for the functions $f \in L_m^p$ *in the following cases:*

(*a*) *if* $\alpha = (n-1)/2$ *then*

$$m \geq -m_o(\alpha, p) \text{ for } 1 \leq p < 4/3\,, \tag{8}$$

$$m > -m_o(\alpha, p) \text{ for } 4/3 \leq p < \infty\,; \tag{9}$$

(*b*) *if* $\alpha > (n-1)/2$ *then*

$$m \geq -m_o(\alpha, p) \text{ for } p \neq 4/3\,, \tag{10}$$

$$m > -m_o(\alpha, p) \text{ for } p = 4/3 \text{ or } p = \infty\,. \tag{11}$$

REMARK 2. The conditions on the parameters α and m in Theorem 2 are optimal. Namely, let $|R_\lambda^\alpha|_{m,p}$ denote the operator norm of R_λ^α as an operator from L_m^p to the space of all continuous functions on a fixed compact. Then there exists a sequence $\lambda \to \infty$ so that

$$|R_\lambda^\alpha|_{m,p} \geq c\left(\lambda^{(n-1)/4-\alpha/2} + \lambda^{-(m+m_o)/2}\right)\,, \tag{12}$$

$$|R_\lambda^\alpha|_{m,p} \geq c\lambda^{-(m+m_o)/2}\log\lambda \text{ if } p = 4/3\,, \tag{13}$$

$$|R_\lambda^\alpha|_{m,p} \geq c(\log\lambda)^{1/p'} \text{ if } \alpha = (n-1)/2,\ \ p > 4/3,\ \ m = -m_o,\ \ 1/p + 1/p' = 1\,. \tag{14}$$

REMARK 3. Theorem 2, case $p = \infty$, means that the relation (7) is true for all functions $f(x) = O(|x|^\beta)$ as $|x| \to \infty$ if $\beta < 2\alpha + 1 - n$. It turns out that we can allow a stronger growth at infinity for the functions having some regularity. Namely, we have

THEOREM 3 (LOCALIZATION). *Let f be locally in L^1 and let the derivatives $\nabla f(x)$ exist for $x^2 > C_f$. If $f(x)$ and $\nabla f(x)$ are $O(|x|^\beta)$ as $|x| \to \infty$ for $\beta < 2\alpha + 2 - n$ and $\alpha \geq (n-1)/2$ then the convergence (7) is fulfilled.*

COROLLARY 2. *Under the conditions of Theorems 2 or 3 the convergence*

$$E_\lambda^\alpha f(y) \to f(y) \text{ as } \lambda \to \infty$$

is fulfilled on the Lebesgue set of the function f if $\alpha > (n-1)/2$.

The main results are announced in [11], where Theorem 1 is proved only for the case $\alpha \geq n/2 - 1/3$ if $m < 0$. The case $\alpha > (n-1)/2$ of Theorems 2 and 3 is considered also in [5].

The further organisation of the paper is as follows. In section 2 the auxiliary results about the asymptotics of the Riesz kernels are stated and the proofs of the Propositions 1, 2, 3 are given in sections 8, 9, 10 respectively. Theorem 1 is proved in section 3. Remark 1 is proved in section 4. Theorem 2 and Remark 2 are proved in sections 5 and 6 respectively. Theorem 3 is proved in section 7.

2. Auxiliary results

From (1) it is clear that the first main problem in our investigations is to have asymptotics or appropriate estimates of the Riesz kernels $I^\alpha e(\lambda, x, y)$ as $\lambda \to \infty$, which must be global with respect to $x, y \in \mathbb{R}^n$. We shall use the following results, proved in [12]. Define

$$e_\alpha(\lambda, x, y) = \lambda^\alpha I^\alpha e(\lambda, x, y),\ \ E_\alpha(\lambda, x, y) = e_\alpha(\lambda, \sqrt{\lambda}x, \sqrt{\lambda}y)\,.$$

THEOREM 4. *([12]) If $x^2 + y^2 < const$ and $\alpha \geq (n-1)/2$, then*

$$|I^\alpha e(\lambda, x, y) - I^\alpha e^o(\lambda, x, y)| \leq c\, \lambda^{(n-1)/2} G_\alpha(\sqrt{\lambda}|x-y|)\,, \tag{15}$$

where $G_\alpha(s) = (1+s)^{-(n+1)/2-\alpha}$, $s \geq 0$ and

$$I^\alpha e^o(\lambda, x, y) = \lambda^{n/2} F_\alpha(\sqrt{\lambda}|x-y|)\,, \tag{16}$$

$$F_\alpha(s) = d_\alpha s^{-n/2-\alpha} J_{n/2+\alpha}(s)\,,\ d_\alpha = 2^\alpha \Gamma(\alpha+1)(2\pi)^{-n/2}\,. \tag{17}$$

THEOREM 5. *([12]) Let $A > 0$, $0 < \delta < 1$ be fixed and $A < x^2 < (1-\delta)\lambda$. There exists $\epsilon > 0$ such that if $|y| < \epsilon|x|$ and $\alpha > 0$ then*

$$E_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^{4} b_k(\lambda, x, y, \alpha) e^{i\lambda\psi_k(x,y)} + |x|^{-(n+1)/2-\alpha} O(\lambda^{-1})\,,$$

where the smooth functions $(x, y) \to b_k$, ψ_k satisfy

$$|b_k| < c|x|^{-(n+1)/2-\alpha}\,,\ |\nabla_x b_k| < c|x|^{-(n+1)/2-\alpha-1} \tag{18}$$

and

$$|\nabla_x \psi_k|^2 = 1 - x^2\,,\ |\partial^2_{x_i x_j} \psi_k|^2 < c(1-x^2)^{-1}\,. \tag{19}$$

THEOREM 6. *([12]) Let $|x^2 - 1| < \delta$, $|y| < \epsilon|x|$ for some small $\epsilon > 0$, $\delta > 0$. Then we have the uniform asymptotics*

$$E_\alpha(\lambda, x, y) = \sum_{k=0}^{\infty} (a_{1k}(\lambda, x, y)\lambda^{-k-1/3} + b_{1k}(\lambda, x, y)\lambda^{-k-2/3})\,,$$

where

$$a_{1k} = (a_k e^{i\lambda A} + b_k e^{-i\lambda A})\, Ai(\lambda^{2/3}B)\,,\ b_{1k} = (c_k e^{i\lambda A} + d_k e^{-i\lambda A})\, Ai'(\lambda^{2/3}B)$$

and the functions $\lambda \to a_k,\ b_k,\ c_k,\ d_k$, or their derivatives with respect to x are bounded. Here Ai is the Airy function, $A(x, y)$, $B(x, y)$ are real and smooth,

$$B(x, y) \sim c(y, \omega)(x^2 - a^2) \text{ as } x^2 \to a^2\,, \tag{20}$$

where $c(y, \omega) > 0$, $x = |x|\omega$ and

$$a^2 = (1-y^2)(1-(\omega y)^2)^{-1}\,. \tag{21}$$

COROLLARY 3. *The asymptotics*

$$E_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^{4} (a_k(a^2-x^2)^{-1/4} + b_k(a^2-x^2)^{1/4}) e^{i\lambda\psi_k} + (a^2-x^2)^{-1} O(\lambda^{-1})\,,$$

is uniform for $x,\ y$ such that $1 - \delta < x^2 < 1 - \lambda^{-2/3+\epsilon}$, $|y| < \epsilon|x|$. Here $\epsilon > 0$, $\delta > 0$ are small and the functions $\lambda \to a_k,\ b_k$, or their derivatives with respect to x, are bounded, and ψ_k satisfy (19).

To prove the bounds (2)–(5) and (12)–(14) we need some additional information about the asymptotics of the Riesz kernels.

PROPOSITION 1. *The uniform asymptotics*

$$E_\alpha(\lambda, x, 0) = \lambda^{-1/2}|x|^{-(n+1)/2-\alpha}(Re(e^{i\lambda\psi(x)}a(x)) + b(\lambda, x)) \tag{22}$$

holds if $\lambda^{-3/2} < x^2 < \epsilon$.

Here $\epsilon > 0$ *is small enough, the functions* $x \to \psi(x)$, $a(x)$, $b(\lambda, x)$ *are smooth and*

$$|\nabla\psi(x)| \asymp 1\,, \tag{23}$$

$$|a(x)| \asymp 1\,,\ |b(\lambda, x)| < 4^{-1}|a(x)| \tag{24}$$

PROPOSITION 2. *Let* $y \in \mathbb{R}^n$ *be fixed,* $y^2 = 1$ *and* $x^2 < \epsilon$, $|xy| < |x|/2$ *for small* $\epsilon > 0$. *Then there exists a sequence* $\lambda \to \infty$ *such that*

$$E_\alpha(\lambda, x, y) = \lambda^{-1/3}Ai(\lambda^{2/3}B(x, y))Re(a(x, y)e^{i\lambda\psi(x,y)}) + O(\lambda^{-1/2})\,, \tag{25}$$

where

$$B(x, y) \asymp x^2 \text{ as } x \to 0\,, \tag{26}$$

$$|a(x, y)| \asymp 1\,,\ |\nabla_x\psi(x, y)| \asymp 1\,, \tag{27}$$

PROPOSITION 3. *Let* $1 - c_1\lambda^{-2/3} < x^2 < 1 - c_2\lambda^{-2/3}$, *where* c_1, c_2 *are positive constants. Then there is a sequence* $\lambda \to \infty$ *such that the uniform asymptotics*

$$E_\alpha(\lambda, x, 0) = \lambda^{-1/3}Ai(\lambda^{2/3}B(x))a(x) + O(\lambda^{-2/3}) \tag{28}$$

holds, where $B(x)$ *and* $a(x)$ *are real, smooth functions, and*

$$B(x) \asymp c(x^2 - 1) \text{ as } x^2 \to 1\,,\ |a(x)| > c > 0\,, \tag{29}$$

$$\psi(x) \stackrel{\text{def}}{=} \frac{2}{3}(-B(x))^{3/2} = \frac{1}{2}\left(|x|\sqrt{1-x^2} - \arcsin\sqrt{1-x^2}\right)\,. \tag{30}$$

Moreover, the asymptotics (28) is valid also for the case $1 - \delta < x^2 < 1 - \delta/2$ *if* $\delta > 0$ *is small and the rest* $O(\lambda^{-2/3})$ *is replaced by* $O(\lambda^{-1/2})$.

