Rend. Sem. Mat. Univ. Pol. Torino
Vol. 56, 3 (1998)

G. E. Karadzhov

RIESZ SUMMABILITY OF MULTIPLE HERMITE SERIES†

Abstract.
Let $E_\lambda^\alpha f(y) = \sum(1-\mu_k/\lambda)^\alpha f_k \phi_k(y)$, $\mu_k < \lambda$ be the Riesz means of order α of the multiple Hermite series of the function f. The following main results are established: (i) The convergence $E_\lambda^\alpha f \to f$ as $\lambda \to \infty$ in the space $L_m^1 = L_m^1(\mathbb{R}^n)$, $n \geq 2$, with a norm $\int(1+|x|)^m |f(x)| dx$ under sharp conditions on the parameters α and m. (ii) The localization principle for the functions in the spaces L_m^p under sharp conditions on the parameters α, m, p. (iii) The convergence $E_\lambda^\alpha f(y) \to f(y)$ on the Lebesgue set of the function f from the spaces L_m^p.

1. Introduction and statement of the results

Consider the multiple Hermite series

$$f(y) \sim \sum f_k \phi_k(y)\,, \quad f_k = \int f(y)\phi_k(y)\,dy\,,$$

and the corresponding spherical partial sum

$$E_\lambda f(y) = \int e(\lambda, x, y) f(x)\,dx\,,$$

where

$$e(\lambda, x, y) = \sum \phi_k(x)\phi_k(y)\,, \quad \mu_k < \lambda$$

and $\mu_k = 2|k| + n$, $\phi_k(x) = H_k(x)\exp(-x^2/2)$ are the eigenvalues and orthonormalized eigenfunctions (Hermite functions) of the operator $-\Delta + x^2$ in $L^2(\mathbb{R}^n)$, $n \geq 2$. Here and later on the integral $\int$ means the integral over $\mathbb{R}^n$. Let

$$E_\lambda^\alpha f(y) = \sum(1-\mu_k/\lambda)^\alpha f_k\phi_k(y)\,, \quad \mu_k < \lambda\,,$$

be the Riesz means of order $\alpha > 0$ of the multiple Hermite series. Then

$$E_\lambda^\alpha f(y) = \int I^\alpha e(\lambda, x, y) f(x)\,dx\,, \tag{1}$$

where

$$I^\alpha e(\lambda, x, y) = \int_0^\lambda (1-\mu/\lambda)^\alpha\,de(\mu, x, y)$$

is the Riesz kernel of order α.

†Research partially supported by the Ministry of Education, Science and Tecnology under Contract MM410/94 and CNR

We are interested in the problem of the convergency $E_\lambda^\alpha f \to f$ as $\lambda \to \infty$. The case of compact manifolds without boundary and general elliptic operators is treated in several publications [2], [3], [4], [8], [9], [14], [17], where this convergence is in L^p norm or almost everywhere. The corresponding problems in $\mathbb{R}^n$ seem to be more difficult because of the arising both local and global obstacles. Some results for Schrödinger operators are obtained in [7], [10], [18], [20] and the classical Laguerre and Hermite series are considered, for example, in [1], [5], [12], [13], [15], [16], [21], [22].

Here we investigate further the multiple Hermite series. Our goal is to clarify the following three problems:

(i) The convergence in the space $L_m^1 = L_m^1(\mathbb{R}^n)$ with a norm $\|f\|_{1,m} = \int (1+|x|)^m |f(x)|\,dx$ for optimal conditions on the parameters α and m. The case $m = 0$ is treated also in [1], [22].

(ii) The localization principle for the functions in the space $L_m^p = L_m^p(\mathbb{R}^n)$ with a norm $\|f\|_{p,m} = (\int (1+|x|)^{mp} |f(x)|^p\,dx)^{1/p}$ for optimal conditions on the parameters α, m.

(iii) The convergence on the Lebesgue set of the function f from the spaces L_m^p. The case of L^p spaces for $p \ge 2$ is considered also in [22].

Let $\|E_\lambda^\alpha\|_{m,p}$ be the corresponding operator norm in the space L_m^p.

THEOREM 1. *If $\alpha > (n-1)/2$ and $-m_1 \le m \le 2\alpha + 1 - n$, where $m_1 = \min(3\alpha + 1 - n/2, 2\alpha + 2/3)$, then $\|E_\lambda^\alpha\|_{m,1} \le c$.*

Here and later on all estimates are uniform with respect to the parameter $\lambda > 1$ and c is a positive constant.

COROLLARY 1. *Under the conditions of Theorem 1, $E_\lambda^\alpha f \to f$ in the space L_m^1 or in the separable subspace $C_m \subset L_m^\infty$, consisting of all continuous functions f on $\mathbb{R}^n$ such that $|x|^m f(x) \to 0$ as $|x| \to \infty$.*

REMARK 1. The restrictions on the parameters α and m in Theorem 1 are optimal. Namely, there exists a sequence $\lambda \to \infty$ such that:

$$\|E_\lambda^\alpha\|_{m,1} \ge c\lambda^{m/2+(n-1)/2-\alpha}, \tag{2}$$

$$\|E_\lambda^\alpha\|_{m,1} \ge c(\lambda^{-m/2-\alpha-1/3} + \lambda^{-m/3-\alpha-1/3+n/6}), \tag{3}$$

$$\|E_\lambda^\alpha\|_{m,1} \ge \lambda^{(n-1)/4-\alpha/2}, \tag{4}$$

$$\|E_\lambda^\alpha\|_{m,1} \ge c\log\lambda \text{ if } \alpha = (n-1)/2. \tag{5}$$

Let

$$m_o = m_o(\alpha, p) = 2\alpha + 1 - n + \min(n/p, n/p + 1 - 4/3p)\,. \tag{6}$$

THEOREM 2 (LOCALIZATION). *Let $\alpha \ge (n-1)/2$, $\delta > 0$. Then as $\lambda \to \infty$*

$$R_\lambda^\alpha f(y) \overset{def}{=} E_\lambda^\alpha f(y) - \int_{|x-y|<\delta} f(x) I^\alpha e(\lambda, x, y)\,dx \to 0, \textit{ locally uniformly}, \tag{7}$$

for the functions $f \in L_m^p$ in the following cases:

(*a*) *if* $\alpha = (n-1)/2$ *then*

$$m \geq -m_o(\alpha, p) \text{ for } 1 \leq p < 4/3\,, \tag{8}$$
$$m > -m_o(\alpha, p) \text{ for } 4/3 \leq p < \infty\,; \tag{9}$$

(*b*) *if* $\alpha > (n-1)/2$ *then*

$$m \geq -m_o(\alpha, p) \text{ for } p \neq 4/3\,, \tag{10}$$
$$m > -m_o(\alpha, p) \text{ for } p = 4/3 \text{ or } p = \infty\,. \tag{11}$$

REMARK 2. The conditions on the parameters α and m in Theorem 2 are optimal. Namely, let $|R^\alpha_\lambda|_{m,p}$ denote the operator norm of R^α_λ as an operator from L^p_m to the space of all continuous functions on a fixed compact. Then there exists a sequence $\lambda \to \infty$ so that

$$|R^\alpha_\lambda|_{m,p} \geq c\left(\lambda^{(n-1)/4-\alpha/2} + \lambda^{-(m+m_o)/2}\right)\,, \tag{12}$$

$$|R^\alpha_\lambda|_{m,p} \geq c\lambda^{-(m+m_o)/2}\log\lambda \text{ if } p = 4/3\,, \tag{13}$$

$$|R^\alpha_\lambda|_{m,p} \geq c(\log\lambda)^{1/p'} \text{ if } \alpha = (n-1)/2,\ \ p > 4/3,\ \ m = -m_o,\ \ 1/p + 1/p' = 1\,. \tag{14}$$

REMARK 3. Theorem 2, case $p = \infty$, means that the relation (7) is true for all functions $f(x) = O(|x|^\beta)$ as $|x| \to \infty$ if $\beta < 2\alpha + 1 - n$. It turns out that we can allow a stronger growth at infinity for the functions having some regularity. Namely, we have

THEOREM 3 (LOCALIZATION). *Let f be locally in L^1 and let the derivatives $\nabla f(x)$ exist for $x^2 > C_f$. If $f(x)$ and $\nabla f(x)$ are $O(|x|^\beta)$ as $|x| \to \infty$ for $\beta < 2\alpha+2-n$ and $\alpha \geq (n-1)/2$ then the convergence (7) is fulfilled.*

COROLLARY 2. *Under the conditions of Theorems 2 or 3 the convergence*

$$E^\alpha_\lambda f(y) \to f(y) \text{ as } \lambda \to \infty$$

is fulfilled on the Lebesgue set of the function f if $\alpha > (n-1)/2$.

The main results are announced in [11], where Theorem 1 is proved only for the case $\alpha \geq n/2 - 1/3$ if $m < 0$. The case $\alpha > (n-1)/2$ of Theorems 2 and 3 is considered also in [5].

The further organisation of the paper is as follows. In section 2 the auxiliary results about the asymptotics of the Riesz kernels are stated and the proofs of the Propositions 1, 2, 3 are given in sections 8, 9, 10 respectively. Theorem 1 is proved in section 3. Remark 1 is proved in section 4. Theorem 2 and Remark 2 are proved in sections 5 and 6 respectively. Theorem 3 is proved in section 7.

2. Auxiliary results

From (1) it is clear that the first main problem in our investigations is to have asymptotics or appropriate estimates of the Riesz kernels $I^\alpha e(\lambda, x, y)$ as $\lambda \to \infty$, which must be global with respect to $x, y \in \mathbb{R}^n$. We shall use the following results, proved in [12]. Define

$$e_\alpha(\lambda, x, y) = \lambda^\alpha I^\alpha e(\lambda, x, y),\ \ E_\alpha(\lambda, x, y) = e_\alpha(\lambda, \sqrt{\lambda}x, \sqrt{\lambda}y)\,.$$

THEOREM 4. *([12]) If $x^2 + y^2 < const$ and $\alpha \geq (n-1)/2$, then*

$$|I^\alpha e(\lambda, x, y) - I^\alpha e^o(\lambda, x, y)| \leq c\,\lambda^{(n-1)/2} G_\alpha(\sqrt{\lambda}|x-y|)\,, \tag{15}$$

where $G_\alpha(s) = (1+s)^{-(n+1)/2-\alpha}$, $s \geq 0$ and

$$I^\alpha e^o(\lambda, x, y) \;=\; \lambda^{n/2} F_\alpha(\sqrt{\lambda}|x-y|)\,, \tag{16}$$

$$F_\alpha(s) \;=\; d_\alpha s^{-n/2-\alpha} J_{n/2+\alpha}(s)\,,\; d_\alpha = 2^\alpha \Gamma(\alpha+1)(2\pi)^{-n/2}\,. \tag{17}$$

