M P E J

Mathematical Physics Electronic Journal

ISSN 1086-6655
Volume 8, 2002

Paper 2
Received: Dec 4, 2001, Accepted: Feb 14, 2002
Editor: R. de la Llave

Biholomorphic maps with linear parts having Jordan blocks: linearization and resonance type phenomena

David DeLatte* and Todor Gramchev†

*Department of Mathematics, University of North Texas, Denton, TX 76203-1430, USA
†Dipartimento di Matematica, Università di Cagliari, via Ospedale 72, 09124 Cagliari, Italy

Abstract

We study the linearizability of biholomorphic maps of $\mathbb{C}^n$ fixing the origin when the Jacobian matrix admits a nontrivial Jordan block. Our main result proves convergence of the linearizing transformation of maps for which the Jordan part of the spectrum lies inside the unit circle and the spectrum satisfies a Rüssmann-type Diophantine condition. Degeneracy of the Jordan block - different geometric and algebraic multiplicity - is allowed. The key to the proof is the decoupling of the homological equation into the Siegel part and Poincaré part. In higher dimensions ($n > 3$) new inhomogeneous Diophantine conditions also appear. We show that quasi-resonance phenomena occur and that when a nontrivial Jordan block is present the homological equation cannot be solved in general due to the accumulated effects of small divisors. In the purely hyperbolic case Jordan blocks are an obstruction to holomorphic linearization - even under Diophantine conditions.

Key words: biholomorphic maps, linearization, Jordan block, homological equation, small divisors, quasi-resonance, near-resonance, decomposition

MSC classification: 37C05, 37F50, 37G05

*Partially supported by UNT FRG 34598 and 63400. E-mail: delatte@alumni.caltech.edu
†Partially supported by INDAM-GNAMPA, Italy and a Local Research Project of the University of Cagliari. E-mail: todor@unica.it

1 Introduction and main results

Let $\Phi(x)$ be a biholomorphic map of $\mathbb{C}^n$ preserving the origin and $\Phi'(0)$ the Jacobian matrix at the origin. Denote by $\operatorname{spec}(\Phi'(0)) = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of $\Phi'(0)$. Clearly $\lambda_j \neq 0$ for all $j = 1, \dots, n$. We define the set of all resonance multiindices of Φ (actually it depends only on $\operatorname{spec}(\Phi'(0))$) as follows:

$$
\begin{aligned}
\operatorname{Res}[\lambda_1, \dots, \lambda_n] &= \bigcup_{j=1}^{n} \operatorname{Res}^j[\lambda_1, \dots, \lambda_n] \\
\operatorname{Res}^j[\lambda_1, \dots, \lambda_n] &= \{\alpha \in \mathbb{Z}_+^n(2) : \lambda^\alpha - \lambda_j = 0\}; \qquad (1.1)
\end{aligned}
$$

where $\mathbb{Z}_+^n(k) := \{\alpha \in \mathbb{Z}_+^n : |\alpha| \geq k\}$, $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\lambda^\alpha = \lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$.

Given $\Lambda \in GL(n, \mathbb{C})$ define $\mathcal{O}[\Lambda]$ as the germ of all local complex analytic diffeomorphisms (biholomorphic maps) $\Phi(x)$ of $\mathbb{C}^n$ with 0 as a fixed point and such that $\Phi'(0)$ is in the $GL(n, \mathbb{C})$-conjugacy class of Λ. We stress that if Λ is diagonalizable (semisimple) then the germ is determined by the spectrum of Λ, i.e., by n nonzero complex numbers $\lambda_1, \dots, \lambda_n$.

We will say that $\Phi(x)$ is formally linearizable if there exists a formal series $u(x) = x + \sum_{\alpha \in \mathbb{Z}_+^n(2)} u_\alpha x^\alpha$ such that

$$
u^{-1} \circ \Phi \circ u = \Phi'(0) \quad \text{formally in } (\mathbb{C}[x])^n, \tag{1.2}
$$

where $\mathbb{C}[x]$ stands for the set of all formal power series with complex coefficients. It is well known (the Poincaré–Dulac theorem [1]) that under the nonresonance hypothesis $\operatorname{Res}[\lambda_1, \dots, \lambda_n] = \emptyset$, i.e.,

$$
\lambda^\alpha - \lambda_j \neq 0, \qquad \alpha \in \mathbb{Z}_+^n(2),\ j = 1, \dots, n, \tag{1.3}
$$

Φ is formally linearizable. In fact, (1.3) is a necessary and sufficient condition in order that every holomorphic $\Phi(x)$ with $\operatorname{spec} \Phi'(0) = \{\lambda_1, \dots, \lambda_n\}$ is formally linearizable. The formal solution of (1.2) involves expressions of the form $(\lambda^\alpha - \lambda_j)^{-1}$; so when $\inf_\alpha |\lambda^\alpha - \lambda_j| = 0$ for some $j \in \{1, \dots, n\}$, the convergence of u becomes a subtle question.

One of the main problems, starting from the pioneering work of C.L. Siegel in 1942 [23], has been (and still is) to find general conditions which guarantee that $u(x)$ converges, i.e., that $\Phi(x)$ is linearizable. We recall the state of the art of this subject. If the linear part $\Phi'(0)$ is semisimple (i.e., $\Phi'(0)$ has no nontrivial Jordan blocks), then it is well known that for convergence we need arithmetic (Diophantine) conditions on

$$
\omega(m) := \min_{2 \leq |\alpha| \leq m, 1 \leq j \leq n} |\lambda^\alpha - \lambda_j|, \qquad m \in \mathbb{Z}_+(2). \tag{1.4}
$$

We refer for the history and references to the survey paper M. Herman [14] (see also J.-C. Yoccoz [27] and E. Zehnder [30]). The best condition that implies linearizability for all maps with a given semisimple linear part is due to A. Bruno [2], and can be expressed as (following [14], p. 143)

$$\sum_{k=1}^{\infty} 2^{-k} \ln(\omega^{-1}(2^{k+1})) < \infty. \tag{1.5}$$

We will use a near-exponential bound on small divisors introduced by H. Rüssmann [21] and used later by different authors (cf. [20], [9], [13], see also [7] and the references therein). We say that spec $(\Phi'(0))$ satisfies the Diophantine condition $(\mathcal{BR}_k)$, where $k \geq 2$, if there exists a nondecreasing function $\psi : [1, +\infty) \longrightarrow (0, +\infty)$ such that

$$\omega(m) \geq e^{-\frac{m}{\psi(m)}}, \qquad m \in \mathbb{Z}_+(k) \tag{1.6}$$

$$\int_1^{+\infty} \frac{1}{t\psi(t)} dt < +\infty. \tag{1.7}$$

If $k = 2$, we recover the original condition of H. Rüssmann.

One observes that $(\mathcal{BR}_2)$ implies the nonresonance condition while for $k \geq 3$ the validity of $(\mathcal{BR}_k)$ does not exclude finite resonances of order $\leq k-1$. Clearly, for any given $k \in \mathbb{Z}_+(2)$, the traditional condition $(\mathcal{BR}_2)$ is equivalent to $(\mathcal{BR}_k)$ and (1.3).

If $n = 1$, the Bruno condition (1.5) is necessary for the linearizability of all holomorphic maps with a given linear part (the Siegel centralizer problem), cf. J.-C. Yoccoz [26], [27] (see also R. Pérez-Marco [19] for nonlinearizibility of commuting maps in $\mathbb{C}$). On the other hand, if we assume the Poincaré condition

$$\max_{1 \leq j \leq n} |\lambda_j| < 1 \tag{1.8}$$

(equivalently, $\min_{1 \leq j \leq n} |\lambda_j| > 1$) then Φ is always analytically equivalent to its linear part $\Phi'(0)$, provided the nonresonance hypothesis holds. More generally, (1.8) implies that there are finitely many resonances and, according to the Poincaré–Dulac theorem, we can find a local biholomorphic change of the variables u bringing Φ to normal form

$$(u^{-1} \circ \Phi \circ u)(x) = \Phi'(0)x + P_{res}(x), \tag{1.9}$$

where the remainder $P_{res}(x)$ is a polynomial map containing only resonant terms.