THEOREM 7. *([12]) If* $x^2 > 1 + \delta$ *and* $y^2 < \epsilon$ *then the uniform estimate*

$$|E_\alpha(\lambda, x, y)| \le c(x^2 - 1)^{-1/4}\lambda^{-1/2}\exp\left(-c\delta\lambda(x^2 - 1)^{1/2}\right)$$

holds for all small $\delta > 0$, $\epsilon > 0$, *where c is a positive constant.*

We need also the following global estimates for the spectral function $e(\lambda, x, y)$.

THEOREM 8. *([13])*

$$e(\lambda, x, x) \le c\lambda^{n/2}e^{-x^2/\lambda}\,, \tag{31}$$

$$e(\lambda, x, x) \le C_N|x|^{-N} \text{ if } x^2 > (1+\delta)\lambda\,,\ \delta > 0\,,\ N > 0\,. \tag{32}$$

3. Proof of Theorem 1

In order to prove the theorem it is sufficient to establish the uniform bound for $\lambda > 1$:

$$g(\lambda, y) \le c(1 + |y|)^m\,, \tag{33}$$

where

$$g(\lambda, y) = \int |I^\alpha e(\lambda, x, y)|(1 + |x|)^m\, dx\,. \tag{34}$$

For every $y \in \mathbb{R}^n$ and for some fixed $\epsilon > 0$ consider the sets:

$$\begin{aligned} G_1(y) &= \{x \in \mathbb{R}^n : \epsilon(1 + |y|) < 1 + |x| < \epsilon^{-1}(1 + |y|)\}\,,\\ G_2(y) &= \{x \in \mathbb{R}^n : 1 + |y| < \epsilon(1 + |x|)\}\,,\\ G_3(y) &= \{x \in \mathbb{R}^n : 1 + |x| < \epsilon(1 + |y|)\} \end{aligned}$$

and the corresponding functions $g_i(\lambda, y)$, given by (34), the integration being taken over $G_i(y)$, $1 \le i \le 3$.

a. Estimate of $g_1(\lambda, y)$. In this case the problem is the same as in L^1 space since

$$g_1(\lambda, y) \le c(1 + |y|)^m \int |I^\alpha e(\lambda, x, y)|\, dx\,. \tag{35}$$

On the other hand in [22] it is proved that $\|E_\lambda^\alpha\|_{0,1} < c$ if $\alpha > (n-1)/2$. Hence

$$\int |I^\alpha e(\lambda, x, y)|\, dx \le c\,. \tag{36}$$

Thus (33) is valid for g_1.

b. Estimate of $g_2(\lambda, y)$. If $m \le 0$ then $g_2(\lambda, y)$ satisfies the estimates (35), (36) for $\alpha > (n-1)/2$, hence it remains for considering only the case $m > 0$. On the other hand, (31) and (32) imply

$$|I^\alpha e(\lambda, x, y)| \le C_N |x|^{-N} \text{ if } x^2 > (1 + \delta)\lambda\,,\ \delta > 0\,,\ N > 0\,. \tag{37}$$

Now (36) and (37) show that it sufficies to estimate the integral

$$I(\lambda, y) \stackrel{\text{def}}{=} \int_G |x|^m |I^\alpha e(\lambda, x, y)|\, dx \text{ for } y^2 < \lambda(1 + \delta)\,, \tag{38}$$

where $G = \{x \in \mathbb{R}^n : 1 < x^2 < \lambda(1 + \delta),\ 1 + |y| < \epsilon(1 + |x|)\}$ and $\epsilon > 0$ is small.

Evidently

$$I(\lambda, \sqrt{\lambda}y) \le c\lambda^{n/2-\alpha+m/2} \int_M |x|^m |E_\alpha(\lambda, x, y)|\, dx\,,\ y^2 < 1 + \delta\,, \tag{39}$$

where $M = \{x \in \mathbb{R}^n : \lambda^{-1} < x^2 < 1 + \delta,\ |y| < \epsilon|x|\}$.

Let $K_j = K_j(\lambda, y)$ be the right hand side of (39), where M is replaced by M_j and $M = M_1 \cup M_2$, $M_1 = \{x \in M : x^2 < 1 - \delta\}$. To estimate K_1 we use Theorem 5 from which it follows

$$|E_\alpha(\lambda, x, y)| \le c\lambda^{-1/2}|x - y|^{-(n+1)/2-\alpha},\ \alpha > 0 \text{ if } x^2 < 1 - \delta,\ |y| < \epsilon|x|\,. \tag{40}$$

Since $|x-y| \asymp |x| > \lambda^{-1/2}$ for $x \in M$ we see that

$$K_1 \leq c \text{ if } \alpha > (n-1)/2,\ m \leq 2\alpha + 1 - n,\ y^2 < 1 + \delta\,. \tag{41}$$

The integral K_2 can be bound by Theorem 6 from which it follows

$$|E_\alpha(\lambda, x, y)| \leq c\lambda^{-1/2}|d(x,y)|^{-1/4} \text{ if } |x^2 - 1| < \delta,\ |y| < \epsilon|x| \tag{42}$$

for small $\epsilon > 0$, $\delta > 0$. Here we have used the asymptotics of the Airy function, the relation (20) and

$$d = (a^2 - x^2)(1 - (y\omega)^2)\,, \tag{43}$$

where a^2 is given by (21). Thus

$$K_2 \leq c \int_{M_2} |d(x,y)|^{-1/4}\, dx \text{ if } m \leq 2\alpha + 1 - n\,. \tag{44}$$

For small $\delta > 0$, $\epsilon > 0$ we have $y^2 < 1/2$ therefore $a^2 = (1-y^2)(1-(y\omega)^2)^{-1}$ satisfies $1/2 < a^2 < 1$. Since in polar coordinates $x = \sigma\omega$,

$$\int_{M_2} |d(x,y)|^{-1/4}\, dx \leq c \int H(y,\omega)\, d\omega\,, \tag{45}$$

where

$$H(y,\omega) = \int_0^c |\sigma - a|^{-1/4} |\sigma + a|^{1/4}\, d\sigma \tag{46}$$

for some $c > 1$, and $H(y,\omega) \leq c(1 + a^{1/2}) \leq$ const we obtain the bound

$$K_2 \leq c \text{ if } m \leq 2\alpha + 1 - n,\ y^2 < 1 + \delta\,. \tag{47}$$

Evidently (38), (39), (41), (47) imply

$$I(\lambda, y) \leq c \text{ if } m \leq 2\alpha + 1 - n\,,\ y^2 < \lambda(1+\delta)\,.$$

Thus

$$g_2(\lambda, y) \leq c(1 + |y|)^m \text{ if } m \leq 2\alpha + 1 - n\,,\ \alpha > (n-1)/2\,. \tag{48}$$

c. Estimate of $g_3(\lambda, y)$. If $m \geq 0$ then g_3 satisfies (35), (36) for $\alpha > (n-1)/2$, therefore we shall suppose $m < 0$. Now we have to estimate $I(\lambda, y)$ given by (38), where $G = \{x \in \mathbb{R}^n : 1 < x^2 < \lambda(1+\delta),\ 1 + |x| < \epsilon(1 + |y|)\}$.

For convenience we shall first consider the case $y^2 < \lambda(1-\delta)$. Since $|y| \asymp |x - y|$ for $x \in G$ we have

$$I(\lambda, y) \leq c(1 + |y|)^m J(\lambda, y)\,,\ m < 0\,, \tag{49}$$

where

$$J(\lambda, y) = \int_G (1 + |x-y|)^{-m} |I^\alpha e(\lambda, x, y)|\, dx\,.$$

Taking into account (36) it sufficies to consider the integral J for $|x-y|>1$. Then (40) implies

$$J(\lambda,\sqrt{\lambda}y)\le c\lambda^{-m/2+(n-1)/2-\alpha}\int_M |x-y|^{-m-(n+1)/2-\alpha}\,dx$$

if $y^2<1-\delta$, $M=\{x\in\mathbb{R}^n : |x|<\epsilon|y|,\ \lambda^{-1/2}<|x-y|<c\}$, hence

$$J(\lambda,y)\le c \text{ if } \alpha>(n-1)/2\,,\ -m\le 2\alpha+1-n\,. \tag{50}$$

Clearly (49) and (50) give

$$I(\lambda,y)\le c(1+|y|)^m \text{ if } \alpha>(n-1)/2\,,\ -m\le 2\alpha+1-n\,. \tag{51}$$

Now we pass to the case $2\alpha+1-n<|m|\le\alpha+(n+1)/2$. Using Theorem 5 we obtain

$$I(\lambda,y)\le c\lambda^{(n-1)/4-\alpha/2}\int_G (1+|x|)^m|x-y|^{-\alpha-(n+1)/2}\,dx \tag{52}$$

if $\alpha>0$, $y^2<\lambda(1-\delta)$. There exists a half-integer $k\ge 1$ such that $|m|\in(k(2\alpha+1-n),(k+1/2)(2\alpha+1-n)]$. Taking $p\ge 0$ so that $n-k(2\alpha+1-n)\le p\le\alpha+(n+1)/2-|m|$ we have $n<|m|+p\le\alpha+(n+1)/2$. Since $|x-y|\asymp|y|\asymp 1+|y|$ for $x\in G$, (52) implies

$$I(\lambda,y)\le c\int(1+|x|)^{-|m|+p}\,dx\,(1+|y|)^{p-\alpha-(n+1)/2} \text{ if } \alpha>(n+1)/2\,,$$

hence

$$I(\lambda,y)\le c(1+|y|)^m \text{ if } \alpha>(n-1)/2\,,\ 2\alpha+1-n<|m|\le\alpha+(n+1)/2\,. \tag{53}$$