THEOREM 5. *([12]) Let $A > 0$, $0 < \delta < 1$ be fixed and $A < x^2 < (1-\delta)\lambda$. There exists $\epsilon > 0$ such that if $|y| < \epsilon|x|$ and $\alpha > 0$ then*

$$E_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^{4} b_k(\lambda, x, y, \alpha) e^{i\lambda\psi_k(x,y)} + |x|^{-(n+1)/2-\alpha} O(\lambda^{-1})\,,$$

where the smooth functions $(x, y) \to b_k$, ψ_k satisfy

$$|b_k| < c|x|^{-(n+1)/2-\alpha}\,,\; |\nabla_x b_k| < c|x|^{-(n+1)/2-\alpha-1} \tag{18}$$

and

$$|\nabla_x \psi_k|^2 = 1 - x^2\,,\; |\partial^2_{x_i x_j} \psi_k|^2 < c(1-x^2)^{-1}\,. \tag{19}$$

THEOREM 6. *([12]) Let $|x^2 - 1| < \delta$, $|y| < \epsilon|x|$ for some small $\epsilon > 0$, $\delta > 0$. Then we have the uniform asymptotics*

$$E_\alpha(\lambda, x, y) = \sum_{k=0}^{\infty} (a_{1k}(\lambda, x, y)\lambda^{-k-1/3} + b_{1k}(\lambda, x, y)\lambda^{-k-2/3})\,,$$

where

$$a_{1k} = (a_k e^{i\lambda A} + b_k e^{-i\lambda A})\, Ai(\lambda^{2/3} B)\,,\; b_{1k} = (c_k e^{i\lambda A} + d_k e^{-i\lambda A})\, Ai'(\lambda^{2/3} B)$$

and the functions $\lambda \to a_k,\ b_k,\ c_k,\ d_k$, or their derivatives with respect to x are bounded. Here Ai is the Airy function, $A(x, y)$, $B(x, y)$ are real and smooth,

$$B(x, y) \sim c(y, \omega)(x^2 - a^2) \text{ as } x^2 \to a^2\,, \tag{20}$$

where $c(y, \omega) > 0$, $x = |x|\omega$ and

$$a^2 = (1-y^2)(1-(\omega y)^2)^{-1}\,. \tag{21}$$

COROLLARY 3. *The asymptotics*

$$E_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^{4} (a_k (a^2 - x^2)^{-1/4} + b_k (a^2 - x^2)^{1/4}) e^{i\lambda\psi_k} + (a^2 - x^2)^{-1} O(\lambda^{-1})\,,$$

is uniform for $x,\ y$ such that $1 - \delta < x^2 < 1 - \lambda^{-2/3+\epsilon}$, $|y| < \epsilon|x|$. Here $\epsilon > 0$, $\delta > 0$ are small and the functions $\lambda \to a_k,\ b_k$, or their derivatives with respect to x, are bounded, and ψ_k satisfy (19).

To prove the bounds (2)–(5) and (12)–(14) we need some additional information about the asymptotics of the Riesz kernels.

PROPOSITION 1. *The uniform asymptotics*

$$E_\alpha(\lambda, x, 0) = \lambda^{-1/2}|x|^{-(n+1)/2-\alpha}(Re(e^{i\lambda\psi(x)}a(x)) + b(\lambda, x)) \tag{22}$$

holds if $\lambda^{-3/2} < x^2 < \epsilon$.

Here $\epsilon > 0$ *is small enough, the functions* $x \to \psi(x)$, $a(x)$, $b(\lambda, x)$ *are smooth and*

$$|\nabla\psi(x)| \asymp 1\,, \tag{23}$$

$$|a(x)| \asymp 1\,,\ |b(\lambda, x)| < 4^{-1}|a(x)| \tag{24}$$

PROPOSITION 2. *Let* $y \in \mathbb{R}^n$ *be fixed,* $y^2 = 1$ *and* $x^2 < \epsilon$, $|xy| < |x|/2$ *for small* $\epsilon > 0$. *Then there exists a sequence* $\lambda \to \infty$ *such that*

$$E_\alpha(\lambda, x, y) = \lambda^{-1/3}Ai(\lambda^{2/3}B(x, y))Re(a(x, y)e^{i\lambda\psi(x,y)}) + O(\lambda^{-1/2})\,, \tag{25}$$

where

$$B(x, y) \asymp x^2 \text{ as } x \to 0\,, \tag{26}$$

$$|a(x, y)| \asymp 1\,,\ |\nabla_x\psi(x, y)| \asymp 1\,, \tag{27}$$

PROPOSITION 3. *Let* $1 - c_1\lambda^{-2/3} < x^2 < 1 - c_2\lambda^{-2/3}$, *where* c_1, c_2 *are positive constants. Then there is a sequence* $\lambda \to \infty$ *such that the uniform asymptotics*

$$E_\alpha(\lambda, x, 0) = \lambda^{-1/3}Ai(\lambda^{2/3}B(x))a(x) + O(\lambda^{-2/3}) \tag{28}$$

holds, where $B(x)$ *and* $a(x)$ *are real, smooth functions, and*

$$B(x) \asymp c(x^2 - 1) \text{ as } x^2 \to 1\,,\ |a(x)| > c > 0\,, \tag{29}$$

$$\psi(x) \overset{\text{def}}{=} \frac{2}{3}(-B(x))^{3/2} = \frac{1}{2}\left(|x|\sqrt{1 - x^2} - \arcsin\sqrt{1 - x^2}\right)\,. \tag{30}$$

Moreover, the asymptotics (28) is valid also for the case $1 - \delta < x^2 < 1 - \delta/2$ *if* $\delta > 0$ *is small and the rest* $O(\lambda^{-2/3})$ *is replaced by* $O(\lambda^{-1/2})$.

THEOREM 7. *([12]) If* $x^2 > 1 + \delta$ *and* $y^2 < \epsilon$ *then the uniform estimate*

$$|E_\alpha(\lambda, x, y)| \le c(x^2 - 1)^{-1/4}\lambda^{-1/2}\exp(-c\delta\lambda(x^2 - 1)^{1/2})$$

holds for all small $\delta > 0$, $\epsilon > 0$, *where* c *is a positive constant.*

We need also the following global estimates for the spectral function $e(\lambda, x, y)$.

THEOREM 8. *([13])*

$$e(\lambda, x, x) \le c\lambda^{n/2}e^{-x^2/\lambda}\,, \tag{31}$$

$$e(\lambda, x, x) \le C_N|x|^{-N} \text{ if } x^2 > (1 + \delta)\lambda\,,\ \delta > 0\,,\ N > 0\,. \tag{32}$$

3. Proof of Theorem 1

In order to prove the theorem it is sufficient to establish the uniform bound for $\lambda > 1$:

$$g(\lambda, y) \leq c(1+|y|)^m \,, \tag{33}$$

where

$$g(\lambda, y) = \int |I^\alpha e(\lambda, x, y)|(1+|x|)^m \, dx \,. \tag{34}$$

For every $y \in \mathbb{R}^n$ and for some fixed $\epsilon > 0$ consider the sets:

$$\begin{aligned} G_1(y) &= \{x \in \mathbb{R}^n \,:\, \epsilon(1+|y|) < 1+|x| < \epsilon^{-1}(1+|y|)\}\,, \\ G_2(y) &= \{x \in \mathbb{R}^n \,:\, 1+|y| < \epsilon(1+|x|)\}\,, \\ G_3(y) &= \{x \in \mathbb{R}^n \,:\, 1+|x| < \epsilon(1+|y|)\} \end{aligned}$$

and the corresponding functions $g_i(\lambda, y)$, given by (34), the integration being taken over $G_i(y)$, $1 \leq i \leq 3$.

a. Estimate of $g_1(\lambda, y)$. In this case the problem is the same as in L^1 space since

$$g_1(\lambda, y) \leq c(1+|y|)^m \int |I^\alpha e(\lambda, x, y)| \, dx \,. \tag{35}$$

On the other hand in [22] it is proved that $\|E^\alpha_\lambda\|_{0,1} < c$ if $\alpha > (n-1)/2$. Hence

$$\int |I^\alpha e(\lambda, x, y)| \, dx \leq c \,. \tag{36}$$

Thus (33) is valid for g_1.

b. Estimate of $g_2(\lambda, y)$. If $m \leq 0$ then $g_2(\lambda, y)$ satisfies the estimates (35), (36) for $\alpha > (n-1)/2$, hence it remains for considering only the case $m > 0$. On the other hand, (31) and (32) imply

$$|I^\alpha e(\lambda, x, y)| \leq C_N |x|^{-N} \text{ if } x^2 > (1+\delta)\lambda \,,\ \delta > 0 \,,\ N > 0 \,. \tag{37}$$

Now (36) and (37) show that it sufficies to estimate the integral

$$I(\lambda, y) \overset{\text{def}}{=} \int_G |x|^m |I^\alpha e(\lambda, x, y)| \, dx \text{ for } y^2 < \lambda(1+\delta) \,, \tag{38}$$

where $G = \{x \in \mathbb{R}^n : 1 < x^2 < \lambda(1+\delta),\ 1+|y| < \epsilon(1+|x|)\}$ and $\epsilon > 0$ is small.

Evidently

$$I(\lambda, \sqrt{\lambda} y) \leq c \lambda^{n/2-\alpha+m/2} \int_M |x|^m |E_\alpha(\lambda, x, y)| \, dx \,,\ y^2 < 1+\delta \,, \tag{39}$$

where $M = \{x \in \mathbb{R}^n : \lambda^{-1} < x^2 < 1+\delta,\ |y| < \epsilon|x|\}$.

Let $K_j = K_j(\lambda, y)$ be the right hand side of (39), where M is replaced by M_j and $M = M_1 \cup M_2$, $M_1 = \{x \in M : x^2 < 1-\delta\}$. To estimate K_1 we use Theorem 5 from which it follows

$$|E_\alpha(\lambda, x, y)| \leq c\lambda^{-1/2}|x-y|^{-(n+1)/2-\alpha},\ \alpha > 0 \text{ if } x^2 < 1-\delta,\ |y| < \epsilon|x| \,. \tag{40}$$

Since $|x - y| \asymp |x| > \lambda^{-1/2}$ for $x \in M$ we see that

$$K_1 \le c \text{ if } \alpha > (n-1)/2,\ m \le 2\alpha + 1 - n,\ y^2 < 1 + \delta\,. \tag{41}$$

The integral K_2 can be bound by Theorem 6 from which it follows

$$|E_\alpha(\lambda, x, y)| \le c\lambda^{-1/2}|d(x, y)|^{-1/4} \text{ if } |x^2 - 1| < \delta,\ |y| < \epsilon|x| \tag{42}$$

for small $\epsilon > 0$, $\delta > 0$. Here we have used the asymptotics of the Airy function, the relation (20) and

$$d = (a^2 - x^2)(1 - (y\omega)^2)\,, \tag{43}$$

where a^2 is given by (21). Thus

$$K_2 \le c \int_{M_2} |d(x, y)|^{-1/4}\, dx \text{ if } m \le 2\alpha + 1 - n\,. \tag{44}$$

For small $\delta > 0$, $\epsilon > 0$ we have $y^2 < 1/2$ therefore $a^2 = (1 - y^2)(1 - (y\omega)^2)^{-1}$ satisfies $1/2 < a^2 < 1$. Since in polar coordinates $x = \sigma\omega$,

$$\int_{M_2} |d(x, y)|^{-1/4}\, dx \le c \int H(y, \omega)\, d\omega\,, \tag{45}$$

where

$$H(y, \omega) = \int_0^c |\sigma - a|^{-1/4} |\sigma + a|^{1/4}\, d\sigma \tag{46}$$

for some $c > 1$, and $H(y, \omega) \le c(1 + a^{1/2}) \le \text{const}$ we obtain the bound

$$K_2 \le c \text{ if } m \le 2\alpha + 1 - n,\ y^2 < 1 + \delta\,. \tag{47}$$

Evidently (38), (39), (41), (47) imply

$$I(\lambda, y) \le c \text{ if } m \le 2\alpha + 1 - n\,,\ y^2 < \lambda(1 + \delta)\,.$$

Thus

$$g_2(\lambda, y) \le c(1 + |y|)^m \text{ if } m \le 2\alpha + 1 - n\,,\ \alpha > (n-1)/2\,. \tag{48}$$

c. Estimate of $g_3(\lambda, y)$. If $m \ge 0$ then g_3 satisfies (35), (36) for $\alpha > (n-1)/2$, therefore we shall suppose $m < 0$. Now we have to estimate $I(\lambda, y)$ given by (38), where $G = \{x \in \mathbb{R}^n : 1 < x^2 < \lambda(1 + \delta),\ 1 + |x| < \epsilon(1 + |y|)\}$.