Recently, J. Ecalle and B. Vallet [11] have studied resonant normal forms of vector fields and biholomorphic maps assuming that the linear part is diagonalizable (see [11], p. 251, the line after the condition (1.3)). T. Gramchev and M. Yoshino [13] have obtained results on simultaneous linearization of d commuting holomorphic maps of $(\mathbb{C}^n, 0)$, not necessarily biholomorphic if $d \geq 2$, $n \geq 2$, while L. Stolovitch [25] has generalized the results of A. Bruno for simultaneous resonant normal forms of commuting holomorphic vector fields, with both papers requiring again semisemplicity of the linear parts at the origin.

Little is known about the (non)linearizability of Φ in the analytic category if the Poincaré condition doesn't hold and the matrix $\Phi'(0)$ has at least one nontrivial Jordan block. When $n = 2$ and Λ has a double eigenvalue $\lambda_1 = \lambda_2$, $|\lambda_1| = 1$ and a nontrivial Jordan block, then in general Φ is not linearizable. This result is contained in [14], Proposition 3, p. 143, which is a consequence of results of Y. Ilyashenko [15] and J.-C. Yoccoz (the latter proved in 1978; for published proofs we refer to [28] appendix, pp. 86-87). It is not difficult to extend this negative result to $\mathbb{C}^n$, $n \geq 3$.

The problem that seems to be open is to study the linearizability of $\Phi(x)$ in the nonresonance case (and more generally for resonant normal forms) under a general near-exponential Diophantine condition $(\mathcal{BR}_k)$, e.g., (1.6) and (1.7), provided that all eigenvalues with a nontrivial Jordan block do not belong to the unit circle. The principal difficulties arise in the study of the inverse of the linearized operator from the homological equation. If $k \geq 3$ one is naturally led to studying resonant normal forms of Φ, provided $(\mathcal{BR}_k)$ is true. Finally, having in mind the results on simultaneous normal forms of commuting vector maps and fields (cf. [3], [9], [13], [16], [18], [25]) one is led to studying the simultaneous linearizability of commuting holomorphic maps of $\mathbb{C}^n$, $n \geq 3$ with parts admitting nontrivial Jordan blocks (compare with [13], where all linear parts are assumed to be diagonalizable).

The main purpose of the present paper is to carry out a detailed study, at least in dimensions $n = 3$ and $n = 4$, of the (non)linearizability of biholomorphic maps with linear part having only one nontrivial Jordan block. We also outline some generalizations and open problems.

Define $\mathcal{J} \subset \{1, \dots, n\}$ as the set of all indices j such that λ_j lies in a nontrivial Jordan block. Our main conjecture is, roughly speaking, that when the spectrum satisfies the arithmetic condition $(\mathcal{BR}_2)$, along with

$$\max_{1 \leq j \leq n} |\lambda_j| \leq 1 \quad \text{and} \quad \max_{j \in \mathcal{J}} |\lambda_j| < 1 \tag{1.10}$$

the formal power series of the formal conjugation converges. If $\min_{1 \leq j \leq n} |\lambda_j| \geq 1$ and $\min_{j \in \mathcal{J}} |\lambda_j| > 1$, the same results can be obtained by considering Φ^{-1}. Note that, in view of the aforementioned result of J. Yoccoz, the unsolved problems occur for dimension $n \geq 3$.

Condition (1.10) leads us to a natural conjecture for a splitting of the spectrum of $\Phi'(0)$, namely, the validity of (1.10) implies that it is enough to impose the arithmetic condition on the spectrum of the linear part $\Phi'(0)$ involving only the eigenvalues not admitting a nontrivial Jordan block. While in space dimension 3 it turns out be the case, when $n \geq 4$ the eigenvalues in the nontrivial Jordan block may be involved (however, finitely many times) in new Diophantine conditions for the eigenvalues outside the nontrivial Jordan block.

Let us now establish some notation and state the main results. If $\Phi'(0)$ is not semisimple, a preliminary linear change of coordinates allows us to assume that it is in Jordan normal form, i.e.,

$$\Phi'(0) = \Lambda^{\varepsilon} = \Lambda + \varepsilon N \tag{1.11}$$

where Λ is the diagonal matrix $\operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, N is a nonzero nilpotent, upper triangular, {0,1}-matrix arising from the nontrivial Jordan blocks, and $\varepsilon \neq 0$ could be made arbitrarily small.

Theorem 1.1 *Let Φ be a biholomorphic map fixing the origin in $\mathbb{C}^3$ and let*

$$\Phi'(0) = \Lambda^{\varepsilon} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & \varepsilon \\ 0 & 0 & \mu \end{pmatrix} \tag{1.12}$$

where $|\lambda| = 1$, $|\mu| < 1$, $\varepsilon \neq 0$. Then spec $(\Phi'(0))$ *is nonresonant iff λ is not a root of unity. If $(\mathcal{BR}_2)$ defined by (1.4), (1.6), and (1.7) holds, there is a transformation u, biholomorphic near the origin, that satisfies*

$$\Phi \circ u = u \circ \Lambda^{\varepsilon}. \tag{1.13}$$

near the origin.

Remark 1.2 *If $|\mu|, |\lambda| < 1$ then the spectrum is in the Poincaré domain and the result is well known. For $|\mu|^{-1}$, $|\lambda| > 1$ a counterexample is constructed below (Theorem 1.3). So, even among hyperbolic germs nonsemisimplicity is an obstruction to linearizability. For positive results, $|\lambda| = 1$ is the most interesting case. Combining our results (Theorem 1.1 and Theorem 1.3) with the previous results on the linearizability of biholomorphic maps with semisimple linear parts, we completely solve the problem of the linearizability of germs of biholomorphic maps of $\mathbb{C}^3$ with a fixed point. The only part that remains is sharpness of the arithmetic condition. The existence of numbers which satisfy the Bruno condition but not the Rüssmann condition was communicated to the authors by M. Herman.*

Next we show that if our nonexpansiveness condition does not hold, the presence of a nontrivial Jordan block in the linear part prevents in general the linearization of hyperbolic biholomorphic maps even if Diophantine conditions are satisfied. This is in contrast to the C^{∞} and C^k results of Sternberg cf. [24] (see also Chen [5]), where the existence of a Jordan block does not influence smooth linearizability.

Theorem 1.3 *Suppose that Λ^{ε} has the form (1.12) where $|\lambda| > 1$, $|\mu| < 1$, $\varepsilon \neq 0$, and it is nonresonant. Then there is a biholomorphic map Φ with linear part $\Phi'(0) = \Lambda^{\varepsilon}$ such that the formal transformation u that conjugates Φ with its linear part diverges.*

Remark 1.4 *Observe that there is no mention of a Diophantine condition in the last theorem. One can require that the spectrum satisfies the classical Siegel–type Diophantine condition in the example. The Diophantine case clearly indicates that it is the nonsemisimplicity that obstructs linearization.*

Next we study maps in $\mathbb{C}^4$.

Definition 1.5 *Let $(\lambda_1, \lambda_2, \mu)$ be distinct complex numbers with $|\lambda_1| = 1$, $0 < |\lambda_2|$, $|\mu| < 1$. Let $\tau = \frac{\ln|\lambda_2|}{\ln|\mu|}$ and $t = \max\{\tau, \tau^{-1}\}$. A near-resonance occurs when (i) $t \in \mathbb{N}$ but (ii) for all $s \in \mathbb{Z}_+$ ($s \in \mathbb{N}$ if $t = 1$) $\lambda_1^s \lambda_2^t \neq \mu$ (respectively, $\lambda_1^s \mu^t \neq \lambda_2$) provided $t = \tau$ (respectively, $t = \tau^{-1}$).*

Remark 1.6 *The opportunity for a near-resonance occurs when three distinct eigenvalues interact. When a Jordan block is present this arises in dimension $n \geq 4$. Control of a near-resonance leads to inhomogeneous Diophantine conditions, cf. [17], [10] for other types of inhomogeneous Diophantine conditions.*