It remains the case $|m|>\alpha+(n+1)/2$. Since $\alpha>(n-1)/2$ we have $|m|>n$ and $\lambda^{(n-1)/4-\alpha/2}|x-y|^{-\alpha-(n+1)/2}\le c(1+|y|)^{-2\alpha-1}$, so the estimate (52) gives

$$I(\lambda,y)\le c(1+|y|)^m \text{ if } \alpha>(n-1)/2\,,\ \alpha+(n+1)/2<|m|\le 2\alpha+1\,. \tag{54}$$

Thus (51), (53) and (54) show that

$$I(\lambda,y)\le c(1+|y|)^m \text{ if } \alpha>(n-1)/2\,,\ -m\le 2\alpha+1\,,\ y^2<\lambda(1-\delta)\,. \tag{55}$$

Now consider the case $\lambda(1-\delta)<y^2<\lambda(1+\delta)$. Using Theorem 6 we can write the uniform estimate

$$|I^\alpha e(\lambda,x,y)|\le c\lambda^{-\alpha-1/3} \tag{56}$$

if $\lambda(1-\delta)<y^2<\lambda(1+\delta)$, $|x|<\epsilon|y|$ for small $\epsilon>0$, $\delta>0$. Therefore

$$I(\lambda,y)\le c(1+|y|)^m \text{ if } -m\le 2\alpha+2/3\,,\ \alpha>n/2-1/3\,. \tag{57}$$

Let now $(n-1)/2<\alpha\le n/2-1/3$. If $x^2<\lambda^{1/3}$ then (56) implies (57) for $-m\le 3\alpha+1-n/2$. In the case $x^2>\lambda^{1/3}$ we obtain (39), where now $M=\{x\in\mathbb{R}^n : \lambda^{-2/3}<x^2<\epsilon\}$ for some small $\epsilon>0$ and $|y^2-1|<\delta$. Then analogously to (42) we have

$$I(\lambda,\sqrt{\lambda}y)\le c\lambda^{(n-1)/2-\alpha+m/2}\int_M |x|^m|d(x,y)|^{-1/4}\,dx\,. \tag{58}$$

Using polar coordinates as after the estimate (44) we get

$$\int_M |x|^m |d(x,y)|^{-1/4}\,dx \le \sum_{j=1}^{2} \int_{F_j} |1-(y\omega)^2|^{-1/4} H_j(\lambda,y,\omega)\,d\omega\,, \tag{59}$$

where $F_2 = \{\omega : (1-y^2)(1-(y\omega)^2)^{-1} \ge 0\}$ and F_1 is the complement of F_2 to the unit sphere. Here

$$H_1(\lambda,y,\omega) = \int_{\lambda^{-1/3}}^{1} \sigma^{m+n-3/2}\,d\sigma \text{ if } \omega \in F_1$$

and for $a \ge 0$, $a^2 = (1-y^2)(1-(y\omega)^2)^{-1}$,

$$H_2(\lambda,y,\omega) = \int_{\lambda^{-1/3}}^{1} \sigma^{m+n-1}|\sigma-a|^{-1/4}|\sigma+a|^{-1/4}\,d\sigma \text{ if } \omega \in F_2\,.$$

We have the estimates:

$$H_j(\lambda,y,\omega) \le c \text{ if } -m \le 3\alpha+1-n/2\,,\ \alpha > (n-1)/2\ (j=1,2)\,. \tag{60}$$

Therefore (58)–(60) imply (57) for the case $-m \le 3\alpha+1-n/2$, $(n-1)/2 < \alpha \le n/2-1/3$. In other words,

$$g_3(\lambda,y) \le c(1+|y|)^m \text{ if } -m \le m_0\,,\ \alpha > (n-1)/2\,, \tag{61}$$

where $m_0 = \min(2\alpha+2/3, 3\alpha+1-n/2)$. Finally, (35), (48) and (61) imply (33) for $\alpha > (n-1)/2$ and $-m_0 \le m \le 2\alpha+1-n$. Theorem 1 is proved.

4. Proof of remark 1

We shall use the property $\|E_\lambda^\alpha\|_{m,1} = \|E_\lambda^\alpha\|_{-m,\infty}$.

a. Proof of bound (2). Let $f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})g(x\lambda^{-1/2})$, where $g(x)$ is the characteristic function of the set $\{x \in \mathbb{R}^n : \epsilon_1 < x^2 < \epsilon\}$ and $a(x)$ satisfies (24). Then $\|f_\lambda\|_{-m,\infty} \le c\lambda^{-m/2}$.

On the other hand

$$\|E_\lambda^\alpha f_\lambda\|_{-m,\infty} \ge |E_\lambda^\alpha f_\lambda(0)|\,, \tag{62}$$

hence it sufficies to prove

$$|E_\lambda^\alpha f_\lambda(0)| \ge c\lambda^{(n-1)/2-\alpha}. \tag{63}$$

Evidently,

$$E_\lambda^\alpha f_\lambda(0) = \lambda^{n/2-\alpha}\int E_\alpha(\lambda,x,0)e^{-i\lambda\psi(x)}\bar{a}(x)g(x)dx\,.$$

Using proposition 1, we obtain for large λ

$$|E_\lambda^\alpha f_\lambda(0)| \ge c\lambda^{(n-1)/2-\alpha}(1-|J(\lambda)|)\,, \tag{64}$$

where

$$J(\lambda) = \int e^{-2i\lambda\psi(x)}(\bar{a}(x))^2|x|^{-(n+1)/2-\alpha}g(x)dx\,.$$

Since $|\nabla\psi| \asymp 1$ we can integrate by parts, whence

$$J(\lambda) = O(\lambda^{-1})\,. \tag{65}$$

Now (64), (65) imply (63). The bound (2) is proved.

b. Proof of bound (3). Let

$$f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2},y_0)}\bar{a}(x\lambda^{-1/2}, y_0)Ai^{-1}(B(x\lambda^{-1/2}, y_0)\lambda^{2/3})g(x)\,,$$

where $y_0^2 = 1$ is fixed and $g(x)$ is the characteristic function of the set $\{x \in \mathbb{R}^n : x^2 < 1,\ |xy_0| < |x|/2\}$ and $B(x, y)$, $a(x, y)$ are given by (26), (27). Then the properties of the Airy function Ai give $\|f_\lambda\|_{-m,\infty} \le c$. Since

$$\|E_\lambda^\alpha f_\lambda\|_{-m,\infty} \ge \lambda^{-m/2}|E_\lambda^\alpha f_\lambda(y_0\lambda^{1/2})|\,,$$

it suffices to prove

$$|E_\lambda^\alpha f_\lambda(y_0\lambda^{1/2})| \ge c\lambda^{-\alpha-1/3}\,. \tag{66}$$

We have

$$E_\lambda^\alpha f_\lambda(y_0\lambda^{1/2}) = \lambda^{n/2-\alpha}\int E_\alpha(\lambda, x, y_0)e^{-i\lambda\psi(x,y_0)}\bar{a}(x, y_0)Ai^{-1}(B\lambda^{2/3})g(x\lambda^{1/2})dx$$

and proposition 2 implies

$$|E_\lambda^\alpha f_\lambda(y_0\lambda^{1/2})| \ge c\lambda^{n/2-\alpha-1/3}\left(\int g(x\lambda^{1/2})dx - |I(\lambda)|\right)\,, \tag{67}$$

where

$$I(\lambda) = \int e^{-2i\lambda\psi(x,y_0)}(\bar{a}(x, y_0))^2 g(x\lambda^{1/2})dx\,.$$

According to (27) we can integrate by parts, whence

$$I(\lambda) = O(\lambda^{-n/2-1/2})\,. \tag{68}$$

On the other hand, $\int g(x\lambda^{1/2})dx = c\lambda^{-n/2}$, therefore (66) is a consequence of (67), (68). The first part of bound (3) is proved. To prove its second part, it is sufficient to replace the function g in the definition of $f_\lambda(x)$ by the characteristic function of the set $\{x \in \mathbb{R}^n : \lambda^{1/3}/2 < x^2 < \lambda^{1/3},\ |xy_0| < |x|/2\}$. Then $\|f_\lambda\|_{-m,\infty} \le c\lambda^{-m/6}$ and $\int g(x\lambda^{1/2})dx = c\lambda^{-n/3}$.

c. Proof of bound (4). Let $f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})g(x)$, where $g(x)$ is the characteristic function of the set $\{x \in \mathbb{R}^n : 1 < x^2 < 2\}$. Then $\|f_\lambda\|_{-m,\infty} \le c$ and according to (62) it sufficies to prove

$$|E_\lambda^\alpha f_\lambda(0)| \ge c\lambda^{(n-1)/4-\alpha/2}. \tag{69}$$

Since

$$E_\lambda^\alpha f_\lambda(0) = \lambda^{n/2-\alpha}\int E_\alpha(\lambda, x, 0)e^{-i\lambda\psi(x)}\bar{a}(x)g(x\lambda^{1/2})dx$$

the proposition 1 gives for large λ

$$|E_\lambda^\alpha f_\lambda(0)| \ge c\lambda^{(n-1)/2-\alpha}\left(\int |x|^{-(n+1)/2-\alpha}g(x\lambda^{1/2})dx - |I(\lambda)|\right)\,, \tag{70}$$

where

$$I(\lambda) = \int e^{-2i\lambda\psi(x)}(\bar{a}(x))^2|x|^{-(n+1)/2-\alpha}g(x\lambda^{1/2})dx\,.$$

Integrating by parts we find

$$I(\lambda) = O(\lambda^{\alpha/2-(n+1)/4}). \tag{71}$$

On the other hand,

$$\int |x|^{-(n+1)/2-\alpha}g(x\lambda^{1/2})dx = c\lambda^{\alpha/2-(n-1)/4}. \tag{72}$$

Thus (69) follows from (70)-(72). The bound (4) is proved.

d. Proof of bound (5). Let $f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})g_\lambda(x)$, where g_λ is the characteristic function of the set $\{x \in \mathbb{R}^n : \lambda^{-1/2} < x^2 < 1\}$. Then $\|f_\lambda\|_{-m,\infty} \le c$ and because of (62) it sufficies to prove

$$|E^\alpha_\lambda f_\lambda(0)| \ge c\log\lambda \text{ for } \alpha = (n-1)/2\,. \tag{73}$$

Since

$$E^\alpha_\lambda f_\lambda(0) = \lambda^{n/2-\alpha}\int E_\alpha(\lambda, x, 0)e^{-i\lambda\psi(x)}\bar{a}(x)g_\lambda(x\lambda^{1/2})\,dx$$

we obtain from proposition 1

$$|E^\alpha_\lambda f_\lambda(0)| \ge c\left(\int |x|^{-n}g_\lambda(x\lambda^{1/2})dx - |J(\lambda)|\right) \tag{74}$$

for large λ, where

$$J(\lambda) = \int e^{-2i\lambda\psi(x)}(\bar{a}(x))^2|x|^{-n}g_\lambda(x\lambda^{1/2})\,dx\,.$$

Integrating by parts we get the bound

$$J(\lambda) = O(\lambda^{-1/4})\,. \tag{75}$$

Noticing that $\int |x|^{-n}g_\lambda(x\lambda^{1/2})dx \ge c\log\lambda$ we see that (73) follows from (74), (75). The bound (5) is proved.