For convenience we shall first consider the case $y^2 < \lambda(1 - \delta)$. Since $|y| \asymp |x - y|$ for $x \in G$ we have

$$I(\lambda, y) \le c(1 + |y|)^m J(\lambda, y)\,,\ m < 0\,, \tag{49}$$

where

$$J(\lambda, y) = \int_G (1 + |x - y|)^{-m} |I^\alpha e(\lambda, x, y)|\, dx\,.$$

Taking into account (36) it sufficies to consider the integral J for $|x-y|>1$. Then (40) implies

$$J(\lambda, \sqrt{\lambda}y) \le c\lambda^{-m/2+(n-1)/2-\alpha} \int_M |x-y|^{-m-(n+1)/2-\alpha}\, dx$$

if $y^2 < 1-\delta$, $M = \{x \in \mathbb{R}^n : |x| < \epsilon|y|,\ \lambda^{-1/2} < |x-y| < c\}$, hence

$$J(\lambda, y) \le c \text{ if } \alpha > (n-1)/2\,,\ -m \le 2\alpha + 1 - n\,. \tag{50}$$

Clearly (49) and (50) give

$$I(\lambda, y) \le c(1+|y|)^m \text{ if } \alpha > (n-1)/2\,,\ -m \le 2\alpha + 1 - n\,. \tag{51}$$

Now we pass to the case $2\alpha + 1 - n < |m| \le \alpha + (n+1)/2$. Using Theorem 5 we obtain

$$I(\lambda, y) \le c\lambda^{(n-1)/4-\alpha/2} \int_G (1+|x|)^m |x-y|^{-\alpha-(n+1)/2}\, dx \tag{52}$$

if $\alpha > 0$, $y^2 < \lambda(1-\delta)$. There exists a half-integer $k \ge 1$ such that $|m| \in (k(2\alpha+1-n), (k+1/2)(2\alpha+1-n)]$. Taking $p \ge 0$ so that $n - k(2\alpha+1-n) \le p \le \alpha + (n+1)/2 - |m|$ we have $n < |m| + p \le \alpha + (n+1)/2$. Since $|x-y| \asymp |y| \asymp 1+|y|$ for $x \in G$, (52) implies

$$I(\lambda, y) \le c \int (1+|x|)^{-|m|+p}\, dx\, (1+|y|)^{p-\alpha-(n+1)/2} \text{ if } \alpha > (n+1)/2\,,$$

hence

$$I(\lambda, y) \le c(1+|y|)^m \text{ if } \alpha > (n-1)/2\,,\ 2\alpha + 1 - n < |m| \le \alpha + (n+1)/2\,. \tag{53}$$

It remains the case $|m| > \alpha + (n+1)/2$. Since $\alpha > (n-1)/2$ we have $|m| > n$ and $\lambda^{(n-1)/4-\alpha/2}|x-y|^{-\alpha-(n+1)/2} \le c(1+|y|)^{-2\alpha-1}$, so the estimate (52) gives

$$I(\lambda, y) \le c(1+|y|)^m \text{ if } \alpha > (n-1)/2\,,\ \alpha + (n+1)/2 < |m| \le 2\alpha + 1\,. \tag{54}$$

Thus (51), (53) and (54) show that

$$I(\lambda, y) \le c(1+|y|)^m \text{ if } \alpha > (n-1)/2\,,\ -m \le 2\alpha + 1\,,\ y^2 < \lambda(1-\delta)\,. \tag{55}$$

Now consider the case $\lambda(1-\delta) < y^2 < \lambda(1+\delta)$. Using Theorem 6 we can write the uniform estimate

$$|I^\alpha e(\lambda, x, y)| \le c\lambda^{-\alpha-1/3} \tag{56}$$

if $\lambda(1-\delta) < y^2 < \lambda(1+\delta)$, $|x| < \epsilon|y|$ for small $\epsilon > 0$, $\delta > 0$. Therefore

$$I(\lambda, y) \le c(1+|y|)^m \text{ if } -m \le 2\alpha + 2/3\,,\ \alpha > n/2 - 1/3\,. \tag{57}$$

Let now $(n-1)/2 < \alpha \le n/2 - 1/3$. If $x^2 < \lambda^{1/3}$ then (56) implies (57) for $-m \le 3\alpha + 1 - n/2$. In the case $x^2 > \lambda^{1/3}$ we obtain (39), where now $M = \{x \in \mathbb{R}^n : \lambda^{-2/3} < x^2 < \epsilon\}$ for some small $\epsilon > 0$ and $|y^2 - 1| < \delta$. Then analogously to (42) we have

$$I(\lambda, \sqrt{\lambda}y) \le c\lambda^{(n-1)/2-\alpha+m/2} \int_M |x|^m |d(x,y)|^{-1/4}\, dx\,. \tag{58}$$

Using polar coordinates as after the estimate (44) we get

$$\int_M |x|^m |d(x,y)|^{-1/4}\,dx \le \sum_{j=1}^{2} \int_{F_j} |1-(y\omega)^2|^{-1/4} H_j(\lambda,y,\omega)\,d\omega\,, \tag{59}$$

where $F_2 = \{\omega : (1-y^2)(1-(y\omega)^2)^{-1} \ge 0\}$ and F_1 is the complement of F_2 to the unit sphere. Here

$$H_1(\lambda,y,\omega) = \int_{\lambda^{-1/3}}^{1} \sigma^{m+n-3/2}\,d\sigma \text{ if } \omega \in F_1$$

and for $a \ge 0$, $a^2 = (1-y^2)(1-(y\omega)^2)^{-1}$,

$$H_2(\lambda,y,\omega) = \int_{\lambda^{-1/3}}^{1} \sigma^{m+n-1}|\sigma-a|^{-1/4}|\sigma+a|^{-1/4}\,d\sigma \text{ if } \omega \in F_2\,.$$

We have the estimates:

$$H_j(\lambda,y,\omega) \le c \text{ if } -m \le 3\alpha+1-n/2\,,\ \alpha > (n-1)/2\ (j=1,2)\,. \tag{60}$$

Therefore (58)–(60) imply (57) for the case $-m \le 3\alpha+1-n/2$, $(n-1)/2 < \alpha \le n/2-1/3$. In other words,

$$g_3(\lambda,y) \le c(1+|y|)^m \text{ if } -m \le m_0\,,\ \alpha > (n-1)/2\,, \tag{61}$$

where $m_0 = \min(2\alpha+2/3, 3\alpha+1-n/2)$. Finally, (35), (48) and (61) imply (33) for $\alpha > (n-1)/2$ and $-m_0 \le m \le 2\alpha+1-n$. Theorem 1 is proved.

4. Proof of remark 1

We shall use the property $\|E_\lambda^\alpha\|_{m,1} = \|E_\lambda^\alpha\|_{-m,\infty}$.

a. Proof of bound (2). Let $f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})g(x\lambda^{-1/2})$, where $g(x)$ is the characteristic function of the set $\{x \in \mathbb{R}^n : \epsilon_1 < x^2 < \epsilon\}$ and $a(x)$ satisfies (24). Then $\|f_\lambda\|_{-m,\infty} \le c\lambda^{-m/2}$.

On the other hand

$$\|E_\lambda^\alpha f_\lambda\|_{-m,\infty} \ge |E_\lambda^\alpha f_\lambda(0)|\,, \tag{62}$$

hence it sufficies to prove

$$|E_\lambda^\alpha f_\lambda(0)| \ge c\lambda^{(n-1)/2-\alpha}. \tag{63}$$

Evidently,

$$E_\lambda^\alpha f_\lambda(0) = \lambda^{n/2-\alpha}\int E_\alpha(\lambda,x,0)e^{-i\lambda\psi(x)}\bar{a}(x)g(x)dx\,.$$

Using proposition 1, we obtain for large λ

$$|E_\lambda^\alpha f_\lambda(0)| \ge c\lambda^{(n-1)/2-\alpha}(1-|J(\lambda)|)\,, \tag{64}$$

where

$$J(\lambda) = \int e^{-2i\lambda\psi(x)}(\bar{a}(x))^2|x|^{-(n+1)/2-\alpha}g(x)dx\,.$$

Since $|\nabla\psi| \asymp 1$ we can integrate by parts, whence

$$J(\lambda) = O(\lambda^{-1}) . \tag{65}$$

Now (64), (65) imply (63). The bound (2) is proved.

b. Proof of bound (3). Let

$$f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2},y_0)}\bar{a}(x\lambda^{-1/2}, y_0)Ai^{-1}(B(x\lambda^{-1/2}, y_0)\lambda^{2/3})g(x) ,$$

where $y_0^2 = 1$ is fixed and $g(x)$ is the characteristic function of the set $\{x \in \mathbb{R}^n : x^2 < 1,\ |xy_0| < |x|/2\}$ and $B(x, y)$, $a(x, y)$ are given by (26), (27). Then the properties of the Airy function Ai give $\|f_\lambda\|_{-m,\infty} \le c$. Since

$$\|E_\lambda^\alpha f_\lambda\|_{-m,\infty} \ge \lambda^{-m/2}|E_\lambda^\alpha f_\lambda(y_0\lambda^{1/2})| ,$$

it suffices to prove

$$|E_\lambda^\alpha f_\lambda(y_0\lambda^{1/2})| \ge c\lambda^{-\alpha-1/3} . \tag{66}$$