Theorem 1.7 *Let Φ be a biholomorphic map fixing the origin in $\mathbb{C}^4$ with*

$$\Phi'(0) = \Lambda^\varepsilon = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu & \varepsilon \\ 0 & 0 & 0 & \mu \end{pmatrix}, \tag{1.14}$$

where the spectrum $\{\lambda_1, \lambda_2, \mu\}$ is nonresonant and $1 = |\lambda_1| \geq |\lambda_2|$, $|\mu| < 1$, $\varepsilon \neq 0$. Then if any of the conditions (i), (ii), or (iii) below holds we can find a linearizing transformation, u, biholomorphic near the origin.

i) if $|\lambda_1| = |\lambda_2| = 1$ and $\{\lambda_1, \lambda_2\}$ satisfies ($\mathcal{BR}_2$) or

ii) if $|\lambda_1| = 1$, λ_1 satisfies ($\mathcal{BR}_2$), and $|\lambda_2| < 1$ but there are no near resonances.

iii) if $|\lambda_1| = 1$, λ_1 satisfies ($\mathcal{BR}_2$), $|\lambda_2| < 1$ and there is a near resonance with μ, let $\tau = \ln|\lambda| / \ln|\mu|$ satisfy $|\lambda||\mu|^\tau = 1$ and $t = \max\{\tau, \tau^{-1}\} \in \mathbb{N}$. Then there exists a Rüssmann type weight function ψ such that if $t = \tau$ (respectively, $t = \tau^{-1}$) then the following inhomogeneous Diophantine condition holds

$$|\lambda_1^s \mu^t - \lambda_2| \quad \geq \quad e^{-\frac{s}{\psi(s)}}, \qquad s \in \mathbb{N}, \tag{1.15}$$

(respectively,

$$|\lambda_1^s \lambda_2^t - \mu| \quad \geq \quad e^{-\frac{s}{\psi(s)}}, \qquad s \in \mathbb{N}). \tag{1.16}$$

Remark 1.8 *Case (ii) includes the possibility of a degenerate three-dimensional Jordan block. The theorem also holds for nondegenerate three-dimensional blocks.*

The paper is organized as follows: In section 2 we define the homological equation, explain the partition into Siegel and Poincaré parts and prove some basic facts. The next section deals with the resolution of the homological equation in dimension three. In section 4 a hyperbolic counterexample to linearization is constructed while in section 5 we consider linearized homological equations and resonance type phenomena for four–dimensional maps. Section 6 outlines the convergence part of the proofs while Section 7 discusses extensions and generalizations of the work presented here.

2 The representation of the homological operator

Let $\mathbb{C}_M[x]$ denote the space of homogeneous polynomials in $x \in \mathbb{C}^n$ of degree M. We have $\mathbb{C}_{[\geq k]}[x] = \bigcup_{M=k}^{\infty} \mathbb{C}_M[x]$. We define the (linear) homological equation

$$\mathcal{L}[f] := f \circ \Lambda^\varepsilon - \Lambda^\varepsilon \circ f = g, \qquad g \in (\mathbb{C}_{[\geq 2]}[x])^n \tag{2.1}$$

Equation (2.1) is the linearization of the conjugacy equation (1.13) at the identity. One notes that $\mathcal{L}$ preserves $(\mathbb{C}_M[x])^n$, $M \in \mathbb{Z}_+(2)$. Next, we define

$$\begin{aligned} \Omega[f] &= f \circ \Lambda - \Lambda \circ f \\ \mathcal{R}^\varepsilon[f] &= f \circ \Lambda^\varepsilon - f \circ \Lambda \end{aligned} \tag{2.2}$$

Recall that $\Lambda^\varepsilon = \Lambda + \varepsilon N$ where Λ is diagonal and N is nilpotent. Thus

$$\mathcal{L} \;=\; \Omega - \varepsilon N + \mathcal{R}^\varepsilon. \tag{2.3}$$

In the case of a semisimple linear part, $\varepsilon = 0$ and $\mathcal{L} = \Omega$.

Definition 2.1 $\mathcal{A} : (\mathbb{C}[x])^n \to (\mathbb{C}[x])^n$ *is a multiplication operator of constant type if*

$$\mathcal{A}[\Sigma_\alpha f_\alpha x^\alpha] = \Sigma_\alpha A_\alpha f_\alpha x^\alpha \tag{2.4}$$

where A_α is a constant matrix (independent of x) for each $\alpha \in \mathbb{Z}_+^n$.

The upshot of the definition is that there is no interaction between terms with different indices. So, the construction of $\mathcal{L}^{-1}$ is immediate. E.g., for $\mathcal{A}$ as above as long as $\det\{A_\alpha\} \neq 0$ for all $\alpha \in \mathbb{Z}_+^n(2)$,

$$\mathcal{A}^{-1}[f] = \Sigma_\alpha A_\alpha^{-1} f_\alpha x^\alpha. \tag{2.5}$$

The main difficulty, in comparison to the semisimple case, is the appearance of operators of nonconstant type in the homological equations, and as a consequence, the necessity to carefully analyze the contributions of the remainder term $\mathcal{R}^\varepsilon$.

For a given operator S, set

$$S_\alpha[v] \;:=\; (S[v])_\alpha = \frac{1}{\alpha!}\partial_x^\alpha(S[v])(0). \tag{2.6}$$

Hence,

$$\Omega_\alpha[v] = \Omega(\alpha)v_\alpha, \quad N_\alpha[v] = Nv_\alpha \tag{2.7}$$

for all $\alpha \in \mathbb{Z}_+^n(2)$ where $\Omega(\alpha) = \mathrm{diag}\,(\lambda^\alpha - \lambda_1, \ldots, \lambda^\alpha - \lambda_n)$. This allows us to rewrite the homological equation (2.1) as

$$\Omega(\alpha)f_\alpha - \varepsilon N f_\alpha = g_\alpha - \mathcal{R}_\alpha^\varepsilon[f], \qquad \alpha \in \mathbb{Z}_+^n(2) \tag{2.8}$$

We point out that in view of the definition of $\Omega(\alpha)$ and the Jordan normal form block structure, the matrices $\Omega(\alpha)$ and N commute for every $\alpha \in \mathbb{Z}_+^n(2)$. Thus if κ denotes the order of nilpotency of N and spec (Λ) is nonresonant we get

$$(\Omega(\alpha) - \varepsilon N)^{-1} = \Omega^{-1}(\alpha) + \sum_{s=1}^{\kappa-1} \varepsilon^s \Omega^{-s-1}(\alpha)N^s, \tag{2.9}$$

which implies that (2.8) is equivalent to

$$f_\alpha = (\Omega(\alpha) - \varepsilon N)^{-1} g_\alpha - (\Omega(\alpha) - \varepsilon N)^{-1}\mathcal{R}_\alpha^\varepsilon[f] \tag{2.10}$$

for $\alpha \in \mathbb{Z}_+^n(2)$. The abstract equation above is important in the verification of KAM-type estimates for the linearizing map. The main difficulty in the Jordan block case is to analyze the contributions of the remainder term $\mathcal{R}^\varepsilon$.

The formal inverse, $\mathcal{L}^{-1}$, is defined by (2.10). For multiindices α where the remainder term, $\mathcal{R}_\alpha$, is zero this is clear while for other multiindices we have, roughly speaking, remainders $\mathcal{R}_\alpha^\varepsilon$ which depend upon quantities definable by a recursion of at most $|\alpha|$ steps (e.g., step one of this recursion determines the f_α whose remainders $\mathcal{R}_\alpha^\varepsilon$ involve only terms whose remainders were zero). These ideas are expressed in more detail during the construction of partitions of $\mathbb{Z}_+^n(2)$ discussed later in this section.