5. Proof of theorem 2

According to (7)

$$R^\alpha_\lambda f(y) = \int_{|x-y|>\delta} f(x)I^\alpha e(\lambda, x, y)\,dx\,. \tag{76}$$

We shall prove the estimate

$$|R^\alpha_\lambda f(y)| \le c\|f\|_{m,p} \tag{77}$$

uniformly for $y^2 < B$, $\lambda > \lambda_o$, if α and m satisfy the conditions of theorem 2 and $\lambda_o > 1$ is sufficiently large. In short, this estimate follows from theorems 4-7 and the Hölder inequality. From theorem 4 it follows

$$|I^\alpha e(\lambda, x, y)| \le c\lambda^{(n-1)/4-\alpha/2} \text{ if } x^2 + y^2 < c,\ |x-y| > \delta,\ \alpha \ge (n-1)/2\,. \tag{78}$$

Therefore

$$|R_\lambda^\alpha f(y)| < c\lambda^{(n-1)/4-\alpha/2}\int_{x^2<A}|f(x)|\,dx + c|K(\lambda,y)|,\ \ y^2 < B, \tag{79}$$

where A is sufficiently large and

$$K(\lambda,y) = \int_{x^2>A} f(x)I^\alpha e(\lambda,x,y)\,dx,\ \ y^2 < B\,. \tag{80}$$

To estimate $K(\lambda, y)$, uniformly for $y^2 < B$, $\lambda > \lambda_o$, it is convenient to consider the functions

$$K_j(\lambda,y) = \int a_j(\lambda,x)f(x)I^\alpha e(\lambda,x,y)\,dx,\ \ 1 \le j \le 5\,, \tag{81}$$

where $a_j(\lambda, x)$ is the characteristic function of the set S_j and for small $\epsilon > 0$, $\delta > 0$,

$$\begin{aligned} &S_1 = \{x : A < x^2 < (1-\delta)\lambda\}, && S_2 = \{x : (1-\delta)\lambda < x^2 < \lambda - \lambda^{1/3}\},\\ &S_3 = \{x : |x^2-\lambda| < \lambda^{1/3}\}, && S_4 = \{x : \lambda + \lambda^{1/3} < x^2 < \lambda + \lambda^{1/3+\epsilon}\},\\ &S_5 = \{x : x^2 > \lambda + \lambda^{1/3+\epsilon}\}. \end{aligned}$$

a. Estimate of $K_1(\lambda, y)$. From theorems 4 and 5 it follows

$$|I^\alpha e(\lambda,x,y)| \le c\lambda^{(n-1)/4-\alpha/2}(1-x^2/\lambda)^{-1/4}|x|^{-(n+1)/2-\alpha} \tag{82}$$

if $x \in S_1$, $y^2 < B$. Therefore the Hölder inequality gives

$$|K_1(\lambda,y)| \le c\lambda^{(n-1)/4-\alpha/2}\|f\|_{m,p}J(\lambda)\,, \tag{83}$$

where $J(\lambda) < c$ if $m > -\alpha - n/p + (n-1)/2$; $J(\lambda) < c(\log\lambda)^{1/p'}$ if $m = -\alpha - n/p + (n-1)/2$, $1/p + 1/p' = 1$; $J(\lambda) < c\lambda^{-(\alpha+m)/2-n/2p+(n-1)/4}$ if $m < -\alpha - n/p + (n-1)/2$. We claim that

$$|K_1(\lambda,y)| \le c\|f\|_{m,p}\,, \tag{84}$$

under the conditions of theorem 2. If $\alpha > (n-1)/2$ then evidently (83) gives (84) for $m \ge -\alpha - n/p + (n-1)/2$. If $m < -\alpha - n/p + (n-1)/2$ and $m \ge -m_o(\alpha, p)$ then (84) follows from (83) as well. Let now $\alpha = (n-1)/2$. If $1 \le p < 4/3$ and $m \ge -m_o(\alpha, p)$ then $m \ge -n/p - 1 + 4/3p > -n/p$ and (83) implies (84). If $4/3 \le p < \infty$ and $m > -m_o(\alpha, p)$ then $m > -n/p$, hence (84) follows from (83).

b. Estimate of $K_2(\lambda, y)$. Now we shall prove

$$|K_2(\lambda,y)| \le c\lambda^{-(m+m_o)/2}\|f\|_{m,p} \text{ if } p \ne 4/3\,, \tag{85}$$

$$|K_2(\lambda,y)| \le c\lambda^{-(m+m_o)/2}(\log\lambda)^{1/4}\|f\|_{m,p} \text{ if } p = 4/3\,. \tag{86}$$

To this end notice that theorem 5 gives the estimate

$$|I^\alpha e(\lambda,x,y)| \le c\lambda^{-\alpha-1/2}(1-x^2/\lambda)^{-1/4} \text{ if } x \in S_2,\ \ y^2 < B\,. \tag{87}$$

Therefore

$$|K_2(\lambda,y)| \le c\lambda^{-\alpha-m/2+(n-1)/2-n/2p}\|f\|_{m,p}J(\lambda)\,, \tag{88}$$

where now

$$J(\lambda) < c \text{ if } p > 4/3\,,$$
$$J(\lambda) < c(\log\lambda)^{1/4} \text{ if } p = 4/3\,,$$
$$J(\lambda) < c\lambda^{-1/2+2/3p} \text{ if } p < 4/3\,.$$

This and the definition of $m_o(\alpha, p)$ (see (6)) imply (85), (86).

c. Estimate of $K_3(\lambda, y)$. From theorem 6 it follows

$$|I^\alpha e(\lambda, x, y)| \le c\lambda^{-\alpha-1/3}, \text{ if } x \in S_3,\ y^2 < B\,. \tag{89}$$

Applying the Hölder inequality we get

$$|K_3(\lambda, y)| \le c\lambda^{-(m+m_o)/2}\|f\|_{m,p}\,. \tag{90}$$

d. Estimate of $K_4(\lambda, y)$. Theorem 7 implies

$$|I^\alpha e(\lambda, x, y)| \le c(x^2/\lambda - 1)^{-1/4}\lambda^{-\alpha-1/2}, \text{ if } x \in S_4,\ y^2 < B\,. \tag{91}$$

Hence

$$|K_4(\lambda, y)| \le c\lambda^{-\alpha-1/2}\|f\|_{m,p}J(\lambda)\,, \tag{92}$$

where now

$$J(\lambda) < c\lambda^{-m/2+n/2-1/2-n/2p+2/3p} \text{ if } p < 4/3\,;$$
$$J(\lambda) < c\lambda^{-m/2+n/2-1/2-n/2p+2/3p}(\log\lambda)^{1/4} \text{ if } p = 4/3\,;$$
$$J(\lambda) < c\lambda^{-m/2+n/2-1/2-n/2p+2/3p+\epsilon(3/4-1/p)} \text{ if } p > 4/3\,.$$

Therefore if $\epsilon > 0$ is small enough then

$$|K_4(\lambda, y)| \le c\lambda^{-(m+m_o)/2}\|f\|_{m,p} \text{ if } p \ne 4/3\,, \tag{93}$$
$$|K_4(\lambda, y)| \le c\lambda^{-(m+m_o)/2}(\log\lambda)^{1/4}\|f\|_{m,p} \text{ if } p = 4/3\,. \tag{94}$$

e. Estimate of $K_5(\lambda, y)$. From theorem 7 it follows

$$|I^\alpha e(\lambda, x, y)| \le c\lambda^{-\alpha-1/3}\exp(-c\lambda^{\epsilon/2}) \text{ if } x^2 > \lambda + \lambda^{1/3+\epsilon},\ y^2 < B\,. \tag{95}$$
$$|I^\alpha e(\lambda, x, y)| \le c\lambda^{-\alpha-1/3}\exp(-c|x|^{1/2}) \text{ if } x^2 > \lambda^2\,. \tag{96}$$

Therefore

$$|K_5(\lambda, y)| \le c\|f\|_{m,p}\,. \tag{97}$$

Evidently (77) follows from (79), (84), (85), (86), (90), (93), (94) and (97).

In the case $\alpha > (n-1)/2$, $p = \infty$ instead of (77) we have $|R^\alpha_\lambda f(y)| < c\lambda^{-\gamma}$ for some $\gamma > 0$.

Finally, since $C^\infty_o(\mathbb{R}^n)$ is a dense set in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$, it is sufficient to prove that

$$R^\alpha_\lambda f(y) \to 0 \text{ locally uniformly,} \tag{98}$$

if $f \in C_o^\infty(\mathbb{R}^n)$ and $\alpha \geq (n-1)/2$.