We have

$$E_\lambda^\alpha f_\lambda(y_0\lambda^{1/2}) = \lambda^{n/2-\alpha}\int E_\alpha(\lambda, x, y_0)e^{-i\lambda\psi(x,y_0)}\bar{a}(x, y_0)Ai^{-1}(B\lambda^{2/3})g(x\lambda^{1/2})dx$$

and proposition 2 implies

$$|E_\lambda^\alpha f_\lambda(y_0\lambda^{1/2})| \ge c\lambda^{n/2-\alpha-1/3}\left(\int g(x\lambda^{1/2})dx - |I(\lambda)|\right) , \tag{67}$$

where

$$I(\lambda) = \int e^{-2i\lambda\psi(x,y_0)}(\bar{a}(x, y_0))^2 g(x\lambda^{1/2})dx .$$

According to (27) we can integrate by parts, whence

$$I(\lambda) = O(\lambda^{-n/2-1/2}) . \tag{68}$$

On the other hand, $\int g(x\lambda^{1/2})dx = c\lambda^{-n/2}$, therefore (66) is a consequence of (67), (68). The first part of bound (3) is proved. To prove its second part, it is sufficient to replace the function g in the definition of $f_\lambda(x)$ by the characteristic function of the set $\{x \in \mathbb{R}^n : \lambda^{1/3}/2 < x^2 < \lambda^{1/3},\ |xy_0| < |x|/2\}$. Then $\|f_\lambda\|_{-m,\infty} \le c\lambda^{-m/6}$ and $\int g(x\lambda^{1/2})dx = c\lambda^{-n/3}$.

c. Proof of bound (4). Let $f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})g(x)$, where $g(x)$ is the characteristic function of the set $\{x \in \mathbb{R}^n : 1 < x^2 < 2\}$. Then $\|f_\lambda\|_{-m,\infty} \le c$ and according to (62) it sufficies to prove

$$|E_\lambda^\alpha f_\lambda(0)| \ge c\lambda^{(n-1)/4-\alpha/2}. \tag{69}$$

Since

$$E_\lambda^\alpha f_\lambda(0) = \lambda^{n/2-\alpha}\int E_\alpha(\lambda, x, 0)e^{-i\lambda\psi(x)}\bar{a}(x)g(x\lambda^{1/2})dx$$

the proposition 1 gives for large λ

$$|E_\lambda^\alpha f_\lambda(0)| \ge c\lambda^{(n-1)/2-\alpha}\left(\int |x|^{-(n+1)/2-\alpha}g(x\lambda^{1/2})dx - |I(\lambda)|\right) , \tag{70}$$

where

$$I(\lambda) = \int e^{-2i\lambda\psi(x)}(\bar{a}(x))^2|x|^{-(n+1)/2-\alpha}g(x\lambda^{1/2})dx\,.$$

Integrating by parts we find

$$I(\lambda) = O(\lambda^{\alpha/2-(n+1)/4}). \tag{71}$$

On the other hand,

$$\int |x|^{-(n+1)/2-\alpha}g(x\lambda^{1/2})dx = c\lambda^{\alpha/2-(n-1)/4}. \tag{72}$$

Thus (69) follows from (70)-(72). The bound (4) is proved.

d. Proof of bound (5). Let $f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})g_\lambda(x)$, where g_λ is the characteristic function of the set $\{x \in \mathbb{R}^n : \lambda^{-1/2} < x^2 < 1\}$. Then $\|f_\lambda\|_{-m,\infty} \le c$ and because of (62) it sufficies to prove

$$|E_\lambda^\alpha f_\lambda(0)| \ge c\log\lambda \text{ for } \alpha = (n-1)/2\,. \tag{73}$$

Since

$$E_\lambda^\alpha f_\lambda(0) = \lambda^{n/2-\alpha}\int E_\alpha(\lambda, x, 0)e^{-i\lambda\psi(x)}\bar{a}(x)g_\lambda(x\lambda^{1/2})\,dx$$

we obtain from proposition 1

$$|E_\lambda^\alpha f_\lambda(0)| \ge c\left(\int |x|^{-n}g_\lambda(x\lambda^{1/2})dx - |J(\lambda)|\right) \tag{74}$$

for large λ, where

$$J(\lambda) = \int e^{-2i\lambda\psi(x)}(\bar{a}(x))^2|x|^{-n}g_\lambda(x\lambda^{1/2})\,dx\,.$$

Integrating by parts we get the bound

$$J(\lambda) = O(\lambda^{-1/4})\,. \tag{75}$$

Noticing that $\int |x|^{-n}g_\lambda(x\lambda^{1/2})dx \ge c\log\lambda$ we see that (73) follows from (74), (75). The bound (5) is proved.

5. Proof of theorem 2

According to (7)

$$R_\lambda^\alpha f(y) = \int_{|x-y|>\delta} f(x)I^\alpha e(\lambda, x, y)\,dx\,. \tag{76}$$

We shall prove the estimate

$$|R_\lambda^\alpha f(y)| \le c\|f\|_{m,p} \tag{77}$$

uniformly for $y^2 < B$, $\lambda > \lambda_o$, if α and m satisfy the conditions of theorem 2 and $\lambda_o > 1$ is sufficiently large. In short, this estimate follows from theorems 4-7 and the Hölder inequality. From theorem 4 it follows

$$|I^\alpha e(\lambda, x, y)| \le c\lambda^{(n-1)/4-\alpha/2} \text{ if } x^2 + y^2 < c,\ |x-y| > \delta,\ \alpha \ge (n-1)/2\,. \tag{78}$$

Therefore

$$|R_\lambda^\alpha f(y)| < c\lambda^{(n-1)/4-\alpha/2}\int_{x^2<A}|f(x)|\,dx + c|K(\lambda,y)|,\ \ y^2 < B, \tag{79}$$

where A is sufficiently large and

$$K(\lambda,y) = \int_{x^2>A} f(x) I^\alpha e(\lambda,x,y)\,dx,\ \ y^2 < B\,. \tag{80}$$

To estimate $K(\lambda, y)$, uniformly for $y^2 < B$, $\lambda > \lambda_o$, it is convenient to consider the functions

$$K_j(\lambda,y) = \int a_j(\lambda,x) f(x) I^\alpha e(\lambda,x,y)\,dx,\ \ 1 \le j \le 5\,, \tag{81}$$

where $a_j(\lambda, x)$ is the characteristic function of the set S_j and for small $\epsilon > 0$, $\delta > 0$,

$$\begin{aligned} &S_1 = \{x : A < x^2 < (1-\delta)\lambda\}, && S_2 = \{x : (1-\delta)\lambda < x^2 < \lambda - \lambda^{1/3}\},\\ &S_3 = \{x : |x^2-\lambda| < \lambda^{1/3}\}, && S_4 = \{x : \lambda + \lambda^{1/3} < x^2 < \lambda + \lambda^{1/3+\epsilon}\},\\ &S_5 = \{x : x^2 > \lambda + \lambda^{1/3+\epsilon}\}. \end{aligned}$$

a. Estimate of $K_1(\lambda, y)$. From theorems 4 and 5 it follows

$$|I^\alpha e(\lambda,x,y)| \le c\lambda^{(n-1)/4-\alpha/2}(1-x^2/\lambda)^{-1/4}|x|^{-(n+1)/2-\alpha} \tag{82}$$

if $x \in S_1$, $y^2 < B$. Therefore the Hölder inequality gives

$$|K_1(\lambda,y)| \le c\lambda^{(n-1)/4-\alpha/2}\|f\|_{m,p} J(\lambda)\,, \tag{83}$$

where $J(\lambda) < c$ if $m > -\alpha - n/p + (n-1)/2$; $J(\lambda) < c(\log\lambda)^{1/p'}$ if $m = -\alpha - n/p + (n-1)/2$, $1/p + 1/p' = 1$; $J(\lambda) < c\lambda^{-(\alpha+m)/2-n/2p+(n-1)/4}$ if $m < -\alpha - n/p + (n-1)/2$. We claim that

$$|K_1(\lambda,y)| \le c\|f\|_{m,p}\,, \tag{84}$$

under the conditions of theorem 2. If $\alpha > (n-1)/2$ then evidently (83) gives (84) for $m \ge -\alpha - n/p + (n-1)/2$. If $m < -\alpha - n/p + (n-1)/2$ and $m \ge -m_o(\alpha, p)$ then (84) follows from (83) as well. Let now $\alpha = (n-1)/2$. If $1 \le p < 4/3$ and $m \ge -m_o(\alpha, p)$ then $m \ge -n/p - 1 + 4/3p > -n/p$ and (83) implies (84). If $4/3 \le p < \infty$ and $m > -m_o(\alpha, p)$ then $m > -n/p$, hence (84) follows from (83).

b. Estimate of $K_2(\lambda, y)$. Now we shall prove

$$|K_2(\lambda,y)| \le c\lambda^{-(m+m_o)/2}\|f\|_{m,p} \text{ if } p \ne 4/3\,, \tag{85}$$

$$|K_2(\lambda,y)| \le c\lambda^{-(m+m_o)/2}(\log\lambda)^{1/4}\|f\|_{m,p} \text{ if } p = 4/3\,. \tag{86}$$

To this end notice that theorem 5 gives the estimate

$$|I^\alpha e(\lambda,x,y)| \le c\lambda^{-\alpha-1/2}(1-x^2/\lambda)^{-1/4} \text{ if } x \in S_2,\ \ y^2 < B\,. \tag{87}$$

Therefore

$$|K_2(\lambda,y)| \le c\lambda^{-\alpha-m/2+(n-1)/2-n/2p}\|f\|_{m,p} J(\lambda)\,, \tag{88}$$

where now

$$J(\lambda) < c \text{ if } p > 4/3\,,$$
$$J(\lambda) < c(\log\lambda)^{1/4} \text{ if } p = 4/3\,,$$
$$J(\lambda) < c\lambda^{-1/2+2/3p} \text{ if } p < 4/3\,.$$

This and the definition of $m_o(\alpha, p)$ (see (6)) imply (85), (86).

c. Estimate of $K_3(\lambda, y)$. From theorem 6 it follows

(89) $$|I^\alpha e(\lambda, x, y)| \le c\lambda^{-\alpha-1/3}, \text{ if } x \in S_3,\ y^2 < B\,.$$

Applying the Hölder inequality we get

(90) $$|K_3(\lambda, y)| \le c\lambda^{-(m+m_o)/2}\|f\|_{m,p}\,.$$

d. Estimate of $K_4(\lambda, y)$. Theorem 7 implies

(91) $$|I^\alpha e(\lambda, x, y)| \le c(x^2/\lambda - 1)^{-1/4}\lambda^{-\alpha-1/2}, \text{ if } x \in S_4,\ y^2 < B\,.$$

Hence

(92) $$|K_4(\lambda, y)| \le c\lambda^{-\alpha-1/2}\|f\|_{m,p}J(\lambda)\,,$$

where now

$$J(\lambda) < c\lambda^{-m/2+n/2-1/2-n/2p+2/3p} \text{ if } p < 4/3\,;$$
$$J(\lambda) < c\lambda^{-m/2+n/2-1/2-n/2p+2/3p}(\log\lambda)^{1/4} \text{ if } p = 4/3\,;$$
$$J(\lambda) < c\lambda^{-m/2+n/2-1/2-n/2p+2/3p+\epsilon(3/4-1/p)} \text{ if } p > 4/3\,.$$

Therefore if $\epsilon > 0$ is small enough then

(93) $$|K_4(\lambda, y)| \le c\lambda^{-(m+m_o)/2}\|f\|_{m,p} \text{ if } p \ne 4/3\,,$$

(94) $$|K_4(\lambda, y)| \le c\lambda^{-(m+m_o)/2}(\log\lambda)^{1/4}\|f\|_{m,p} \text{ if } p = 4/3\,.$$

e. Estimate of $K_5(\lambda, y)$. From theorem 7 it follows

(95) $$|I^\alpha e(\lambda, x, y)| \le c\lambda^{-\alpha-1/3}\exp(-c\lambda^{\epsilon/2}) \text{ if } x^2 > \lambda + \lambda^{1/3+\epsilon},\ y^2 < B\,.$$

(96) $$|I^\alpha e(\lambda, x, y)| \le c\lambda^{-\alpha-1/3}\exp(-c|x|^{1/2}) \text{ if } x^2 > \lambda^2\,.$$

Therefore

(97) $$|K_5(\lambda, y)| \le c\|f\|_{m,p}\,.$$

Evidently (77) follows from (79), (84), (85), (86), (90), (93), (94) and (97).