We introduce two Banach algebras; first $H(T) = \{f(x) = \sum_{\alpha\in\mathbb{Z}_+^n(2)} f_\alpha x^\alpha;\ f_\alpha \in \mathbb{C}^n : |\!|f|\!|_T < +\infty\}$ with the norm

$$|\!|f|\!|_T = \sum_{\alpha\in\mathbb{Z}_+^n(2)} |f_\alpha| T^{|\alpha|} \tag{2.11}$$

and second $\mathcal{H}(T) = \{f \in C(\overline{B(T)}) \bigcap (\mathcal{O}_2(B(T))^n : \|f\|_T < +\infty\})$ with the norm

$$\|f\|_T = \sup_{z\in B(T)} |f(z)|, \tag{2.12}$$

where $B(T) = \{z \in \mathbb{C}^n : |z| = \max_{j=1,\dots,n} |z_j| < T\}$ and $\mathcal{O}_2(B(T))$ is the space of all holomorphic functions f in $B(T)$ such that $f(0) = 0$, $\partial_{z_j} f(0) = 0$, $j = 1, \dots, n$. In order to apply a KAM type scheme, it is enough to prove that the inverse of $\mathcal{L}$ satisfies the following estimates

$$|\mathcal{L}^{-1}\varphi|_{T_1} \leq \Gamma(\delta, q)|\varphi|_T, \qquad \varphi \in H(T),\ T > T_1 = Te^{-\delta} > 0. \tag{2.13}$$

Here $\Gamma(\delta, q)$, $\delta > 0$, $q \in \mathbb{Z}_+$ is a (Rüssmann type) weight function defined by

$$\Gamma(\delta, q) = \sum_m e^{-\delta m} e^{\frac{qm}{\psi(m)}} < +\infty. \tag{2.14}$$

We propose an abstract approach, which turns out to be essential in the Jordan block case. The homological equation, $\mathcal{L}f = g$, can be solved for f on fixed-height slices of $\mathbb{Z}_+^n(2)$, i.e., $|\alpha| = M$ for $\alpha \in \mathbb{Z}_+^n(2)$ and $M \in \mathbb{Z}_+(2)$. Transverse to this partition we have a decoupling into Poincaré and Siegel parts for which the norm $|\cdot|_T$ is particularly well suited. After a preliminary linear coordinate change we can suppose that Λ^ε has the block form

$$\mathrm{diag}\,(\lambda_S, \lambda_P) \times \text{Jordan blocks} \tag{2.15}$$

where λ_P is a vector of eigenvalues each of magnitude less than one and λ_S is a vector of Siegel type eigenvalues - magnitude one, not a root of unity and satisfying an arithmetic condition. (It could happen that λ_P is empty as in the three dimensional case.) Write $\alpha = (\alpha_S, \alpha_P, \alpha_J)$ to denote the corresponding indices.

Definition 2.2 *A multiindex $\alpha \succeq \beta$ (read "depends upon") if inverting $f_\alpha = (\mathcal{L}^{-1}g)_\alpha$ involves terms including g_β.*

The mixing of terms with different indices arises due to the Jordan blocks. The next lemma makes this more precise for two-dimensional blocks.

Lemma 2.3 *Let α_J be two-dimensional and $|\alpha_J| \geq 2$.*

$$(\alpha_S, \alpha_P, \alpha_{n-1}, \alpha_n) \succeq \bigcup_{r=0}^{\alpha_n} (\alpha_S, \alpha_P, \alpha_{n-1} + r, \alpha_n - r) \tag{2.16}$$

Proof: Homogeneous polynomials are $\mathcal{L}$ and $\mathcal{L}^{-1}$-invariant, so the dependent indices all satisfy $|\beta| = |\alpha|$. For simplicity we prove the three-dimensional case. After writing down the homological equation one sees that given $\alpha \in \mathbb{Z}_+^3(2)$ the dependence is determined by terms of the form

$$x^{\beta_1}(y + \varepsilon z)^{\beta_2} z^{\beta_3} \tag{2.17}$$

Several conditions on β must hold: $\beta_1 = \alpha_1$, $\alpha_2 + \alpha_3 \geq \beta_2 \geq \alpha_2$, and $0 \leq \beta_3 \leq \alpha_3$. If $\beta_2 = \alpha_2 + r$, then $\beta_3 = \alpha_3 - r$ and the corresponding monomial is

$$\binom{\alpha_2 + r}{r} x^{\alpha_1} \varepsilon^r y^{\alpha_2} z^r z^{\alpha_3 - r}, \tag{2.18}$$

which concludes the proof. $\square$

Definition 2.4 *Let S and P be two disjoint subsets of $\mathbb{Z}_+^n(2)$ such that*

$$\mathbb{Z}_+^n(2) = S \cup P \tag{2.19}$$

We call the decomposition (S,P) an $\mathcal{L}$–decomposition if the following conditions hold:

i) Invariance: $\mathcal{L}^{\pm 1}S \subset S$, $\mathcal{L}^{\pm 1}P \subset P$.

ii) S is contained in a finite union of affine spaces of the type $\alpha_J =$ constant vector. So there is an integer $m \geq 2$ such that for each $\alpha = (\alpha_S, \alpha_P, \alpha_J) \in S$, $|\alpha_J| \leq m$.

iii) S has the following graded structure: $S = \bigcup_{i=0}^m S_i$, where $S_j \succeq \bigcup_{i=0}^{j-1} S_i$ for each $j = 1 \dots m$,. Here $S_0 = \{\alpha \mid \mathcal{R}_\alpha = 0\}$.

iv) Writing $\mathcal{L}_S := \mathcal{L}|_S = (\Omega - \varepsilon N)|_S$, $\mathcal{L}_S$ satisfies a Rüssmann type Diophantine condition for $\alpha \in S$.

v) The operator $\mathcal{L}_P = \mathcal{L}|_P$ satisfies a Poincaré type condition, namely there exist $C > 0$ and $\rho \in]0,1[$ such that

$$|\mathcal{L}_\alpha^{-1}| \leq C\rho^{|\alpha|}, \qquad \alpha \in P. \tag{2.20}$$

The partition above is transverse to the partition into homogeneous indices (corresponding to the splitting of a biholomorphic function into the sum of homogeneous polynomials). Since the second partition is also $\mathcal{L}$-invariant, the effect of small divisors can accumulate at most finitely many times. For example when $n = 3$, $S = \bigcup_{|\alpha_J| \leq 1}(\alpha_S, \alpha_J)$. Consider a homogeneous slice $|\alpha| = k+1$. For all k, $\mathcal{R}^\varepsilon_{(k+1,0,0)}$ and $\mathcal{R}^\varepsilon_{(k,0,1)}$ are zero while $\mathcal{R}^\varepsilon_{(k,1,0)}$ has one term corresponding to the dependence on $(k,0,1)$ implied by Lemma 2.3. If the Jordan block is larger there are more levels in the graded structure of S. Assuming uniform nonresonance, m depends only on the algebraic multiplicity of the block. An exceptional case is described in definition 1.5 when there is a particular type of near resonance. When $\mathcal{R}^\varepsilon$ is not zero the bound on the number of terms is uniform for all $\alpha \in S$ and is determined by $\operatorname{spec}(\Phi'(0))$ and the size of the Jordan block.

The idea of the $\mathcal{L}$–decomposition is to pick out the part of the lattice where Diophantine estimates occur (S), while in the remaining part (P) uniform estimates are found. Solutions are built by proceeding in an organized manner through subpartitions of each of the fundamental partitions (into Siegel and Poincaré parts). As we will see below, this decomposition turns out to be essential in the study of the inverse operator $\mathcal{L}$ in the presence of nontrivial Jordan blocks.

Set

$$\omega_S(m) := \min_{2 \leq |\alpha| \leq m, \alpha \in S, 1 \leq j \leq d} |\lambda^\alpha - \lambda_j|, \qquad m \in \mathbb{Z}_+(2). \tag{2.21}$$

Next, we introduce the following Diophantine condition in S, namely

$$\omega_S(m) \ \geq \ e^{-\frac{m}{\psi(m)}}, \qquad m \in \mathbb{Z}_+(k), \tag{2.22}$$

where ψ satisfies (1.7).

Following the ideas in [8], in view of the $\mathcal{L}$–decomposition, for any $f(x) \in (\mathbb{C}_2[x])^n$ we can define

$$f^P(x) := \sum_{\alpha\in P} f_\alpha x^\alpha, \quad f^S(x) := \sum_{\alpha\in S} f_\alpha x^\alpha \tag{2.23}$$

The norm $\lfloor\cdot\rfloor_T$ is well suited to partitions since

$$\lfloor f\rfloor_T = \lfloor f^S + f^P\rfloor_T = \lfloor f^S\rfloor_T + \lfloor f^P\rfloor_T. \tag{2.24}$$

We also have (using Cauchy estimates to get the second inequality)

$$\|f\|_{T_1} \le \lfloor f\rfloor_{T_1} \le \frac{1}{(1-r)^n}\|f\|_T \tag{2.25}$$

where $r = \frac{T_1}{T}$ and n is the dimension.