If $\alpha > (n-1)/2$ then (98) follows directly from (76) and (78). Let now $\alpha = (n-1)/2$. Using theorem 4 we can write

$$|I^\alpha e(\lambda, x, y) - I^\alpha e^o(\lambda, x, y)| \leq c\lambda^{-1/2+\epsilon} G_{\alpha,\epsilon}(\sqrt{\lambda}|x-y|)\lambda^{n/2}, \tag{99}$$

for $x^2 + y^2 < c$, $\alpha \geq (n-1)/2$, $\epsilon \geq 0$, where $G_{\alpha,\epsilon}(s) = (1+s)^{-(n+1)/2-\alpha-\epsilon}$, $s > 0$. For $\epsilon > 0$ we have $G_{\alpha,\epsilon}(|x|) \in L^1(\mathbb{R}^n)$ and

$$\left| \int f(x)[I^\alpha e(\lambda, x, y) - I^\alpha e^o(\lambda, x, y)]\, dx \right| \leq c\lambda^{-1/2+\epsilon}. \tag{100}$$

On the other hand, (16), (17) and the asymptotics of the Bessel function give as $\lambda \to \infty$,

$$I^\alpha e^o(\lambda, x, y) \sim a_\alpha \cos(\sqrt{\lambda}|x-y| + b_\alpha) \text{ if } |x-y| > \delta,\ x^2 + y^2 < c, \tag{101}$$

where $\alpha = (n-1)/2$ and a_α, b_α are constants. Since $f \in C_o^\infty(\mathbb{R}^n)$ it follows

$$\int_{|x-y|>\delta} f(x) I^\alpha e^o(\lambda, x, y)\, dx \to 0 \text{ locally uniformly}. \tag{102}$$

Evidently (98) for $\alpha = (n-1)/2$ follows from (100) and (102). Theorem 2 is proved.

6. Proof of remark 2

a. Proof of bound (12). Let $f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})g(x)$, where $g(x)$ is the characteristic function of the set $\{x \in \mathbb{R}^n : 1 < x^2 < 2\}$ and $a(x)$ satisfies (24). Then

$$\|f_\lambda\|_{m,\infty} \leq c \tag{103}$$

and (76) implies for small $\delta > 0$,

$$R_\lambda^\alpha f_\lambda(0) = \lambda^{n/2-\alpha} \int E_\alpha(\lambda, x, 0) f_\lambda(\sqrt{\lambda}x)\, dx. \tag{104}$$

Thus $R_\lambda^\alpha f_\lambda(0) = E_\lambda^\alpha f_\lambda(0)$ and (69) gives

$$|R_\lambda^\alpha f_\lambda(0)| \geq c\lambda^{(n-1)/4-\alpha/2}. \tag{105}$$

Therefore the first part of (12) follows from (103) and (105). To prove its second part we shall use proposition 3. Let

$$f_\lambda(x) = Ai^{-1}(B(x\lambda^{-1/2}, y_0)\lambda^{2/3})a(x\lambda^{-1/2})g_\lambda(x\lambda^{-1/2}),$$

where $g_\lambda(x)$ is the characteristic function of the set $S = \{x : \epsilon\lambda^{-2/3} < 1 - x^2 < 2\epsilon\lambda^{-2/3}\}$ for small $\epsilon > 0$. In particular, $|\lambda^{2/3}B(x\lambda^{-1/2})| < c\epsilon$ if $x\lambda^{-1/2} \in S$. Therefore

$$\|f_\lambda\|_{m,p} \leq c\lambda^{m/2+n/2p-2/3p}. \tag{106}$$

Using proposition 2 we find $Ai(B(x\lambda^{-1/2}, y_0)\lambda^{2/3}) > c > 0$ for $x\lambda^{-1/2} \in S$ and

$$E_\alpha(\lambda, x, 0) f_\lambda(x\lambda^{1/2}) = \lambda^{-1/3}|a(x)|^2 g_\lambda(x)(1 + O(\lambda^{-1/3})).$$

Hence (104) implies

$$|R_\lambda^\alpha f_\lambda(0)| \geq c\lambda^{n/2-\alpha-1/3} \int g_\lambda(x)\, dx\, (1 + O(\lambda^{-1/3}))\,.$$

Since $\int g_\lambda(x)\, dx \asymp \lambda^{-2/3}$ it follows

$$|R_\lambda^\alpha f_\lambda(0)| > c\lambda^{n/2-\alpha-1}\,. \tag{107}$$

Now (107) and (106) give

$$|R_\lambda^\alpha|_{m,p} \geq c\lambda^{-m/2+(n-1)/2-\alpha-n/2p-1/2+2/3p}\,. \tag{108}$$

Further, we shall use the asymptotics (28) for $\lambda^{-2/3+\epsilon} < 1 - x^2 < \delta$ and small $\epsilon > 0$, $\delta > 0$. According to (29) we have $\lambda^{2/3}B(x) < -c\lambda^\epsilon$, $c > 0$, and the asymptotics of the Airy function gives

$$\begin{aligned} Ai(\lambda^{2/3}B(x)) =& \lambda^{-1/6}(1 - x^2)^{-1/4} Re(e^{i\lambda\psi(x)}a(x)) \\ &+ (1 - x^2)^{-1/4} O(\lambda^{-1/6-\epsilon})\,, \end{aligned} \tag{109}$$

where a satisfies (29) and ψ is given by (30). In particular,

$$|\nabla\psi| = (1 - x^2)^{1/2},\ |\partial^2\psi| < c(1 - x^2)^{-1/2}\,. \tag{110}$$

Set

$$f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})g_\lambda(x\lambda^{-1/2})\,, \tag{111}$$

where $g_\lambda(x)$ is the characteristic function of the set $\{x : \lambda^{-2/3+\epsilon} < 1 - x^2 < \delta\}$. Then

$$\|f_\lambda\|_{m,p} \leq c\lambda^{m/2+n/2p}\,. \tag{112}$$

On the other hand, (28) and (109) imply

$$E_\alpha(\lambda, x, 0) = \lambda^{-1/2}(1 - x^2)^{-1/4} Re(e^{i\lambda\psi(x)}a(x)) + (1 - x^2)^{-1/4} O(\lambda^{-1/2-\epsilon})\,. \tag{113}$$

Hence by (104),

$$|R_\lambda^\alpha f_\lambda(0)| \geq c\lambda^{(n-1)/2-\alpha}[I(\lambda) - |J(\lambda)|]\,, \tag{114}$$

where $I(\lambda) = \int g_\lambda(x)(1 - x^2)^{-1/4}\, dx$, $J(\lambda) = \int g_\lambda(x)\bar{a}^2(x)(1 - x^2)^{-1/4}e^{-2i\lambda\psi(x)}\, dx$. We have

$$I(\lambda) > c\delta \text{ for } \lambda > \lambda_o\,. \tag{115}$$

In the integral $J(\lambda)$ we can integrate by parts using (110). Thus

$$\begin{aligned} |J(\lambda)| &< c\lambda^{-1}\left[\int g_\lambda(x)(1 - x^2)^{-7/4}\, dx + O(\lambda^{1/6})\right] \\ &< c\lambda^{-1/2}\int g_\lambda(x)(1 - x^2)^{-1}\, dx + O(\lambda^{-1/2}) \\ &< c\lambda^{-1/2}\log\lambda\,. \end{aligned}$$

This and (114), (115) imply

$$|R_\lambda^\alpha f_\lambda(0)| \geq c\lambda^{(n-1)/2-\alpha} \text{ for } \lambda > \lambda_o\,. \tag{116}$$

From (112) and (116) it follows

$$|R_\lambda^\alpha|_{m,p} > c\lambda^{-m/2+(n-1)/2-\alpha-n/2p}\,. \tag{117}$$

Evidently the second part of the estimate (12) follows from (108) and (117) taking into account the definition (6).

b. Proof of bound (13). Set

$$f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})(1-x^2/\lambda)^{-3/4}g_\lambda(x\lambda^{-1/2})\,, \tag{118}$$

where the function $g_\lambda(x)$ is the same as in (111). Then

$$\|f_\lambda\|_{m,4/3} < c\lambda^{m/2+3n/8}(\log\lambda)^{3/4}\,. \tag{119}$$

Using (113) and (104) we obtain (114) again where now

$$I(\lambda) = \int g_\lambda(x)(1-x^2)^{-1}\,dx\,,\ \ J(\lambda) = \int g_\lambda(x)\bar{a}^2(x)(1-x^2)^{-1}e^{-2i\lambda\psi(x)}\,dx\,.$$

We have

$$I(\lambda) > c\log\lambda\,. \tag{120}$$

In the integral $J(\lambda)$ we can integrate by parts using (110). Analogously to the estimates preceding (116) we get $|J(\lambda)| < c\lambda^{-3\epsilon/2}\log\lambda$, therefore

$$|R_\lambda^\alpha f_\lambda(0)| \geq c\lambda^{(n-1)/2-\alpha}\log\lambda\,. \tag{121}$$

Thus (121) and (119) imply the estimate

$$|R_\lambda^\alpha|_{m,4/3} > c\lambda^{-m/2+(n-1)/2-\alpha-3n/8}(\log\lambda)^{1/4}\,,$$

which coincides with (13).

c. Proof of bound (14). Having in mind proposition 1 we set

$$f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})g_\lambda(x\lambda^{-1/2})\,,$$

where $g_\lambda(x)$ is the characteristic function of the set $\{x : 1/\lambda < x^2 < \epsilon\}$ for small $\epsilon > 0$. Let $\alpha = (n-1)/2$, $p > 4/3$ and $m = -m_o$. Then by (6) we have $m_o = n/p$ and therefore

$$\|f_\lambda\|_{m,p} < c(\log\lambda)^{1/p}\,. \tag{122}$$

On the other hand, analogously to (114) we have

$$|R_\lambda^\alpha f_\lambda(0)| \geq c[I(\lambda) - |J(\lambda)|]\,, \tag{123}$$

where $I(\lambda) = \int g_\lambda(x)|x|^{-n}\,dx$, $J(\lambda) = \int g_\lambda(x)\bar{a}^2(x)|x|^{-n}e^{-2i\lambda\psi(x)}\,dx$. Since $I(\lambda) = c\log\lambda$, $J(\lambda) = O(\lambda^{-1})$ we get

$$|R_\lambda^\alpha f(0)| > c\log\lambda\,. \tag{124}$$

Now the estimate (14) follows from (122) and (124).