In the case $\alpha > (n-1)/2$, $p = \infty$ instead of (77) we have $|R_\lambda^\alpha f(y)| < c\lambda^{-\gamma}$ for some $\gamma > 0$.

Finally, since $C_o^\infty(\mathbb{R}^n)$ is a dense set in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$, it is sufficient to prove that

(98) $$R_\lambda^\alpha f(y) \to 0 \text{ locally uniformly,}$$

if $f \in C_o^\infty(\mathbb{R}^n)$ and $\alpha \geq (n-1)/2$.

If $\alpha > (n-1)/2$ then (98) follows directly from (76) and (78). Let now $\alpha = (n-1)/2$. Using theorem 4 we can write

$$|I^\alpha e(\lambda, x, y) - I^\alpha e^o(\lambda, x, y)| \leq c\lambda^{-1/2+\epsilon} G_{\alpha,\epsilon}(\sqrt{\lambda}|x-y|)\lambda^{n/2}\,, \tag{99}$$

for $x^2 + y^2 < c$, $\alpha \geq (n-1)/2$, $\epsilon \geq 0$, where $G_{\alpha,\epsilon}(s) = (1+s)^{-(n+1)/2-\alpha-\epsilon}$, $s > 0$. For $\epsilon > 0$ we have $G_{\alpha,\epsilon}(|x|) \in L^1(\mathbb{R}^n)$ and

$$\left| \int f(x)[I^\alpha e(\lambda, x, y) - I^\alpha e^o(\lambda, x, y)]\,dx \right| \leq c\lambda^{-1/2+\epsilon}\,. \tag{100}$$

On the other hand, (16), (17) and the asymptotics of the Bessel function give as $\lambda \to \infty$,

$$I^\alpha e^o(\lambda, x, y) \sim a_\alpha \cos(\sqrt{\lambda}|x-y| + b_\alpha) \text{ if } |x-y| > \delta,\ x^2 + y^2 < c\,, \tag{101}$$

where $\alpha = (n-1)/2$ and a_α, b_α are constants. Since $f \in C_o^\infty(\mathbb{R}^n)$ it follows

$$\int_{|x-y|>\delta} f(x) I^\alpha e^o(\lambda, x, y)\,dx \to 0 \text{ locally uniformly}\,. \tag{102}$$

Evidently (98) for $\alpha = (n-1)/2$ follows from (100) and (102). Theorem 2 is proved.

6. Proof of remark 2

a. Proof of bound (12). Let $f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})g(x)$, where $g(x)$ is the characteristic function of the set $\{x \in \mathbb{R}^n : 1 < x^2 < 2\}$ and $a(x)$ satisfies (24). Then

$$\|f_\lambda\|_{m,\infty} \leq c \tag{103}$$

and (76) implies for small $\delta > 0$,

$$R_\lambda^\alpha f_\lambda(0) = \lambda^{n/2-\alpha} \int E_\alpha(\lambda, x, 0) f_\lambda(\sqrt{\lambda}x)\,dx\,. \tag{104}$$

Thus $R_\lambda^\alpha f_\lambda(0) = E_\lambda^\alpha f_\lambda(0)$ and (69) gives

$$|R_\lambda^\alpha f_\lambda(0)| \geq c\lambda^{(n-1)/4-\alpha/2}\,. \tag{105}$$

Therefore the first part of (12) follows from (103) and (105). To prove its second part we shall use proposition 3. Let

$$f_\lambda(x) = Ai^{-1}(B(x\lambda^{-1/2}, y_0)\lambda^{2/3})a(x\lambda^{-1/2})g_\lambda(x\lambda^{-1/2})\,,$$

where $g_\lambda(x)$ is the characteristic function of the set $S = \{x : \epsilon\lambda^{-2/3} < 1 - x^2 < 2\epsilon\lambda^{-2/3}\}$ for small $\epsilon > 0$. In particular, $|\lambda^{2/3}B(x\lambda^{-1/2})| < c\epsilon$ if $x\lambda^{-1/2} \in S$. Therefore

$$\|f_\lambda\|_{m,p} \leq c\lambda^{m/2+n/2p-2/3p}\,. \tag{106}$$

Using proposition 2 we find $Ai(B(x\lambda^{-1/2}, y_0)\lambda^{2/3}) > c > 0$ for $x\lambda^{-1/2} \in S$ and

$$E_\alpha(\lambda, x, 0) f_\lambda(x\lambda^{1/2}) = \lambda^{-1/3}|a(x)|^2 g_\lambda(x)(1 + O(\lambda^{-1/3}))\,.$$

Hence (104) implies

$$|R_\lambda^\alpha f_\lambda(0)| \geq c\lambda^{n/2-\alpha-1/3} \int g_\lambda(x)\, dx\, (1 + O(\lambda^{-1/3}))\,.$$

Since $\int g_\lambda(x)\, dx \asymp \lambda^{-2/3}$ it follows

$$|R_\lambda^\alpha f_\lambda(0)| > c\lambda^{n/2-\alpha-1}\,. \tag{107}$$

Now (107) and (106) give

$$|R_\lambda^\alpha|_{m,p} \geq c\lambda^{-m/2+(n-1)/2-\alpha-n/2p-1/2+2/3p}\,. \tag{108}$$

Further, we shall use the asymptotics (28) for $\lambda^{-2/3+\epsilon} < 1 - x^2 < \delta$ and small $\epsilon > 0$, $\delta > 0$. According to (29) we have $\lambda^{2/3}B(x) < -c\lambda^\epsilon$, $c > 0$, and the asymptotics of the Airy function gives

$$\begin{aligned} Ai(\lambda^{2/3}B(x)) =& \lambda^{-1/6}(1 - x^2)^{-1/4} Re(e^{i\lambda\psi(x)}a(x)) \\ &+ (1 - x^2)^{-1/4} O(\lambda^{-1/6-\epsilon})\,, \end{aligned} \tag{109}$$

where a satisfies (29) and ψ is given by (30). In particular,

$$|\nabla\psi| = (1 - x^2)^{1/2},\ |\partial^2\psi| < c(1 - x^2)^{-1/2}\,. \tag{110}$$

Set

$$f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})g_\lambda(x\lambda^{-1/2})\,, \tag{111}$$

where $g_\lambda(x)$ is the characteristic function of the set $\{x : \lambda^{-2/3+\epsilon} < 1 - x^2 < \delta\}$. Then

$$\|f_\lambda\|_{m,p} \leq c\lambda^{m/2+n/2p}\,. \tag{112}$$

On the other hand, (28) and (109) imply

$$E_\alpha(\lambda, x, 0) = \lambda^{-1/2}(1 - x^2)^{-1/4} Re(e^{i\lambda\psi(x)}a(x)) + (1 - x^2)^{-1/4} O(\lambda^{-1/2-\epsilon})\,. \tag{113}$$

Hence by (104),

$$|R_\lambda^\alpha f_\lambda(0)| \geq c\lambda^{(n-1)/2-\alpha}[I(\lambda) - |J(\lambda)|]\,, \tag{114}$$

where $I(\lambda) = \int g_\lambda(x)(1 - x^2)^{-1/4}\, dx$, $J(\lambda) = \int g_\lambda(x)\bar{a}^2(x)(1 - x^2)^{-1/4}e^{-2i\lambda\psi(x)}\, dx$. We have

$$I(\lambda) > c\delta \text{ for } \lambda > \lambda_o\,. \tag{115}$$

In the integral $J(\lambda)$ we can integrate by parts using (110). Thus

$$\begin{aligned} |J(\lambda)| &< c\lambda^{-1}\left[\int g_\lambda(x)(1 - x^2)^{-7/4}\, dx + O(\lambda^{1/6})\right] \\ &< c\lambda^{-1/2}\int g_\lambda(x)(1 - x^2)^{-1}\, dx + O(\lambda^{-1/2}) \\ &< c\lambda^{-1/2}\log\lambda\,. \end{aligned}$$

This and (114), (115) imply

(116) $$|R_\lambda^\alpha f_\lambda(0)| \geq c\lambda^{(n-1)/2-\alpha} \text{ for } \lambda > \lambda_o\,.$$

From (112) and (116) it follows

(117) $$|R_\lambda^\alpha|_{m,p} > c\lambda^{-m/2+(n-1)/2-\alpha-n/2p}\,.$$

Evidently the second part of the estimate (12) follows from (108) and (117) taking into account the definition (6).

b. Proof of bound (13). Set

(118) $$f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})(1-x^2/\lambda)^{-3/4}g_\lambda(x\lambda^{-1/2})\,,$$

where the function $g_\lambda(x)$ is the same as in (111). Then

(119) $$\|f_\lambda\|_{m,4/3} < c\lambda^{m/2+3n/8}(\log\lambda)^{3/4}\,.$$

Using (113) and (104) we obtain (114) again where now

$$I(\lambda) = \int g_\lambda(x)(1-x^2)^{-1}\,dx\,,\ J(\lambda) = \int g_\lambda(x)\bar{a}^2(x)(1-x^2)^{-1}e^{-2i\lambda\psi(x)}\,dx\,.$$

We have

(120) $$I(\lambda) > c\log\lambda\,.$$

In the integral $J(\lambda)$ we can integrate by parts using (110). Analogously to the estimates preceding (116) we get $|J(\lambda)| < c\lambda^{-3\epsilon/2}\log\lambda$, therefore

(121) $$|R_\lambda^\alpha f_\lambda(0)| \geq c\lambda^{(n-1)/2-\alpha}\log\lambda\,.$$

Thus (121) and (119) imply the estimate

$$|R_\lambda^\alpha|_{m,4/3} > c\lambda^{-m/2+(n-1)/2-\alpha-3n/8}(\log\lambda)^{1/4}\,,$$

which coincides with (13).

c. Proof of bound (14). Having in mind proposition 1 we set

$$f_\lambda(x) = e^{-i\lambda\psi(x\lambda^{-1/2})}\bar{a}(x\lambda^{-1/2})g_\lambda(x\lambda^{-1/2})\,,$$

where $g_\lambda(x)$ is the characteristic function of the set $\{x : 1/\lambda < x^2 < \epsilon\}$ for small $\epsilon > 0$. Let $\alpha = (n-1)/2$, $p > 4/3$ and $m = -m_o$. Then by (6) we have $m_o = n/p$ and therefore

(122) $$\|f_\lambda\|_{m,p} < c(\log\lambda)^{1/p}\,.$$

On the other hand, analogously to (114) we have

(123) $$|R_\lambda^\alpha f_\lambda(0)| \geq c[I(\lambda) - |J(\lambda)|]\,,$$

where $I(\lambda) = \int g_\lambda(x)|x|^{-n}\,dx$, $J(\lambda) = \int g_\lambda(x)\bar{a}^2(x)|x|^{-n}e^{-2i\lambda\psi(x)}\,dx$. Since $I(\lambda) = c\log\lambda$, $J(\lambda) = O(\lambda^{-1})$ we get

(124) $$|R_\lambda^\alpha f(0)| > c\log\lambda\,.$$

Now the estimate (14) follows from (122) and (124).