We outline the main advantage of the decomposition approach for estimating the solutions to the linearized homological equations.

Lemma 2.5 *If (S,P) is a $\mathcal{L}$–decomposition, then there exist $C>0$ and a weight function $\Gamma(\delta,q)$ such that $f=\mathcal{L}^{-1}g$ satisfies the following estimates:*

$$\begin{aligned} \lfloor f^S\rfloor_{T_1} &\le \Gamma(\delta,q)\lfloor g^S\rfloor_T, & (2.26)\\ \lfloor f^P\rfloor_T &\le C\lfloor g^P\rfloor_T + \Gamma(\delta,q)\|g^S_{nr}\|_T. & (2.27)\end{aligned}$$

for all $T_1 = Te^{-\delta} < T$, where g^S_{nr} is a projection of g^S on smaller subset of indices.

When there is no near resonance or the near resonance is of the form $|\lambda^\gamma| = |\mu|$ the subset of indices is empty and the term can be omitted. When there are near resonances involving μ on the left hand side of the last equality, one has a finite set of such indices. Since the second term in the RHS of (2.27) is present only if a near–resonance occurs, it arises when $n\ge 4$.

Proof: We use the graded decomposition of S. By the definition of $\mathcal{L}$–decomposition, on S_0 we can cancel the term $R^\varepsilon_\alpha[g]$ in (2.10) and write

$$f_\alpha = (\Omega(\alpha)-\varepsilon N)^{-1}g_\alpha = \sum_{j=0}^{\kappa-1}\varepsilon^j\Omega^{-j-1}(\alpha)N^j g_\alpha \tag{2.28}$$

for all $\alpha\in S$. Therefore on S_0, $\mathcal{L}^{-1}$ is an operator of constant type, and repeating (at most) κ times the direct estimates as in the semisimple case, we derive (2.26) on S_0. Now one can estimate $\mathcal{L}^{-1}$ on $S_1,\dots,S_m$. As to (2.27), it is obtained from (2.20) and the representation of the homological operator on P (e.g., see Lemma 3.2). For the estimates on the Poincaré part, a larger combinatorial complexity appears due to the off diagonal terms and unbounded number of accumulations captured in the remainder term $\mathcal{R}^\varepsilon$. $\square$

As to the decomposition, we will carry out the proof for a single Jordan block in $\mathbb{C}^n$. More precisely, we suppose that

$$\max_{j=1,\dots,n} |\lambda_j| =: \lambda_{max} \leq 1. \tag{2.29}$$

If the expansion condition holds, we consider Φ^{-1} instead of Φ.

Now we consider matrices admitting only one Jordan block of size ≥ 2. Hence, without loss of generality we may assume that for some $1 \leq m \leq n-2$ we have $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|$, $\lambda_{m+1} = \cdots = \lambda_n = \mu$, $|\mu| < 1$, and

$$\Lambda(\varepsilon) \quad = \quad \begin{pmatrix} \lambda_1 & 0 & \dots & 0_{1\times k} \\ 0 & \lambda_2 & \dots & 0_{1\times k} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \mu I_k + \varepsilon N_k \end{pmatrix} \tag{2.30}$$

where $k := n - m$. In this notation, if the Poincaré condition is violated, necessarily $|\lambda_1| = 1$.

Lemma 2.6 *Under the hypotheses (2.29), (2.30) we define*

$$K = K(\Lambda) := \min\left\{p \in \mathbb{N} : \, p \geq \max\left\{2, \left[\frac{\ln(|\lambda_m|)}{\ln|\mu|}\right] + 1\right\}\right\} \tag{2.31}$$

and $C = |\lambda_m| - |\mu|^K$. *We claim that*

$$\|\Omega(\alpha)^{-1}\| := \max_{j=1,\dots,n} |\lambda^\alpha - \lambda_j|^{-1} \leq \max\left\{\frac{1}{C}, \frac{1}{|\mu|(1-|\mu|)}\right\} \tag{2.32}$$

for all $\alpha \in \mathbb{Z}^n_+(2)$ *such that* $|\alpha_J| \geq K$.

Proof: Set $\alpha = (\alpha_S, \alpha_P, \alpha_J)$ and $\lambda = (\lambda_S, \lambda_P, \lambda_J)$. We note that $\lambda^\alpha = \lambda_S^{\alpha_S}\lambda_P^{\alpha_P}\lambda_J^{\alpha_J}$. Next assuming $|\alpha_J| \geq 2$, we get

$$|\lambda_S^{\alpha_S}\lambda_P^{\alpha_P}\mu^{|\alpha_J|} - \lambda_j| \;\geq\; |\lambda_j| - |\mu|^{|\alpha_J|} \geq |\lambda_m| - |\mu|^{|\alpha_J|}, \qquad j = 1,\dots,m, \tag{2.33}$$

$$|\lambda_S^{\alpha_S}\lambda_P^{\alpha_P}\mu^{|\alpha_J|} - \mu| \;\geq\; |\mu|(1 - |\mu|^{|\alpha_J|-1}). \tag{2.34}$$

Clearly (2.33) and (2.34) yield the desired estimate (2.32). $\square$

Remark 2.7 *We point out that using the arguments in the proof of Lemma 2.6, we show that for every* $C \in]0, |\lambda_m|[$ *the estimate (2.32) holds for all* $\alpha \in \mathbb{Z}^n_+(2)$ *such that* $|\alpha_J| \geq \max\left\{2, \left[\frac{\ln(|\lambda_m|-C)}{\ln|\mu|}\right] + 1\right\}$.

3 One Jordan block in $\mathbb{C}^3$

In this section we begin the proof of Theorem 1.1. Recall that $\Phi : \mathbb{C}^3 \to \mathbb{C}^3$, $\Phi(0) = 0$, is biholomorphic near the origin and $\Phi'(0)$ is defined by (1.12), with $|\lambda| = 1$, λ is not a root of unity, $0 < |\mu| < 1$, and $0 < |\varepsilon| \ll 1$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha|_{2^+} = \alpha_2 + \alpha_3$ and $|\alpha|_{2^-} = \alpha_1 + \alpha_2$. Here $\lambda_P = 0$. With the notation above we have

$$\Omega(\alpha) = \mathrm{diag}\,(\lambda^{\alpha_1}\mu^{|\alpha|_{2^+}} - \lambda, \lambda^{\alpha_1}\mu^{|\alpha|_{2^+}} - \mu, \lambda^{\alpha_1}\mu^{|\alpha|_{2^+}} - \mu).$$

If we partition the lattice of indices as follows

$$\mathbb{Z}^3_+(2) = S \cup P = \{\alpha \in \mathbb{Z}^3_+(2) |\, \alpha_2 + \alpha_3 < 2\} \cup \{\alpha \in \mathbb{Z}^3_+(2) |\, \alpha_2 + \alpha_3 \geq 2\}, \tag{3.1}$$

then for indices $\alpha \in P$ we have the Poincaré-type estimates

$$\Omega(\alpha)^{-1} \leq \frac{1}{|\mu|(1 - |\mu|)},$$

whereas for indices $\alpha \in S$ we have the Rüssmann-type estimates

$$\Omega(\alpha)^{-1} \leq (\omega_S(|\alpha|))^{-1} \leq e^{\frac{|\alpha|}{\psi(|\alpha|)}}$$

where ω is the modulus that estimates the small divisors in S defined in (2.21).