7. Proof of theorem 3

We start with (79). Let $K_j(\lambda, y)$ be defined by (81) for $1 \leq j \leq 4$, where $a_j(\lambda, x)$ is the characteristic function of the set B_j and $B_1 = A_1$, $B_2 = \{x : (1-\delta)\lambda < x^2 < \lambda - \lambda^{1/3+\epsilon}\}$, $B_3 = \{x : |x^2 - \lambda| < \lambda^{1/3+\epsilon}\}$, $B_4 = A_5$. The estimates (95) and (96) show that

$$K_4(\lambda, y) \to 0 \text{ locally uniformly.} \tag{125}$$

According to (89) and the bound

$$|f(x)| < c(1+|x|)^{\beta}, \ \beta < 2\alpha + 2 - n\,, \tag{126}$$

we have

$$|K_3(\lambda, y)| < c\lambda^{-\alpha-1/3}\int a_3(\lambda, x)|f(x)|\,dx < c\lambda^{-\alpha-1/3+\beta/2}\int a_3(\lambda, x)\,dx\,.$$

On the other hand, $\int a_3(\lambda, x)dx < c\lambda^{n/2-1+\epsilon+1/3}$, hence

$$|K_3(\lambda, y)| < c\lambda^{n/2-1-\alpha+\beta/2+\epsilon}, \text{ where } 0 < \epsilon < \alpha + 1 - n/2 - \beta/2\,. \tag{127}$$

It remains to bound $K_1(\lambda, y)$ and $K_2(\lambda, y)$.

a. Estimate of $K_1(\lambda, y)$. Let $B_j(\lambda, y) = K_j(\lambda, \sqrt{\lambda}y)$ $(j = 1, 2)$. Then

$$B_j(\lambda, y) = \lambda^{n/2-\alpha}\int a_j(\lambda, \sqrt{\lambda}x) f(\sqrt{\lambda}x) E_\alpha(\lambda, x, y)\,dx\,. \tag{128}$$

According to theorem 6,

$$\begin{aligned} B_1(\lambda, y) =& \lambda^{(n-1)/2-\alpha}\sum_{k=1}^{4}\int e^{i\lambda\psi_k(x,y)} a_1(\lambda, \sqrt{\lambda}x) b_k(\lambda, x, y, \alpha) f(\sqrt{\lambda}x)\,dx \\ &+ RO(\lambda^{n/2-\alpha-1})\,, \end{aligned} \tag{129}$$

where

$$R = \int a_1(\lambda, \sqrt{\lambda}x)|f(\sqrt{\lambda}x)||x|^{-(n+1)/2-\alpha}\,dx\,.$$

Using (126) we get

$$|R| < c\lambda^{\beta/2}J(\lambda)\,, \ J(\lambda) = \int_{\lambda^{-1/2}}^{1}\sigma^{(n-1)/2-\alpha+\beta-1}\,d\sigma\,. \tag{130}$$

Denote by $I(\lambda)$ the integral in (129). Integrating by parts and using (19) and the estimate (126) for f and ∇f, we obtain

$$|I(\lambda)| < c[\lambda^{\beta/2-1/2}J(\lambda) + \lambda^{\beta/2-1} + \lambda^{-(n+1)/4)+\alpha/2}]\,. \tag{131}$$

Since

$$J(\lambda) < c \text{ if } \beta > \alpha - (n-1)/2\,;$$
$$J(\lambda) < c\log\lambda \text{ if } \beta = \alpha - (n-1)/2\,;$$
$$J(\lambda) < c\lambda^{-(n-1)/4-\beta/2+\alpha/2} \text{ if } \beta < \alpha - (n-1)/2$$

we have from (129)–(131) if $\alpha \geq (n-1)/2$,

$$|B_1(\lambda, y)| < c[\lambda^{n/2-\alpha-1+\beta/2} + \lambda^{-1/2}\log\lambda],$$

or

$$|K_1(\lambda, y)| < c[\lambda^{n/2-\alpha-1+\beta/2} + \lambda^{-1/2}\log\lambda], \quad \text{where } \beta < 2\alpha + 2 - n,\ \alpha \geq (n-1)/2. \tag{132}$$

b. Estimate of $K_2(\lambda, y)$. Using corollary 3 and (128) we can write

$$\begin{aligned} B_2(\lambda, y) = & \lambda^{(n-1)/2-\alpha} \sum_{k=1}^{4} \int e^{i\lambda\psi_k(x,y)} a_2(\lambda, \sqrt{\lambda}x) g_k(\lambda, x, y) f(\sqrt{\lambda}x)\, dx \\ & + R_1 O(\lambda^{n/2-\alpha-1}), \end{aligned} \tag{133}$$

where

$$\begin{aligned} g_k(\lambda, x, y) &= a_k(\lambda, x, y)(a^2 - x^2)^{-1/4} + b_k(\lambda, x, y)(a^2 - x^2)^{1/4}, \\ R_1 &= \int a_2(\lambda, \sqrt{\lambda}x)|f(\sqrt{\lambda}x)|(1 - x^2)^{-1}\, dx. \end{aligned}$$

Evidently,

$$R_1 < c\lambda^{\beta/2}\log\lambda. \tag{134}$$

Let $I(\lambda)$ be the integral in (133).

Integrating by parts we obtain analogously to (131),

$$|I(\lambda)| < c[\lambda^{-1} I_1(\lambda) + \lambda^{-1/2} I_2(\lambda)], \tag{135}$$

where

$$\begin{aligned} I_1 &= \int a_2(\lambda, \sqrt{\lambda}x)|f(\sqrt{\lambda}x)|(1 - x^2)^{-7/4}\, dx, \\ I_2 &= \int a_2(\lambda, \sqrt{\lambda}x)|(\nabla f)(\sqrt{\lambda}x)|(1 - x^2)^{-3/4}\, dx. \end{aligned}$$

To estimate $I_1(\lambda)$ we use that $(1 - x^2)^{-3/4} < \lambda^{1/2}$ on the support of the integrand. Since $|f(\sqrt{\lambda}x)| + |(\nabla f)(\sqrt{\lambda}x)| < c\lambda^{\beta/2}$ we get

$$|I_1(\lambda)| < c\lambda^{\beta/2+1/2}\log\lambda,\ |I_2(\lambda)| < c\lambda^{\beta/2}. \tag{136}$$

Now (133)–(136) imply

$$|B_2(\lambda, y)| < c\lambda^{\beta/2+n/2-\alpha-1}\log\lambda,$$

hence

$$|K_2(\lambda, y)| < c\lambda^{\beta/2+n/2-\alpha-1}\log\lambda, \text{ where } \beta < 2\alpha + 2 - n. \tag{137}$$

Finally, (79)–(81), (125), (127), (132), (137) show that

$$|R_\lambda^\alpha f(y)| < c\int_{x^2<A} |f(x)|\, dx + o(1), \text{ locally uniformly if } \alpha \geq (n-1)/2,\ \beta < 2\alpha + 2 - n.$$

This and (98) give the theorem.

8. Proof of proposition 1

We shall use the formula

$$e_\alpha(\lambda, x, y) = \Gamma(\alpha+1)(2\pi i)^{-1} \int_S e^{\lambda p} V(p, x, y) H_\alpha(\lambda+n, p)\, dp\,, \tag{138}$$

where $S = (\delta - i\pi/2, \delta + i\pi/2)$, $\delta > 0$, $\alpha > 0$ and the function $s \to H_\alpha(s, p)$ is 2-periodic,

$$H_\alpha(s, p) = \sum_{k=-\infty}^{\infty} e^{isk\pi} (p + ik\pi)^{-\alpha-1},\ \ p \in S,\ \ \alpha > 0\,. \tag{139}$$

For proving (138) we notice that $p^{-\alpha-1}\Gamma(\alpha+1)V(p, x, y)$ is the Laplace transform of the function $\lambda \to e_\alpha(\lambda, x, y)$, where

$$V(p, x, y) = \int_0^\infty e^{-\lambda p} de(\lambda, x, y),\ \ Re\, p > 0,$$

in particular, $V(p + ik\pi, x, y) = e^{ik\pi n} V(p, x, y)$. Applying the inverse Laplace formula we get (138). Note that for $Re\, p > 0$ ([21]),

$$V(p, x, y) = (2\pi \sinh 2p)^{-n/2} \exp\left(-\frac{1}{2}(x^2 + y^2)\coth 2p + xy(\sinh 2p)^{-1}\right).$$

Having in mind the singularities at the points $p = 0$, $p = \pm i\pi/2$ we write for $E_\alpha(\lambda, x) = E_\alpha(\lambda, x, 0)$:

$$E_\alpha(\lambda, x) = \sum_{j=1}^{3} E_j(\lambda, x, \delta)\,, \tag{140}$$

where

$$E_j(\lambda, x, \delta) = c \int e^{\lambda p} V(p, \sqrt{\lambda}x, 0) H_\alpha(\lambda+n, p) g_j(p)\, dp$$

and g_j are C^∞ functions, $1 = g_1(p) + g_2(p) + g_3(p)$ for $p \in S$, and $supp\, g_1 \subset \{p \in S : |Im\, p| < \epsilon\}$, $supp\, g_2 \subset \{p \in S : |Im\, p| > \pi/2 - \epsilon\}$ for some small $\epsilon > 0$, and g_2 is $i\pi$-periodic.