7. Proof of theorem 3

We start with (79). Let $K_j(\lambda, y)$ be defined by (81) for $1 \le j \le 4$, where $a_j(\lambda, x)$ is the characteristic function of the set B_j and $B_1 = A_1$, $B_2 = \{x : (1-\delta)\lambda < x^2 < \lambda - \lambda^{1/3+\epsilon}\}$, $B_3 = \{x : |x^2 - \lambda| < \lambda^{1/3+\epsilon}\}$, $B_4 = A_5$. The estimates (95) and (96) show that

$$K_4(\lambda, y) \to 0 \text{ locally uniformly.} \tag{125}$$

According to (89) and the bound

$$|f(x)| < c(1+|x|)^\beta, \ \beta < 2\alpha + 2 - n\,, \tag{126}$$

we have

$$|K_3(\lambda, y)| < c\lambda^{-\alpha-1/3} \int a_3(\lambda, x)|f(x)|\,dx < c\lambda^{-\alpha-1/3+\beta/2} \int a_3(\lambda, x)\,dx\,.$$

On the other hand, $\int a_3(\lambda, x)dx < c\lambda^{n/2-1+\epsilon+1/3}$, hence

$$|K_3(\lambda, y)| < c\lambda^{n/2-1-\alpha+\beta/2+\epsilon}, \text{ where } 0 < \epsilon < \alpha + 1 - n/2 - \beta/2\,. \tag{127}$$

It remains to bound $K_1(\lambda, y)$ and $K_2(\lambda, y)$.

a. Estimate of $K_1(\lambda, y)$. Let $B_j(\lambda, y) = K_j(\lambda, \sqrt{\lambda} y)$ $(j = 1, 2)$. Then

$$B_j(\lambda, y) = \lambda^{n/2-\alpha} \int a_j(\lambda, \sqrt{\lambda}x) f(\sqrt{\lambda}x) E_\alpha(\lambda, x, y)\,dx\,. \tag{128}$$

According to theorem 6,

$$\begin{aligned} B_1(\lambda, y) =& \lambda^{(n-1)/2-\alpha} \sum_{k=1}^{4} \int e^{i\lambda\psi_k(x,y)} a_1(\lambda, \sqrt{\lambda}x) b_k(\lambda, x, y, \alpha) f(\sqrt{\lambda}x)\,dx \\ &+ RO(\lambda^{n/2-\alpha-1})\,, \end{aligned} \tag{129}$$

where

$$R = \int a_1(\lambda, \sqrt{\lambda}x)|f(\sqrt{\lambda}x)||x|^{-(n+1)/2-\alpha}\,dx\,.$$

Using (126) we get

$$|R| < c\lambda^{\beta/2} J(\lambda)\,, \ J(\lambda) = \int_{\lambda^{-1/2}}^{1} \sigma^{(n-1)/2-\alpha+\beta-1}\,d\sigma\,. \tag{130}$$

Denote by $I(\lambda)$ the integral in (129). Integrating by parts and using (19) and the estimate (126) for f and ∇f, we obtain

$$|I(\lambda)| < c[\lambda^{\beta/2-1/2} J(\lambda) + \lambda^{\beta/2-1} + \lambda^{-(n+1)/4)+\alpha/2}]\,. \tag{131}$$

Since

$$J(\lambda) < c \text{ if } \beta > \alpha - (n-1)/2\,;$$
$$J(\lambda) < c\log\lambda \text{ if } \beta = \alpha - (n-1)/2\,;$$
$$J(\lambda) < c\lambda^{-(n-1)/4-\beta/2+\alpha/2} \text{ if } \beta < \alpha - (n-1)/2$$

we have from (129)–(131) if $\alpha \geq (n-1)/2$,

$$|B_1(\lambda, y)| < c[\lambda^{n/2-\alpha-1+\beta/2} + \lambda^{-1/2}\log\lambda],$$

or

$$|K_1(\lambda, y)| < c[\lambda^{n/2-\alpha-1+\beta/2} + \lambda^{-1/2}\log\lambda], \quad \text{where } \beta < 2\alpha + 2 - n,\ \alpha \geq (n-1)/2. \tag{132}$$

b. Estimate of $K_2(\lambda, y)$. Using corollary 3 and (128) we can write

$$\begin{aligned} B_2(\lambda, y) =& \lambda^{(n-1)/2-\alpha} \sum_{k=1}^{4} \int e^{i\lambda\psi_k(x,y)} a_2(\lambda, \sqrt{\lambda}x) g_k(\lambda, x, y) f(\sqrt{\lambda}x)\, dx \\ &+ R_1 O(\lambda^{n/2-\alpha-1}), \end{aligned} \tag{133}$$

where

$$\begin{aligned} g_k(\lambda, x, y) &= a_k(\lambda, x, y)(a^2 - x^2)^{-1/4} + b_k(\lambda, x, y)(a^2 - x^2)^{1/4}, \\ R_1 &= \int a_2(\lambda, \sqrt{\lambda}x)|f(\sqrt{\lambda}x)|(1 - x^2)^{-1}\, dx. \end{aligned}$$

Evidently,

$$R_1 < c\lambda^{\beta/2}\log\lambda. \tag{134}$$

Let $I(\lambda)$ be the integral in (133).

Integrating by parts we obtain analogously to (131),

$$|I(\lambda)| < c[\lambda^{-1} I_1(\lambda) + \lambda^{-1/2} I_2(\lambda)], \tag{135}$$

where

$$\begin{aligned} I_1 &= \int a_2(\lambda, \sqrt{\lambda}x)|f(\sqrt{\lambda}x)|(1 - x^2)^{-7/4}\, dx, \\ I_2 &= \int a_2(\lambda, \sqrt{\lambda}x)|(\nabla f)(\sqrt{\lambda}x)|(1 - x^2)^{-3/4}\, dx. \end{aligned}$$

To estimate $I_1(\lambda)$ we use that $(1 - x^2)^{-3/4} < \lambda^{1/2}$ on the support of the integrand. Since $|f(\sqrt{\lambda}x)| + |(\nabla f)(\sqrt{\lambda}x)| < c\lambda^{\beta/2}$ we get

$$|I_1(\lambda)| < c\lambda^{\beta/2+1/2}\log\lambda,\ |I_2(\lambda)| < c\lambda^{\beta/2}. \tag{136}$$

Now (133)–(136) imply

$$|B_2(\lambda, y)| < c\lambda^{\beta/2+n/2-\alpha-1}\log\lambda,$$

hence

$$|K_2(\lambda, y)| < c\lambda^{\beta/2+n/2-\alpha-1}\log\lambda,\ \text{where } \beta < 2\alpha + 2 - n. \tag{137}$$

Finally, (79)–(81), (125), (127), (132), (137) show that

$$|R_\lambda^\alpha f(y)| < c\int_{x^2<A} |f(x)|\, dx + o(1),\ \text{locally uniformly if } \alpha \geq (n-1)/2,\ \beta < 2\alpha + 2 - n.$$

This and (98) give the theorem.

8. Proof of proposition 1

We shall use the formula

$$e_\alpha(\lambda, x, y) = \Gamma(\alpha+1)(2\pi i)^{-1}\int_S e^{\lambda p} V(p, x, y) H_\alpha(\lambda+n, p)\, dp\,, \tag{138}$$

where $S = (\delta - i\pi/2, \delta + i\pi/2)$, $\delta > 0$, $\alpha > 0$ and the function $s \to H_\alpha(s, p)$ is 2-periodic,

$$H_\alpha(s, p) = \sum_{k=-\infty}^{\infty} e^{isk\pi}(p + ik\pi)^{-\alpha-1},\ \ p \in S,\ \ \alpha > 0\,. \tag{139}$$

For proving (138) we notice that $p^{-\alpha-1}\Gamma(\alpha+1)V(p, x, y)$ is the Laplace transform of the function $\lambda \to e_\alpha(\lambda, x, y)$, where

$$V(p, x, y) = \int_0^\infty e^{-\lambda p} de(\lambda, x, y),\ \ Re\, p > 0,$$

in particular, $V(p + ik\pi, x, y) = e^{ik\pi n} V(p, x, y)$. Applying the inverse Laplace formula we get (138). Note that for $Re\, p > 0$ ([21]),

$$V(p, x, y) = (2\pi \sinh 2p)^{-n/2} \exp\left(-\frac{1}{2}(x^2 + y^2)\coth 2p + xy(\sinh 2p)^{-1}\right).$$

Having in mind the singularities at the points $p = 0$, $p = \pm i\pi/2$ we write for $E_\alpha(\lambda, x) = E_\alpha(\lambda, x, 0)$:

$$E_\alpha(\lambda, x) = \sum_{j=1}^{3} E_j(\lambda, x, \delta)\,, \tag{140}$$

where

$$E_j(\lambda, x, \delta) = c\int e^{\lambda p} V(p, \sqrt{\lambda}x, 0) H_\alpha(\lambda+n, p) g_j(p)\, dp$$

and g_j are C^∞ functions, $1 = g_1(p) + g_2(p) + g_3(p)$ for $p \in S$, and $supp\, g_1 \subset \{p \in S : |Im\, p| < \epsilon\}$, $supp\, g_2 \subset \{p \in S : |Im\, p| > \pi/2 - \epsilon\}$ for some small $\epsilon > 0$, and g_2 is $i\pi$-periodic.