To estimate $\mathcal{R}^\varepsilon(\alpha)$ in the presence of the off-diagonal terms from the Jordan block one must take into account other terms of order $M = |\alpha| \geq 2$.

$$\begin{aligned}
R^\varepsilon_M(v) &= \sum_{|\alpha|=M} v_\alpha \lambda^{\alpha_1} x_1^{\alpha_1} \mu^{\alpha_3} x_3^{\alpha_3} [(\mu x_2 + \varepsilon x_3)^{\alpha_2} - (\mu x_2)^{\alpha_2}] && (3.2)\\
&= \sum_{|\alpha|=M} v_\alpha \lambda^{\alpha_1} x_1^{\alpha_1} \mu^{\alpha_3} x_3^{\alpha_3} \sum_{s=0}^{\alpha_2-1} \binom{\alpha_2}{s} \mu^s \varepsilon^{\alpha_2-s} x_2^s x_3^{\alpha_2-s} && (3.3)\\
&= \sum_{\alpha_1=0}^{M-1} \sum_{\alpha_2=0}^{M-\alpha_1} \sum_{s=0}^{\alpha_2-1} v_\alpha \lambda^{\alpha_1} x_1^{\alpha_1} \mu^{M-|\alpha|_2^-} x_3^{M-|\alpha|_2^-} \binom{\alpha_2}{s} \mu^s \varepsilon^{\alpha_2-s} x_2^s x_3^{\alpha_2-s} && (3.4)\\
&= \sum_{\alpha_1=0}^{M-1} \sum_{s=0}^{M-\alpha_1-1} \sum_{\alpha_2=s+1}^{M-\alpha_1} v_\alpha \lambda^{\alpha_1} x_1^{\alpha_1} \mu^{M-|\alpha|_2^-} x_3^{M-|\alpha|_2^-} \binom{\alpha_2}{s} \mu^s \varepsilon^{\alpha_2-s} x_2^s x_3^{\alpha_2-s} && (3.5)\\
&= \sum_{\alpha_1=0}^{M-1} \sum_{s=0}^{M-\alpha_1-1} V^M_{\alpha_1,s}(\varepsilon)\; x_1^{\alpha_1} x_2^s x_3^{M-\alpha_1-s} && (3.6)
\end{aligned}$$

where $\nu = \alpha_2 - s$ and

$$V^M_{\alpha_1,s}(\varepsilon) = \sum_{\nu=1}^{M-\alpha_1-s} v_{\alpha_1,s+\nu,M-\alpha_1-s-\nu} \binom{s+\nu}{s} \lambda^{\alpha_1} \mu^{M-\alpha_1-\nu} \varepsilon^\nu. \tag{3.7}$$

Set $\ell = \alpha_1$, $k = M - \ell$, $s = k - r$, and $R^{\varepsilon}_{\ell,k-r,r}[v] := V^M_{\ell,k-r}(\varepsilon)$. Rewriting (3.7) and changing the index of summation $\nu \to k - s - \nu$ we get

$$\begin{aligned} R^{\varepsilon}_{\ell,k-r,r}[v] &= \lambda^\ell \sum_{\nu=0}^{r-1} v_{\ell,k-\nu,\nu} \binom{k-\nu}{k-r} \mu^{k-r+\nu} \varepsilon^{r-\nu} \\ &= \lambda^\ell \mu^k R'_{\ell,k-r,r}[v] + \varepsilon^r \lambda^\ell \mu^{k-r} \binom{k}{k-r} v_{\ell,k,0}, \end{aligned} \tag{3.8}$$

where, setting $\rho = \varepsilon/\mu$, we obtain

$$R'_{\ell,k-r,r}[v] = {R'_{\ell,k-r,r}}^{\varepsilon}[v] = \sum_{\nu=1}^{r-1} v_{\ell,k-\nu,\nu} \binom{k-\nu}{k-r} \rho^{r-\nu}, \qquad r \geq 2 \tag{3.9}$$

with the convention $R'_{\ell,k-1,1}[v] = 0$. Set $\Omega_{\ell,k} = \operatorname{diag}\{\lambda^\ell\mu^k - \lambda, \lambda^\ell\mu^k - \mu, \lambda^\ell\mu^k - \mu\}$.

Thus we can rewrite the homological equation (2.8) in the following way:

$$(\Omega_{\ell,0} - \varepsilon N) v_{\ell,0,0} = g_{\ell,0,0} \tag{3.10}$$

$$(\Omega_{\ell,1} - \varepsilon N) v_{\ell,1,0} = g_{\ell,1,0} \tag{3.11}$$

$$(\Omega_{\ell,1} - \varepsilon N) v_{\ell,0,1} = g_{\ell,0,1} - \rho\lambda^\ell \mu v_{\ell,1,0} \tag{3.12}$$

$$(\Omega_{\ell,k} - \varepsilon N) v_{\ell,k,0} = g_{\ell,k,0}, \tag{3.13}$$

$$\begin{aligned} (\Omega_{\ell,k} - \varepsilon N) v_{\ell,k-r,r} &= g_{\ell,k-r,r} - \varepsilon^r \lambda^\ell \mu^{k-r} \binom{k}{k-r} v_{\ell,k,0} \\ &\quad - \lambda^\ell \mu^k R'_{\ell,k-r,r}[v], \quad r = 1, \dots, k \end{aligned} \tag{3.14}$$

for all $k \in \mathbb{Z}_+(2)$, $\ell \in \mathbb{Z}_+$.

We decompose $v(x) = v_P(x) + v_S(x)$ where

$$v_P(x) = \sum_{\alpha \in P} v_\alpha x^\alpha, \qquad v_S(x) = \sum_{\alpha \in S} v_\alpha x^\alpha.$$

Clearly we can rewrite v_S as follows

$$v_S(x) = \sum_{\ell=2}^{\infty} v_{(\ell,0,0)} x_1^\ell + \sum_{\ell=1}^{\infty} v_{(\ell,1,0)} x_1^\ell x_2 + \sum_{\ell=1}^{\infty} v_{(\ell,0,1)} x_1^\ell x_3$$

while for v_P we have $v_P(x) = v_P^0(x) + v'_P(x)$, where

$$v_P^0(x) = \sum_{\ell=0}^{\infty} \sum_{k=2}^{\infty} v_{(\ell,k,0)} x_1^\ell x_2^k \tag{3.15}$$

$$v'_P(x) = \sum_{\ell=0}^{\infty} \sum_{k=2}^{\infty} v_{\vec{\ell,k}}(x) \tag{3.16}$$

$$v_{\vec{\ell,k}}(x) = \sum_{r=1}^{k} v_{(\ell,k-r,r)} x_1^\ell x_2^{k-r} x_3^r \tag{3.17}$$

Hence $\|v_P\|_T = \|v_P^0\|_T + \|v_P'\|_T$, where

$$\|v_P^0\|_T = \sum_{\ell=0}^{\infty}\sum_{k=2}^{\infty} T^{\ell+k}|v_{(\ell,k,0)}|_{\ell^\infty}, \tag{3.18}$$

$$\|v_P'\|_T = \sum_{\ell=0}^{\infty}\sum_{k=2}^{\infty} \|\vec{v_{\ell,k}}\|_T = \sum_{\ell=0}^{\infty}\sum_{k=2}^{\infty} T^{\ell+k}\sum_{r=1}^{k}|v_{(\ell,k-r,r)}|_{\ell^\infty}, \tag{3.19}$$

We set $\omega_\lambda(\ell) := \min_{2\le j\le \ell}|\lambda^j - \lambda|$, $\ell \in \mathbb{Z}_+(2)$.

Proposition 3.1 *Under the hypotheses of Theorem 1.1 in the introduction, we can find a Rüssmann type weight function* $\Gamma(\delta, q)$ *such that*

$$\|v_S\|_{T_1} \le \Gamma(\delta, q)\,\|g_S\|_T, \qquad T > T_1 = Te^{-\delta}, \delta > 0, g \in H(T), \tag{3.20}$$

Moreover, for every $\delta_0 \in]0,1[$ *there exists* $\varepsilon_0 > 0$*, depending on* μ, δ_0 *such that*

$$\|v_P\|_T \le \frac{1}{1-\delta_0}\frac{1}{|\mu| - |\mu|^2}\|g_P\|_T \tag{3.21}$$

provided $0 < |\varepsilon| < \varepsilon_0$.

Proof: Since

$$\max\left\{\|\Omega_{\ell,0}^{-1}\|, \|\Omega_{\ell,1}^{-1}\|\right\} \le \frac{1}{(|\mu| - |\mu|^2)\omega_\lambda(\ell)}, \qquad \ell \in \mathbb{Z}_+(2), \tag{3.22}$$

we get by (3.10)–(3.12), the estimate (3.22), and the Rüssmann condition on λ that, following [13], the estimate (3.20) is true.