To evaluate E_1 we begin with the representations for $Re\, p > 0$:

$$V(p, x, 0) = (2\pi)^{-n} \int \exp\left(-\frac{1}{2}\xi^2 \sinh 2p - \frac{1}{2}x^2 \tanh p + ix\xi\right) d\xi$$

and

$$1 = c_\alpha (\sinh 2p)^{\alpha+1} \int \eta^{2\alpha} \exp\left(-\frac{1}{2}\eta^2 \sinh 2p\right) d\eta,\ \ \eta \in \mathbb{R}^2\,.$$

Since $H_\alpha(\lambda+n, p) = p^{-\alpha-1} + h_\alpha(\lambda+n, p)$ and h_α has no singularities on S we obtain

$$E_1(\lambda, x, \delta) \sim I_1(\lambda, x, \delta) + I_2(\lambda, x, \delta)\,, \tag{141}$$

where

$$\begin{aligned} I_1 &= \lambda^{n/2+\alpha+1} \int e^{\lambda\phi_1} q_1\, d\xi\, d\eta\, dp\,, \\ \phi_1 &= p - \frac{1}{2}(\xi^2 + \eta^2)\sinh 2p - \frac{1}{2}x^2 \tanh p + ix\xi \end{aligned}$$

and $q_1 = c(p/\sinh 2p)^{-\alpha-1} g(\xi,\eta) g_1(p)\eta^{2\alpha}$. Here g is a cutoff function due to an integration by parts using the property $|\partial_p \phi_1| > c(\xi^2+\eta^2)$ for large $\xi^2+\eta^2$. The equivalence "$a(\lambda, x, \delta) \sim b(\lambda, x, \delta)$" here means that $a - b = e^{c\lambda\delta} O(\lambda^{-\infty})$. Further,

$$I_2 = \lambda^{n/2} \int e^{\lambda\phi_2} q_2 \, d\xi \, dp \,,$$

where

$$\phi_2 = p - \frac{1}{2}\xi^2 \sinh 2p - \frac{1}{2}x^2 \tanh p + ix\xi$$

and $q_2 = c\, h_\alpha(\lambda, p) g(\xi) g_1(p)$ for some cutoff function g.

To represent E_2 we use the periodicity of the integrand in (138) and the formula

$$V(p + i\pi/2, x, 0) = (4\pi \cosh 2p)^{-n/2} \int \exp\left(-\frac{1}{2}\xi^2 \tanh 2p + ix\xi\right) d\xi \,.$$

Thus

$$E_2(\lambda, x, \delta) \sim \lambda^{n/2} \int e^{\lambda\phi} q \, d\xi \, dp \,, \tag{142}$$

where

$$\phi = p - \frac{1}{2}\xi^2 \tanh 2p + ix\xi$$

and $q = c(\cosh 2p)^{-n/2} H_\alpha(\lambda + n, p + i\pi/2) g(\xi) g_2(p + i\pi/2) e^{i\lambda\pi/2}$.

Therefore letting $\delta \to 0$ in (140)–(142) we obtain

$$E_\alpha(\lambda, x) = \sum_{j=1}^{3} E_j(\lambda, x) \,, \tag{143}$$

where $E_1 \sim I_1(\lambda, x) + I_2(\lambda, x)$,

$$\begin{aligned}
I_1 &= \lambda^{n/2+\alpha+1} \int e^{i\lambda\phi_1} q_1 \, dt \, d\xi \, d\eta \,, & I_2 &= \lambda^{n/2} \int e^{i\lambda\phi_2} q_2 \, dt \, d\xi \,, \\
\phi_1 &= t - \frac{\xi^2+\eta^2}{2} \sin 2t - \frac{x^2}{2} \tan t + x\xi \,, & \phi_2 &= t - \frac{\xi^2}{2} \sin 2t - \frac{x^2}{2} \tan t + x\xi \,, \\
q_1 &= c(t/\sin 2t)^{-\alpha-1} g(\xi,\eta) g_1(it) \eta^{2\alpha} \,, & q_2 &= c(\cos 2t)^{-n/2} h_\alpha(\lambda, it) g(\xi) g_1(it) \,.
\end{aligned}$$

The integral E_2 is analogous to I_2 and

$$\begin{aligned}
E_3 &= \int e^{i\lambda\phi_3} q_3 \, dt \,, \\
\phi_3 &= t + \frac{1}{2}x^2 \cot 2t \,, \\
q_3 &= (2\pi)^{-1} \Gamma(\alpha+1) (2\pi i \sin 2t)^{-n/2} H_\alpha(\lambda + n, it) g_3(it) \,.
\end{aligned} \tag{144}$$

Further, to find the uniform asymptotics of the integrals E_j in the domain $\{x : \lambda^{-5/2} < x^2 < \epsilon\}$ we shall apply the method of the stationary phase.

a. Asymptotics of I_1. First, in polar coordinates $(\xi, \eta) = \sigma(\omega, \theta)$, $\sigma > 0$, $\omega \in \mathbb{R}^n$, $\omega^2 + \theta^2 = 1$ the integral I_1 becomes

$$I_1 = \lambda |x|^{-n/2-\alpha} \int_0^\infty \int e^{i\lambda\psi_0} \sigma^{n/2+\alpha+1} J_{n/2+\alpha}(\lambda|x|\sigma) q(t, \sigma) \, dt \, d\sigma \,, \tag{145}$$

taking into account the relation

$$\int e^{i\lambda x\omega\sigma}\theta^{2\alpha}\,d(\theta,\omega) = c_\alpha(\lambda|x|\sigma)^{-n/2-\alpha}J_{n/2+\alpha}(\lambda|x|\sigma)\,.$$

Here $c_\alpha = (2\pi)^{n/2+1}2^\alpha\Gamma(\alpha+1)$ and

$$\psi_0 = t - \frac{1}{2}\sigma^2\sin 2t - \frac{x^2}{2}\tan t\,. \tag{146}$$

On the other hand, $|\partial_t\psi| > c > 0$ if σ is close to zero and then we can integrate by parts. Using the bound $|J_\beta(s)| < c|s|^{-1/2}$ to estimate the remainder we obtain

$$\begin{aligned} I_1 = {} & (\lambda|x|)^{-n/2-\alpha}\int\int e^{i\lambda\psi_0}J_{n/2+\alpha}(\lambda|x|\sigma)q(t,\sigma)\,dt\,d\sigma \\ & + (\lambda|x|)^{-(n+1)/2-\alpha}O(\lambda^{-\infty})\,, \end{aligned} \tag{147}$$

where $q \in C_o^\infty(\mathbb{R}^n \times (0,\infty))$. We shall use the asymptotics of the Bessel function,

$$J_{n/2+\alpha}(\sigma) = \sum_{j=0}^{2}(c_j e^{i\sigma} + \bar{c}_j e^{-i\sigma})\sigma^{-1/2-j} + O(\sigma^{-7/2})\,. \tag{148}$$

Since $|x| > \lambda^{-5/4}$ and $n \geq 2$ it follows that

$$I_1 = \lambda^{1/2}|x|^{-(n+1)/2-\alpha}\sum_{j=0}^{2}(\lambda|x|)^{-j}(K_j^+ + K_j^-) + |x|^{-(n+1)/2-\alpha}O(\lambda^{-3/4})\,, \tag{149}$$

where

$$K_j^\pm = \int_0^\infty\int e^{i\lambda\psi_\pm}q(t,\sigma)\,dt\,d\sigma,\ \ q \in C_0^\infty$$

and $\psi_\pm = t - \frac{1}{2}\sigma^2\sin 2t - \frac{1}{2}x^2\tan t \pm |x|\sigma$.

We have the following critical points: $(t_k, 1)$ for ψ_+ and $(-t_k, 1)$ for ψ_- $(k = 1, 2)$, where $t_k \in (0, \pi/2)$, $\cos 2t_1 = \sqrt{1-x^2}$, $\cos 2t_2 = -\sqrt{1-x^2}$, for which $\det\psi''(t_k, 1) = -4\cos 2t_k$. Since $x^2 < \epsilon$ it follows that $t_1 \sim 1$, $t_2 \sim \pi/2$ and hence on the support of the integrand in K_j the phase ψ_+ has only one critical point $(t_1, 1)$ and the phase ψ_- has only one critical point $(-t_1, 1)$. Therefore the method of the stationary phase implies

$$K_0^+ + K_0^- = \lambda^{-1}a(\lambda, x) + O(\lambda^{-2})\,, \tag{150}$$

where

$$a(\lambda, x) = Re\,(c\,e^{i\lambda\psi(x)})(t_1^{-1}\sin 2t_1)^{\alpha+1}(1-x^2)^{-1/4},\ \ \psi = \psi_+(t_1, 1)\,. \tag{151}$$

Note that

$$|\nabla\psi(x)|^2 = 1 - x^2\,. \tag{152}$$

We have also

$$K_j^\pm = O(\lambda^{-1}),\ \ j = 1, 2\,. \tag{153}$$

b. Asymptotics of I_2. In polar coordinates $\xi = \sigma\omega$, $\sigma > 0$, we obtain that the integral I_2 has the same form as in (145), where α is formally replaced by -1. Therefore we have the estimate

$$I_2 = |x|^{-(n-1)/2} O(\lambda^{-1/2})\,. \tag{154}$$

Note that E_2 has the same bound (154) and analogously

$$E_3 = O(\lambda^{-1/2})\,. \tag{155}$$

Now (143), (149)–(155) imply

$$E_\alpha(\lambda, x) = \lambda^{-1/2}|x|^{-(n+1)/2-\alpha}(a(\lambda, x) + |x|^{\alpha+1}O(1) + O(\lambda^{-1/4}))\,. \tag{156}$$

Finally, it is clear that (22)–(24) follow from (156) with $a(x) = c(t_1^{-1}\sin 2t_1)^{\alpha+1}(1-x^2)^{-1/4}$, $b(\lambda, x) = |x|^{\alpha+1}O(1) + O(\lambda^{-1/4})$.