To evaluate E_1 we begin with the representations for $Re\, p > 0$:

$$V(p, x, 0) = (2\pi)^{-n}\int \exp\left(-\frac{1}{2}\xi^2 \sinh 2p - \frac{1}{2}x^2 \tanh p + ix\xi\right) d\xi$$

and

$$1 = c_\alpha(\sinh 2p)^{\alpha+1}\int \eta^{2\alpha}\exp\left(-\frac{1}{2}\eta^2\sinh 2p\right) d\eta,\ \ \eta \in \mathbb{R}^2\,.$$

Since $H_\alpha(\lambda+n, p) = p^{-\alpha-1} + h_\alpha(\lambda+n, p)$ and h_α has no singularities on S we obtain

$$E_1(\lambda, x, \delta) \sim I_1(\lambda, x, \delta) + I_2(\lambda, x, \delta)\,, \tag{141}$$

where

$$\begin{aligned} I_1 &= \lambda^{n/2+\alpha+1}\int e^{\lambda\phi_1} q_1\, d\xi\, d\eta\, dp\,, \\ \phi_1 &= p - \frac{1}{2}(\xi^2 + \eta^2)\sinh 2p - \frac{1}{2}x^2\tanh p + ix\xi \end{aligned}$$

and $q_1 = c(p/\sinh 2p)^{-\alpha-1} g(\xi,\eta) g_1(p)\eta^{2\alpha}$. Here g is a cutoff function due to an integration by parts using the property $|\partial_p \phi_1| > c(\xi^2+\eta^2)$ for large $\xi^2+\eta^2$. The equivalence "$a(\lambda, x, \delta) \sim b(\lambda, x, \delta)$" here means that $a - b = e^{c\lambda\delta} O(\lambda^{-\infty})$. Further,

$$I_2 = \lambda^{n/2} \int e^{\lambda\phi_2} q_2 \, d\xi \, dp \,,$$

where

$$\phi_2 = p - \frac{1}{2}\xi^2 \sinh 2p - \frac{1}{2}x^2 \tanh p + ix\xi$$

and $q_2 = c\, h_\alpha(\lambda, p) g(\xi) g_1(p)$ for some cutoff function g.

To represent E_2 we use the periodicity of the integrand in (138) and the formula

$$V(p + i\pi/2, x, 0) = (4\pi \cosh 2p)^{-n/2} \int \exp\left(-\frac{1}{2}\xi^2 \tanh 2p + ix\xi\right) d\xi \,.$$

Thus

$$E_2(\lambda, x, \delta) \sim \lambda^{n/2} \int e^{\lambda\phi} q \, d\xi \, dp \,, \tag{142}$$

where

$$\phi = p - \frac{1}{2}\xi^2 \tanh 2p + ix\xi$$

and $q = c(\cosh 2p)^{-n/2} H_\alpha(\lambda + n, p + i\pi/2) g(\xi) g_2(p + i\pi/2) e^{i\lambda\pi/2}$.

Therefore letting $\delta \to 0$ in (140)–(142) we obtain

$$E_\alpha(\lambda, x) = \sum_{j=1}^{3} E_j(\lambda, x) \,, \tag{143}$$

where $E_1 \sim I_1(\lambda, x) + I_2(\lambda, x)$,

$$\begin{aligned}
I_1 &= \lambda^{n/2+\alpha+1} \int e^{i\lambda\phi_1} q_1 \, dt \, d\xi \, d\eta \,, & I_2 &= \lambda^{n/2} \int e^{i\lambda\phi_2} q_2 \, dt \, d\xi \,, \\
\phi_1 &= t - \frac{\xi^2 + \eta^2}{2} \sin 2t - \frac{x^2}{2} \tan t + x\xi \,, & \phi_2 &= t - \frac{\xi^2}{2} \sin 2t - \frac{x^2}{2} \tan t + x\xi \,, \\
q_1 &= c(t/\sin 2t)^{-\alpha-1} g(\xi, \eta) g_1(it) \eta^{2\alpha} \,, & q_2 &= c(\cos 2t)^{-n/2} h_\alpha(\lambda, it) g(\xi) g_1(it) \,.
\end{aligned}$$

The integral E_2 is analogous to I_2 and

$$\begin{aligned}
E_3 &= \int e^{i\lambda\phi_3} q_3 \, dt \,, \\
\phi_3 &= t + \frac{1}{2}x^2 \cot 2t \,, \\
q_3 &= (2\pi)^{-1} \Gamma(\alpha + 1)(2\pi i \sin 2t)^{-n/2} H_\alpha(\lambda + n, it) g_3(it) \,.
\end{aligned} \tag{144}$$

Further, to find the uniform asymptotics of the integrals E_j in the domain $\{x : \lambda^{-5/2} < x^2 < \epsilon\}$ we shall apply the method of the stationary phase.

a. Asymptotics of I_1. First, in polar coordinates $(\xi, \eta) = \sigma(\omega, \theta)$, $\sigma > 0$, $\omega \in \mathbb{R}^n$, $\omega^2 + \theta^2 = 1$ the integral I_1 becomes

$$I_1 = \lambda |x|^{-n/2-\alpha} \int_0^\infty \int e^{i\lambda\psi_0} \sigma^{n/2+\alpha+1} J_{n/2+\alpha}(\lambda |x| \sigma) q(t, \sigma) \, dt \, d\sigma \,, \tag{145}$$

taking into account the relation

$$\int e^{i\lambda x\omega\sigma}\theta^{2\alpha}\,d(\theta,\omega) = c_\alpha(\lambda|x|\sigma)^{-n/2-\alpha}J_{n/2+\alpha}(\lambda|x|\sigma)\,.$$

Here $c_\alpha = (2\pi)^{n/2+1}2^\alpha\Gamma(\alpha+1)$ and

$$\psi_0 = t - \frac{1}{2}\sigma^2\sin 2t - \frac{x^2}{2}\tan t\,. \tag{146}$$

On the other hand, $|\partial_t\psi| > c > 0$ if σ is close to zero and then we can integrate by parts. Using the bound $|J_\beta(s)| < c|s|^{-1/2}$ to estimate the remainder we obtain

$$\begin{aligned} I_1 &= (\lambda|x|)^{-n/2-\alpha}\int\int e^{i\lambda\psi_0}J_{n/2+\alpha}(\lambda|x|\sigma)q(t,\sigma)\,dt\,d\sigma \\ &\quad + (\lambda|x|)^{-(n+1)/2-\alpha}O(\lambda^{-\infty})\,, \end{aligned} \tag{147}$$

where $q \in C_o^\infty(\mathbb{R}^n\times(0,\infty))$. We shall use the asymptotics of the Bessel function,

$$J_{n/2+\alpha}(\sigma) = \sum_{j=0}^{2}(c_j e^{i\sigma} + \bar{c}_j e^{-i\sigma})\sigma^{-1/2-j} + O(\sigma^{-7/2})\,. \tag{148}$$

Since $|x| > \lambda^{-5/4}$ and $n \geq 2$ it follows that

$$I_1 = \lambda^{1/2}|x|^{-(n+1)/2-\alpha}\sum_{j=0}^{2}(\lambda|x|)^{-j}(K_j^+ + K_j^-) + |x|^{-(n+1)/2-\alpha}O(\lambda^{-3/4})\,, \tag{149}$$

where

$$K_j^\pm = \int_0^\infty\int e^{i\lambda\psi_\pm}q(t,\sigma)\,dt\,d\sigma,\ q\in C_0^\infty$$

and $\psi_\pm = t - \frac{1}{2}\sigma^2\sin 2t - \frac{1}{2}x^2\tan t \pm |x|\sigma$.

We have the following critical points: $(t_k,1)$ for ψ_+ and $(-t_k,1)$ for ψ_- $(k=1,2)$, where $t_k\in(0,\pi/2)$, $\cos 2t_1 = \sqrt{1-x^2}$, $\cos 2t_2 = -\sqrt{1-x^2}$, for which $\det\psi''(t_k,1) = -4\cos 2t_k$. Since $x^2 < \epsilon$ it follows that $t_1\sim 1$, $t_2\sim\pi/2$ and hence on the support of the integrand in K_j the phase ψ_+ has only one critical point $(t_1,1)$ and the phase ψ_- has only one critical point $(-t_1,1)$. Therefore the method of the stationary phase implies

$$K_0^+ + K_0^- = \lambda^{-1}a(\lambda,x) + O(\lambda^{-2})\,, \tag{150}$$

where

$$a(\lambda,x) = Re\,(c\,e^{i\lambda\psi(x)})(t_1^{-1}\sin 2t_1)^{\alpha+1}(1-x^2)^{-1/4},\ \psi = \psi_+(t_1,1)\,. \tag{151}$$

Note that

$$|\nabla\psi(x)|^2 = 1 - x^2\,. \tag{152}$$

We have also

$$K_j^\pm = O(\lambda^{-1}),\ j = 1,2\,. \tag{153}$$

b. Asymptotics of I_2. In polar coordinates $\xi = \sigma\omega$, $\sigma > 0$, we obtain that the integral I_2 has the same form as in (145), where α is formally replaced by -1. Therefore we have the estimate

$$I_2 = |x|^{-(n-1)/2} O(\lambda^{-1/2}) . \tag{154}$$

Note that E_2 has the same bound (154) and analogously

$$E_3 = O(\lambda^{-1/2}) . \tag{155}$$

Now (143), (149)–(155) imply

$$E_\alpha(\lambda, x) = \lambda^{-1/2}|x|^{-(n+1)/2-\alpha}(a(\lambda, x) + |x|^{\alpha+1}O(1) + O(\lambda^{-1/4})) . \tag{156}$$

Finally, it is clear that (22)–(24) follow from (156) with $a(x) = c(t_1^{-1}\sin 2t_1)^{\alpha+1}(1-x^2)^{-1/4}$, $b(\lambda, x) = |x|^{\alpha+1}O(1) + O(\lambda^{-1/4})$.

9. Proof of proposition 2

We have the formula

$$E_\alpha(\lambda, x, y) = \int_S e^{\lambda\phi} q \, dp , \tag{157}$$

where $\phi = p - 2^{-1}(x^2 + y^2)\coth 2p + xy(\sinh 2p)^{-1}$ and

$$q = \Gamma(\alpha + 1)(2\pi i)^{-1}(2\pi \sinh 2p)^{-n/2} H_\alpha(\lambda + n, p) . \tag{158}$$

Evidently (157) follows from (138), (139). To find the uniform asymptotics of the integral (157) we shall apply the saddle point method. The phase function ϕ has the critical points $p_\pm$, $\bar{p}_\pm$, satisfying $\cosh 2p = xy + \sqrt{d}$, where $d = d(x, y) = (xy)^2 + 1 - x^2 - y^2$. Let $x = r\omega$, $|\omega| = 1$. Since $d = -r^2(1 - (\omega y)^2)$ and $|\omega y| < 1/2$, $r^2 < \epsilon$, the critical points degenerate if $r = 0$, coinciding with $p_0 = i\pi/4$ and $\bar{p}_0$ respectively. Near the critical points the integrand in (157) is a holomorphic function and $\partial_p^3\phi \neq 0$, $\partial_{p,r}^2\phi \neq 0$ for $p = \pm i\pi/4$. Therefore we can apply Lemma 2.3 ([6, p. 343]) and conclude that there exists a holomorphic change of variables $p = p(z, x, y)$, defined in a neighbourhood of the points $z = 0$, $r = 0$, such that

$$\phi(p(z, x, y), x, y) = A(x, y) - B(x, y)z + z^3/3, \;\; p(0, 0, y) = p_0 ,$$

for every fixed ω, y. In addition,

$$A = A(x, y) = \frac{1}{2}(\phi(p_+, x, y) + \phi(p_-, x, y)) , \tag{159}$$

$$B = B(x, y) = \left(\frac{3}{4}(\phi(p_+, x, y) - \phi(p_-, x, y))\right)^{2/3} . \tag{160}$$

To use this change of variables in the integral (157) we first notice that

$$E_\alpha(\lambda, x, y) \sim \int_L e^{\lambda\phi} q \, dp, \;\; L = L_1 \cup L_2 , \tag{161}$$

where L_1 is the segment $(\delta + i(\pi/4 - 2\delta), \delta + i(\pi/4 + 2\delta))$ and L_2- the segment $(\delta - i(\pi/4 + 2\delta), \delta + i(-\pi/4 + 2\delta))$ for $\delta > 0$ small enough. The equivalence relation "$a(\lambda, x, y) \sim$

$b(\lambda, x, y)$" here means that $a - b = O(e^{-c\lambda})$, $c > 0$. Indeed, it sufficies to use the bound $Re\,\phi(p, x, y) \le -c < 0$ for $p \in S \setminus L$, which follows from the definition of ϕ if $\delta > 0$ is small enough.