Next, we focus on the equations (3.13), (3.14) in the Poincaré "slice" P. First of all, we observe that $|\mu| < 1$ and $|\lambda| = 1$ yield

$$\|\Omega_{\ell,k}^{-1}\| \le \max\left\{\frac{1}{|\mu| - |\mu|^2}, \frac{1}{1 - |\mu|^2}\right\} = \frac{1}{|\mu| - |\mu|^2}, \qquad \ell \in \mathbb{Z}_+, k \in \mathbb{Z}_+(2). \tag{3.23}$$

Hence by inverting (3.13), we obtain that

$$\|v_P^0\|_T \le \left(\frac{1}{|\mu| - |\mu|^2} + \frac{|\varepsilon|}{(|\mu| - |\mu|^2)^2}\right)\|g_P^0\|_T, \qquad T > 0. \tag{3.24}$$

Define the following tensor

$$\mathcal{V}_{\ell,k} = (v_{(\ell,k-1,1)}, \dots, v_{(\ell,0,k)}) \in (\mathbb{C}^3)^k. \tag{3.25}$$

with the norm

$$|\mathcal{V}_{\ell,k}|_{\ell^1} = \sum_{r=1}^{k}|v_{(\ell,k-r,r)}|_{\ell^\infty}, \qquad \ell \in \mathbb{Z}_+, k \in \mathbb{Z}_+(2), \tag{3.26}$$

where

$$|v_{(\ell,k-r,r)}|_{\ell^\infty} = \max_{1\le j\le 3} |v_{(\ell,k-r,r);j}| \tag{3.27}$$

In view of the system (3.14) it is natural to define the operator

$$\begin{aligned}
\mathcal{R}'_{\ell,k} : (\mathbb{C}^3)^k &\longrightarrow (\mathbb{C}^3)^k, && (3.28)\\
\mathcal{R}'_{\ell,k}[\mathcal{V}_{\ell,k}] &= (\mathcal{R}'_{\ell,k}[\mathcal{V}_{\ell,k}]_1, \dots, \mathcal{R}'_{\ell,k}[\mathcal{V}_{\ell,k}]_k) && (3.29)\\
\mathcal{R}'_{\ell,k}[\mathcal{V}_{\ell,k}]_r &= R'_{\ell,k-r,r}[v] && (3.30)
\end{aligned}$$

Thus we can rewrite (3.14) as

$$\begin{aligned}
(\Omega_{\ell,k} - \varepsilon N)\mathcal{V}_{\ell,k} &= \mathcal{G}_{\ell,k} - \varepsilon^r \lambda^\ell \mu^{k-r} \begin{pmatrix} k \\ k-r \end{pmatrix} \mathbf{1}_k v_{\ell,k,0} \\
&\quad - \lambda^\ell \mu^k \mathcal{R}'_{\ell,k}[\mathcal{V}_{\ell,k}], && (3.31)
\end{aligned}$$

where $\mathbf{1}_k$ stands for a tensor with k rows with the 3×3 identity matrix $\mathbf{1}$ in each row. Equivalently, inverting $(\Omega_{\ell,k} - \varepsilon N)$, we obtain

$$\begin{aligned}
\mathcal{V}_{\ell,k} &= \mathcal{H}^\varepsilon_{\ell,k} - \varepsilon^r \lambda^\ell \mu^{k-r} \begin{pmatrix} k \\ k-r \end{pmatrix} (\Omega_{\ell,k} - \varepsilon N)^{-1} \mathbf{1}_k v_{\ell,k,0} \\
&\quad - \lambda^\ell \mu^k (\Omega_{\ell,k} - \varepsilon N)^{-1} \mathcal{R}'_{\ell,k}[\mathcal{V}_{\ell,k}], && (3.32)
\end{aligned}$$

where

$$\mathcal{H}^\varepsilon_{\ell,k} = (\Omega_{\ell,k} - \varepsilon N)^{-1} \mathcal{G}_{\ell,k}. \tag{3.33}$$

We now derive a crucial lemma.

Lemma 3.2 *We have*

$$|\mathcal{R}_{\ell,k}[\mathcal{V}_{\ell,k}]|_{\ell^1} \le ((1+|\rho|)^{k-1} - 1)|\mathcal{V}_{\ell,k}|_{\ell^1}, \tag{3.34}$$

for all $\rho = |\varepsilon|/|\mu|$, $\mathcal{V}_{\ell,k} \in (\mathbb{C}^3)^k$.

Proof: We have

$$\begin{aligned}
|\mathcal{R}_{\ell,k}[\mathcal{V}_{\ell,k}]|_{\ell^1} &\le \sum_{r=2}^{k} |R'_{\ell,k-r,r}[v]| \\
&\le \sum_{r=2}^{k}\sum_{\nu=1}^{r-1} |v_{\ell,k-\nu,\nu}| \left(\begin{array}{c} k-\nu \\ k-r \end{array}\right) |\rho|^{r-\nu} \\
&\le \sum_{\beta=1}^{k-1} |\rho|^{\beta} \sum_{r=\beta+1}^{k} |v_{\ell,k-r+\beta,r-\beta}| \left(\begin{array}{c} k-r+\beta \\ k-r \end{array}\right) \\
&\le \sum_{\beta=1}^{k-1} |\rho|^{\beta} \left(\max_{\beta+1\le r\le k} \left(\begin{array}{c} k-r+\beta \\ k-r \end{array}\right)\right) \sum_{r=\beta+1}^{k} |v_{\ell,k-r+\beta,r-\beta}| \\
&\le \sum_{\beta=1}^{k-1} |\rho|^{\beta} \left(\begin{array}{c} k-1 \\ k-1-\beta \end{array}\right) |\mathcal{V}_{\ell,k}|_{\ell^1} \\
&= ((1+|\rho|)^{k-1}-1)|\mathcal{V}_{\ell,k}|_{\ell^1}. \qquad (3.35)
\end{aligned}$$

We have used the identities

$$\max_{\beta+1\le r\le k} \left(\begin{array}{c} k-r+\beta \\ k-r \end{array}\right) = \left(\begin{array}{c} k-1 \\ k-1-\beta \end{array}\right),$$

$$\sum_{\beta=1}^{k-1} |\rho|^{\beta} \left(\begin{array}{c} k-1 \\ k-1-\beta \end{array}\right) = ((1+|\rho|)^{k-1}-1).$$

The proof of the lemma is complete. □

Combining the lemma above and the estimate (3.22) we obtain from (3.32) the following estimates

$$\begin{aligned}
|\mathcal{V}_{\ell,k}|_{\ell^1} &\le \frac{|\mu|^{k-1}((1+|\rho|)^{k-1}-1)}{1-|\mu|}|\mathcal{V}_{\ell,k}|_{\ell^1} + |\mathcal{H}^{\varepsilon}_{\ell,k}|_{\ell^1} \\
&\quad + \sum_{r=1}^{k} \varepsilon^r |\mu|^{k-r} \left(\begin{array}{c} k \\ k-r \end{array}\right) \frac{1}{|\mu|-|\mu|^2}|v_{\ell,k,0}| \\
&\le \frac{|\mu|^{k-1}((1+|\rho|)^{k-1}-1)}{1-|\mu|}|\mathcal{V}_{\ell,k}|_{\ell^1} + |\mathcal{H}^{\varepsilon}_{\ell,k}|_{\ell^1} \\
&\quad + \frac{(\varepsilon+|\mu|)^k-|\mu|^k}{|\mu|-|\mu|^2}|v_{\ell,k,0}|. \qquad (3.36)
\end{aligned}$$

Since $|\mu|<1$ and we have the freedom to choose ε and $\rho=\varepsilon/\mu$ as small as we like, we obtain that for every $z\in]0,1[$ we can find $\varepsilon_{z,\mu}>0$ such that

$$\max\left\{\frac{|\mu|^{k-1}((1+|\rho|)^{k-1}-1)}{1-|\mu|}, \frac{(\varepsilon+|\mu|)^k-|\mu|^k}{|\mu|-|\mu|^2}\right\} \le z \qquad (3.37)$$

for all $|\varepsilon| < \varepsilon_{z,\mu}, k \in \mathbb{Z}_+(2)$. The estimate (3.21) follows readily from (3.36), (3.37), combined with (3.18), (3.19). The proof is complete. $\square$