9. Proof of proposition 2

We have the formula

$$E_\alpha(\lambda, x, y) = \int_S e^{\lambda\phi} q\, dp\,, \tag{157}$$

where $\phi = p - 2^{-1}(x^2 + y^2)\coth 2p + xy(\sinh 2p)^{-1}$ and

$$q = \Gamma(\alpha + 1)(2\pi i)^{-1}(2\pi \sinh 2p)^{-n/2} H_\alpha(\lambda + n, p)\,. \tag{158}$$

Evidently (157) follows from (138), (139). To find the uniform asymptotics of the integral (157) we shall apply the saddle point method. The phase function ϕ has the critical points $p_\pm$, $\bar{p}_\pm$, satisfying $\cosh 2p = xy + \sqrt{d}$, where $d = d(x, y) = (xy)^2 + 1 - x^2 - y^2$. Let $x = r\omega$, $|\omega| = 1$. Since $d = -r^2(1 - (\omega y)^2)$ and $|\omega y| < 1/2$, $r^2 < \epsilon$, the critical points degenerate if $r = 0$, coinciding with $p_0 = i\pi/4$ and $\bar{p}_0$ respectively. Near the critical points the integrand in (157) is a holomorphic function and $\partial_p^3\phi \neq 0$, $\partial_{p,r}^2\phi \neq 0$ for $p = \pm i\pi/4$. Therefore we can apply Lemma 2.3 ([6, p. 343]) and conclude that there exists a holomorphic change of variables $p = p(z, x, y)$, defined in a neighbourhood of the points $z = 0$, $r = 0$, such that

$$\phi(p(z, x, y), x, y) = A(x, y) - B(x, y)z + z^3/3,\ \ p(0, 0, y) = p_0\,,$$

for every fixed ω, y. In addition,

$$A \;=\; A(x, y) = \frac{1}{2}(\phi(p_+, x, y) + \phi(p_-, x, y))\,, \tag{159}$$

$$B \;=\; B(x, y) = \left(\frac{3}{4}(\phi(p_+, x, y) - \phi(p_-, x, y))\right)^{2/3}\,. \tag{160}$$

To use this change of variables in the integral (157) we first notice that

$$E_\alpha(\lambda, x, y) \sim \int_L e^{\lambda\phi} q\, dp,\ \ L = L_1 \cup L_2\,, \tag{161}$$

where L_1 is the segment $(\delta + i(\pi/4 - 2\delta), \delta + i(\pi/4 + 2\delta))$ and L_2- the segment $(\delta - i(\pi/4 + 2\delta), \delta + i(-\pi/4 + 2\delta))$ for $\delta > 0$ small enough. The equivalence relation "$a(\lambda, x, y) \sim$

$b(\lambda, x, y)$" here means that $a - b = O(e^{-c\lambda})$, $c > 0$. Indeed, it sufficies to use the bound $Re\,\phi(p, x, y) \le -c < 0$ for $p \in S \setminus L$, which follows from the definition of ϕ if $\delta > 0$ is small enough.

Now (157)–(161) yield

$$E_\alpha(\lambda, x, y) \sim \sum_{j=1}^{2} e^{\lambda A_j} \int_{L_{1j}} e^{\lambda(-Bz+z^3/3)} q_j(z, \lambda)\, dz\,, \tag{162}$$

where $A_1 = A$, $A_2 = \bar{A}$, $q_1(z, \lambda) = q(p(z, x, y), x, y) \cdot \partial p/\partial z$, $q_2(z, \lambda) = q(\bar{p}(\bar{z}, x, y)) \cdot \partial \bar{p}(\bar{z}, x, y)/\partial z$, L_{1j} being the image of the segment L_j. Notice that $L_{1j} \subset \{z : Re\, z > 0\}$ and that the endpoints a_j, b_j of L_{1j} satisfy: $\arg a_j \in (-\pi/2, -\pi/6)$, $\arg b_j \in (\pi/6, \pi/2)$.

Using the Weierstrass preparation theorem,

$$q_j(z, \lambda) = r_j + r_{1j} z + (z^2 - B) q_{1j}(z, \lambda)$$

and the following representation of the Airy function

$$\begin{aligned} Ai(s) &= (2\pi i)^{-1} \int_M e^{-sz+z^3/3}\, dz,\ M = M_1 \cup M_2\,,\\ M_1 : z &= re^{i\theta},\ r \in (0, +\infty),\ \theta \in (-\pi/2, -\pi/6)\,,\\ M_2 : z &= re^{i\theta},\ r \in (0, +\infty),\ \theta \in (\pi/6, \pi/2) \end{aligned}$$

in the integral (162), we obtain the uniform asymptotics

$$E_\alpha(\lambda, x, y) = \lambda^{-1/3} Ai(B\lambda^{2/3})\, Re(a(\lambda, x, y)e^{\lambda A}) + Ai'(B\lambda^{2/3}) O(\lambda^{-2/3})\,, \tag{163}$$

where

$$a(\lambda, x, y) = c(x, y)(q(p_+, x, y) + q(p_-, x, y)),\ |c(x, y)| \asymp 1 \tag{164}$$

and q is given by (158). Note also that (159), (160) imply (26) and

$$A(x, y) = i\psi(x, y),\ \psi(0, y) = \pi/4,\ \partial_x \psi(0, y) = -y\,. \tag{165}$$

Let $\lambda = \lambda_m$, where $\lambda_m = 2m - n$ for positive integers m. Then one can see that $a(\lambda_m, x, y)$ does not depend on m for large m, and $|a(\lambda_m, x, y)| \asymp 1$. Therefore (25)–(27) follow from (163)–(165) and the estimate

$$|Ai'(B\lambda^{2/3})| < c\lambda^{1/6}\,. \tag{166}$$

10. Proof of proposition 3

We start with formula (157). Thus we have

$$E_\alpha(\lambda, x, 0) = \int_S e^{\lambda\phi} q(p, \lambda)\, dp\,, \tag{167}$$

where

$$\phi = \phi(p, x, 0) = p - 2^{-1} x^2 \coth 2p \tag{168}$$

and $q(p,\lambda)$ is given by (158). The critical points of ϕ satisfy $\cosh^2 2p = 1-x^2$. If $1-\delta < x^2 < 1-\delta/2$ and $\delta > 0$ is small we have four critical points $p_\pm = it_\pm$ and $\bar{p}_\pm$, where $\sin 2t_\pm = |x|$, hence $t_\pm = \pi/4 \pm \gamma$, $\gamma > 0$. If $x^2 = 1$ the critical points degenerate and $\partial_p^3\phi \neq 0$, $\partial_{p,r}^2\phi \neq 0$ for $p = \pm i\pi/4$, where $r = |x|$. Therefore, analogously to the proof of proposition 2, we obtain

$$E_\alpha(\lambda,x,0) = \lambda^{-1/3} Ai(B(x)\lambda^{2/3}) a(\lambda,x) + Ai'(B(x)\lambda^{2/3}) O(\lambda^{-2/3})\,, \tag{169}$$

where $a(\lambda,x) = Re(e^{\lambda A(x)} r(\lambda,x))$,

$$r(\lambda,x) = ic(x)(q(p_+,\lambda)+q(p_-,\lambda)),\ |c(x)| \asymp 1,\ c(x) \text{ is real}$$

and $A(x) = A(x,0)$, $B(x) = B(x,0)$ are given by (159), (160). From (165) we have $A(x) = i\pi/4$ and it is not hard to see that (29), (30) are valid.

To compute $r(\lambda,x)$ we use

$$r(\lambda,x) = c_1(x) e^{-i\pi/4}[H_\alpha(\lambda+n, it_+) + H_\alpha(\lambda+n, it_-)]\,,$$

where $c_1(x)$ is real and $|c_1(x)| \asymp 1$ as $x^2 \to 1$. From (139) it follows

$$\begin{aligned} a(\lambda,x) &= c_1(x)\sum_{k\geq 0}((t_+ + k\pi)^{-\alpha-1} + (t_- + k\pi)^{-\alpha-1})\,\cdot \\ &\quad \cdot \cos\left(\frac{(\lambda-n)\pi}{4} + (\lambda+n)k\pi - \frac{(\alpha+1)\pi}{2}\right) \\ &\quad + c_1(x)\sum_{k\geq 1}((k\pi - t_+)^{-\alpha-1} + (k\pi - t_-)^{-\alpha-1})\,\cdot \\ &\quad \cdot \cos\left(\frac{(\lambda-n)\pi}{4} + (\lambda+n)k\pi - \frac{(\alpha+1)\pi}{2}\right). \end{aligned}$$

Now we see that there is a sequence $\lambda_m \to \infty$ so that $a(x) = a(\lambda_m, x)$ does not depend on λ_m and for $x^2 = 1$,

$$a(x) = c_2(x)\left(\sum_{k\geq 0}(\pi/4 + k\pi)^{-\alpha-1} + \sum_{k\geq 1}(k\pi - \pi/4)^{-\alpha-1}\right),\ c_2(x) \neq 0\,.$$

Thus for this choice of $\lambda \to \infty$ we have

$$E_\alpha(\lambda,x,0) = \lambda^{-1/3} Ai(B(x)\lambda^{2/3} a(x) + Ai'(B(x)\lambda^{2/3}) O(\lambda^{-2/3})\,. \tag{170}$$

If $1 - c_1\lambda^{-2/3} < x^2 < 1 - c_2\lambda^{-2/3}$ then $|Ai'(B(x)\lambda^{2/3})| < c$ and (170) implies (28). If $1-\delta < x^2 < 1-\delta/2$ then we have only the estimate (166). Thus proposition 3 is proved.

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AMS Subject Classification: 42C10, 42C15.

G. E. KARADZHOV
Institute of Mathematics
Bulgarian Academy of Sciences
1113 Sofia, Bulgaria
geremika@math.bas.bg

Lavoro pervenuto in redazione il 30.6.1997.

Rendiconti del Seminario Matematico
dell'Università e del Politecnico di Torino

Volume 56, N. 3 (1998)

CONTENTS

ISSN 0373-1243 EDITRICE UNIVERSITARIA LEVROTTO & BELLA - TORINO