Now (157)–(161) yield

$$E_\alpha(\lambda, x, y) \sim \sum_{j=1}^{2} e^{\lambda A_j} \int_{L_{1j}} e^{\lambda(-Bz+z^3/3)} q_j(z, \lambda)\, dz\,, \tag{162}$$

where $A_1 = A$, $A_2 = \bar{A}$, $q_1(z, \lambda) = q(p(z, x, y), x, y) \cdot \partial p/\partial z$, $q_2(z, \lambda) = q(\bar{p}(\bar{z}, x, y)) \cdot \partial \bar{p}(\bar{z}, x, y)/\partial z$, L_{1j} being the image of the segment L_j. Notice that $L_{1j} \subset \{z : Re\,z > 0\}$ and that the endpoints a_j, b_j of L_{1j} satisfy: $\arg a_j \in (-\pi/2, -\pi/6)$, $\arg b_j \in (\pi/6, \pi/2)$.

Using the Weierstrass preparation theorem,

$$q_j(z, \lambda) = r_j + r_{1j} z + (z^2 - B) q_{1j}(z, \lambda)$$

and the following representation of the Airy function

$$\begin{aligned} Ai(s) &= (2\pi i)^{-1} \int_M e^{-sz+z^3/3}\, dz,\ M = M_1 \cup M_2\,, \\ M_1 : z &= re^{i\theta},\ r \in (0, +\infty),\ \theta \in (-\pi/2, -\pi/6)\,, \\ M_2 : z &= re^{i\theta},\ r \in (0, +\infty),\ \theta \in (\pi/6, \pi/2) \end{aligned}$$

in the integral (162), we obtain the uniform asymptotics

$$E_\alpha(\lambda, x, y) = \lambda^{-1/3} Ai(B\lambda^{2/3})\, Re(a(\lambda, x, y)e^{\lambda A}) + Ai'(B\lambda^{2/3}) O(\lambda^{-2/3})\,, \tag{163}$$

where

$$a(\lambda, x, y) = c(x, y)(q(p_+, x, y) + q(p_-, x, y)),\ |c(x, y)| \asymp 1 \tag{164}$$

and q is given by (158). Note also that (159), (160) imply (26) and

$$A(x, y) = i\psi(x, y),\ \psi(0, y) = \pi/4,\ \partial_x \psi(0, y) = -y\,. \tag{165}$$

Let $\lambda = \lambda_m$, where $\lambda_m = 2m - n$ for positive integers m. Then one can see that $a(\lambda_m, x, y)$ does not depend on m for large m, and $|a(\lambda_m, x, y)| \asymp 1$. Therefore (25)–(27) follow from (163)–(165) and the estimate

$$|Ai'(B\lambda^{2/3})| < c\lambda^{1/6}\,. \tag{166}$$

10. Proof of proposition 3

We start with formula (157). Thus we have

$$E_\alpha(\lambda, x, 0) = \int_S e^{\lambda\phi} q(p, \lambda)\, dp\,, \tag{167}$$

where

$$\phi = \phi(p, x, 0) = p - 2^{-1} x^2 \coth 2p \tag{168}$$

and $q(p,\lambda)$ is given by (158). The critical points of ϕ satisfy $\cosh^2 2p = 1-x^2$. If $1-\delta < x^2 < 1-\delta/2$ and $\delta > 0$ is small we have four critical points $p_\pm = it_\pm$ and $\bar{p}_\pm$, where $\sin 2t_\pm = |x|$, hence $t_\pm = \pi/4 \pm \gamma$, $\gamma > 0$. If $x^2 = 1$ the critical points degenerate and $\partial_p^3\phi \neq 0$, $\partial_{p,r}^2\phi \neq 0$ for $p = \pm i\pi/4$, where $r = |x|$. Therefore, analogously to the proof of proposition 2, we obtain

$$E_\alpha(\lambda,x,0) = \lambda^{-1/3} Ai(B(x)\lambda^{2/3}) a(\lambda,x) + Ai'(B(x)\lambda^{2/3}) O(\lambda^{-2/3})\,, \tag{169}$$

where $a(\lambda,x) = Re(e^{\lambda A(x)} r(\lambda,x))$,

$$r(\lambda,x) = ic(x)(q(p_+,\lambda) + q(p_-,\lambda)),\ |c(x)| \asymp 1,\ c(x) \text{ is real}$$

and $A(x) = A(x,0)$, $B(x) = B(x,0)$ are given by (159), (160). From (165) we have $A(x) = i\pi/4$ and it is not hard to see that (29), (30) are valid.

To compute $r(\lambda,x)$ we use

$$r(\lambda,x) = c_1(x) e^{-i\pi/4}[H_\alpha(\lambda+n, it_+) + H_\alpha(\lambda+n, it_-)]\,,$$

where $c_1(x)$ is real and $|c_1(x)| \asymp 1$ as $x^2 \to 1$. From (139) it follows

$$\begin{aligned} a(\lambda,x) \quad = \quad & c_1(x)\sum_{k\geq 0}((t_+ + k\pi)^{-\alpha-1} + (t_- + k\pi)^{-\alpha-1})\cdot \\ & \cdot\cos\left(\frac{(\lambda-n)\pi}{4} + (\lambda+n)k\pi - \frac{(\alpha+1)\pi}{2}\right) \\ & + c_1(x)\sum_{k\geq 1}((k\pi - t_+)^{-\alpha-1} + (k\pi - t_-)^{-\alpha-1})\cdot \\ & \cdot\cos\left(\frac{(\lambda-n)\pi}{4} + (\lambda+n)k\pi - \frac{(\alpha+1)\pi}{2}\right). \end{aligned}$$

Now we see that there is a sequence $\lambda_m \to \infty$ so that $a(x) = a(\lambda_m, x)$ does not depend on λ_m and for $x^2 = 1$,

$$a(x) = c_2(x)\left(\sum_{k\geq 0}(\pi/4 + k\pi)^{-\alpha-1} + \sum_{k\geq 1}(k\pi - \pi/4)^{-\alpha-1}\right),\ c_2(x) \neq 0\,.$$

Thus for this choice of $\lambda \to \infty$ we have

$$E_\alpha(\lambda,x,0) = \lambda^{-1/3} Ai(B(x)\lambda^{2/3} a(x) + Ai'(B(x)\lambda^{2/3}) O(\lambda^{-2/3})\,. \tag{170}$$

If $1 - c_1\lambda^{-2/3} < x^2 < 1 - c_2\lambda^{-2/3}$ then $|Ai'(B(x)\lambda^{2/3})| < c$ and (170) implies (28). If $1-\delta < x^2 < 1-\delta/2$ then we have only the estimate (166). Thus proposition 3 is proved.

References

[1] ASKEY R., WAINGER S., *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. **87** (1965), 695–708.

[2] BERARD P. H., *Riesz means on Riemannian manifolds*, Proc. Symp. Pure Math. **36** (1980), 1–12.

[3] CHRIST F. M., SOGGE C. D., *The weak type L^1 convergence of eigenfunction expansions for pseudodifferential operators*, Invent. math. **94** (1988), 421–453.

[4] COLZANI L., TRAVAGLINI G., *Estimates for Riesz kernels of eigenfunction expansions of elliptic differential operators on compact manifolds*, J. Funct. Anal. **96** (1991), 1–30.

[5] EL-ADAD E. E., *About Riesz summability of multiple Hermite series*, C.r. de l'Acad. bulgare des Sci. **49** (1996), 13–16.

[6] FEDORJUK M., *Method perevala*, Nauka, Moscow 1977 (Russian).

[7] HEBISCH W., *Almost everywhere summability of eigenfunction expansions associated to elliptic operators*, Studia Math. **46** (1990), 263–275.

[8] HÖRMANDER L., *On the Riesz means of spectral functions and eigenfunction expansions for elliptic differential operators*, in "Some recent advances in the basic sciences", 155–202, Yeshiva Univ., New York 1966.

[9] HÖRMANDER L., *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218.

[10] HULANICKI A., JENKINS J. W., *Nilpotent Lie groups and summability of eigenfunction expansions of Schrödinger operators*, Studia Math. **80** (1984), 235–244.

[11] KARADZHOV G. E., *Sommabilite de Riesz des series d'Hermite multiples*, C. R. Acad. Sci. Paris, Serie I, **317** (1993), 1023–1028.

[12] KARADZHOV G. E., EL-ADAD E. E., *Localization and summability of multiple Hermite series*, Intern. J. of Math. and Math. Sci. **20** (1997), 61–74.

[13] KARADZHOV G. E., *Riesz summability of multiple Hermite series in L^p spaces*, Math. Z. **219** (1995), 107–118.

[14] KON M., *Summation and spectral theory of eigenfunction expansions*, in "Discourses in math. and its applications" **1** (1991), 49–76, Dept. Math., Texas A&M University.

[15] MUCKENHOUPT B., *Mean convergence of Hermite and Laguerre series I*, Trans. Amer. Math. Soc. **147** (1970), 419–431.

[16] MUCKENHOUPT B., *Mean convergence of Hermite and Laguerre series II*, Trans. Amer. Math. Soc. **147** (1970), 433–462.

[17] SOGGE C. D., *Fourier integrals in classical analysis*, Cambridge Tracts in Math. **105**, Cambridge Univ. Press 1993.

[18] STEIN E. M., *Beijing lectures on Fourier analysis*, Ann. Math. Studies, Princeton Univ. Press 1986.

[19] STEIN E. M., WEISS G., *Introduction to Fourier analysis on Eucleadian spaces*, Princeton Univ. Press 1981.

[20] STRICHARTZ R., *Harmonic analysis as spectral theory of Laplacians*, J. Funct. Anal. **87** (1989), 51–148.

[21] SZEGÖ G., *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ. **23**, Providence, R. I. 1967.

[22] THANGAVELU S., *Lectures on Hermite and Laguerre expansions*, Math. notes, Princeton Univ. Press 1993.

AMS Subject Classification: 42C10, 42C15.

G. E. KARADZHOV
Institute of Mathematics
Bulgarian Academy of Sciences
1113 Sofia, Bulgaria
geremika@math.bas.bg

Lavoro pervenuto in redazione il 30.6.1997.