4 Quasi-resonance phenomena

We consider a hyperbolic matrix Λ^ε defined by (1.12), where

$$|\lambda| > 1, \ |\mu| < 1 \text{ and } 0 < |\varepsilon| \ll 1 \tag{4.1}$$

If $|\lambda| < 1$ and $|\mu| > 1$ we deal with Λ^{-1}. We assume that Λ^ε is nonresonant, which in view of (1.12), (4.1) means

$$(\lambda^{\alpha_1}\mu^{\alpha_2+\alpha_3} - \lambda)(\lambda^{\alpha_1}\mu^{\alpha_2+\alpha_3} - \mu) \neq 0 \quad \forall \alpha \in \mathbb{Z}_+^3(2). \tag{4.2}$$

Let $\tau = -\ln|\lambda|/\ln|\mu| > 0$ be the solution to $|\lambda||\mu|^\tau = 1$. Set $2\pi\rho :=\arg(\lambda)$, $2\pi\theta :=\arg(\mu)$, $\rho, \theta \in [0,1]$. We have

$$\begin{aligned}
\lambda^\alpha\mu^\beta - 1 &= |\lambda|^\alpha|\mu|^\beta e^{2\pi i(\alpha\rho+\beta\theta)} - 1 \\
&= (|\lambda|^\alpha|\mu|^\beta - 1)e^{i2\pi(\alpha\rho+\beta\theta)} + e^{i2\pi(\alpha\rho+\beta\theta)} - 1 \\
&= e^{-(\alpha\tau-\beta)\ln|\mu|}e^{i2\pi(\alpha\rho+\beta\theta)} \\
&+ (e^{i2\pi\alpha(\tau\theta+\rho)} - 1)e^{-i2\pi\theta(\alpha\tau-\beta)} \\
&+ e^{-i2\pi\theta(\alpha\tau-\beta)} - 1
\end{aligned} \tag{4.3}$$

for all $\alpha, \beta \in \mathbb{Z}_+$. We introduce the Diophantine quasinorm

$$|||\alpha\omega||| = \inf_{q\in\mathbb{Z}} |\alpha\omega - q|, \qquad \omega \in \mathbb{R}.$$

In view of (4.3) we can find $C_0 > 0$ such that

$$\inf_{\beta\in\mathbb{N}} |\lambda^\alpha\mu^\beta - 1| \ \leq \ C_0(|||\alpha\tau||| + |||\alpha(\tau\theta+\rho)|||) \tag{4.4}$$

for all $\alpha \in \mathbb{N}$. We readily obtain that the nonresonance of $\{\lambda, \mu, \mu\}$ is equivalent to the requirement that at least one of the two numbers τ and $\rho + \tau\theta$ is irrational. Thus, by well known results for simultaneous approximations in metric number theory (e.g., cf. W. Schmidt [22], see also [18]) the estimate (4.4) implies the existence of a sequence $\{(\alpha_p, \beta_p)\}_{p=1}^\infty \subset \mathbb{N}^2$ such that $\lim_{p\to\infty} \alpha_p = \lim_{p\to\infty} \beta_p = \infty$ and

$$\lim_{p\to\infty} (\lambda^{\alpha_p}\mu^{\beta_p} - 1) = 0 \tag{4.5}$$

Thus, setting $\ell_p = \alpha_p + 1$, $k_p = \beta_p$, we get

$$\lim_{p\to\infty} (\lambda^{\ell_p}\mu^{k_p} - \lambda) = 0 \tag{4.6}$$

Hence setting

$$\gamma_p = \frac{\lambda^{\ell_p}\mu^{k_p}}{\lambda^{\ell_p}\mu^{k_p} - \lambda} \tag{4.7}$$

we get

$$\lim_{p\to\infty} |\gamma_p| = \infty. \tag{4.8}$$

We shall write the representation (3.6) in a different way.

Choose g in the right–hand side of the homological equation in the following way: $g_{\alpha_1,0,0;1} = 0$, $\alpha_1 \geq 2$; $g_{\alpha;2} = g_{\alpha;3} = 0$, $|\alpha| \geq 2$; $g_{\ell_p,k_p,0;1} = 1$ and $g_{\ell,k,0;j} = 0$ otherwise ($j = 2,3$); $g_{\ell,k-r,r} = 0$ for all $r \geq 1$. Therefore, by the choice of g_α, $\alpha \in \mathbb{Z}^3_+(2)$, and (3.14) we obtain that the nonzero terms v_α of the formal solution $v(x) = \sum_{\alpha\in\mathbb{Z}^3_+(2)} v_\alpha x^\alpha$ of the linear homological equation $\mathcal{L}v = g$ are given by

$$v_{\ell_p,k_p-r,r;1} \quad = \quad -\gamma_p R'_{\ell_p,k_p-r,r;1}[v] + h_{p;r}, \qquad r = 1,\ldots,k_p, \tag{4.9}$$

for all $p \in \mathbb{N}$, where, in view of (3.13) and (3.14) for $k = k_p$, $\ell = \ell_p$ and $g_{\ell_p,k_p,0;1} = 1$, we have

$$\begin{aligned} h_{p;r} \quad &= \quad -\gamma_p\rho^r \binom{k_p}{k_p - r} v_{\ell_p,k_p,0;1} \\ &= \quad -\gamma_p^2\rho^r\binom{k_p}{k_p - r} \end{aligned} \tag{4.10}$$

We readily obtain that there exists $C > 0$ such that

$$|h_{\ell_p,k_p-r,r;1}| \quad \leq \quad 2^{k_p}\rho^r\gamma_p^2, \qquad r = 1,\ldots,k_p,\, p\in\mathbb{N}. \tag{4.11}$$

However, we observe that (4.8), (4.11), $0 < |\varepsilon| \ll 1$, and inductive arguments applied to (4.9) lead to

$$v_{\ell_p,0,k_p;1} \quad = \quad (-\gamma_p)^{k_p}\varepsilon^{k_p}(k_p - 1)!(1 + o(1)), \qquad p\to\infty \tag{4.12}$$

Since $\ell_p \sim k_p \sim \ell_p + k_p$ as $p \to \infty$ and $|\gamma_p|^{\ell_p}$ grows more rapidly than C^{ℓ_p} for any $C > 1$, for any fixed $\varepsilon \neq 0$ we obtain by (4.12) that

$$\sum_{p=1}^{\infty} v_{\ell_p,0,k_p;1} x_1^{\ell_p} x_3^{k_p}$$

diverges for all $(x_1,x_3) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Therefore, the homological equation for the matrix Λ^ε in (1.12) satisfying (4.1) is not solvable in the space all holomorphic functions near the origin, i.e., we can find a convergent series g on the RHS such that the unique formal solution diverges. We conclude the proof following the arguments similar to the ones used in [14], [15], [28]. $\square$

Remark 4.1 *By well known estimates in metric number theory, if τ and $\tau\theta+\rho$ are simultaneously badly approximable we can choose a sequence satisfying $|\gamma_p| \geq c\sqrt{l_p+k_p}$ for some positive constant c, while if a simultaneous Siegel (respectively, Rüssman type) condition holds, namely for some $\nu>0$, $C_0>0$ (respectively, a nondecreasing positive function $\psi(m)$ satisfying (1.7))*

$$\max\{|||m\tau|||, |||m(\tau\theta+\rho)|||\} \geq \frac{C_0}{m^{1/2+\nu}}, \qquad m \in \mathbb{N} \tag{4.13}$$

(respectively,

$$\max\{|||m\tau|||, |||m(\tau\theta+\rho)|||\} \geq e^{-\frac{m}{\psi(m)}}, \qquad m \in \mathbb{N}) \tag{4.14}$$

we can choose the sequence such that

$$|\gamma_p| \geq (\ell_p+k_p)^{1/2+\nu}, \qquad p \in \mathbb{N} \tag{4.15}$$

(respectively,

$$|\gamma_p| \geq e^{(\ell_p+k_p)/\psi(\ell_p+k_p)}, \qquad p \in \mathbb{N}). \tag{4.16}$$

We point out that if (4.13) holds, then in view of (4.15) our estimates (4.12) imply that the divergent formal power series which transform formally the hyperbolic map Φ, $\Phi'(0)=\Lambda^\varepsilon$, is not even in the formal Gevrey spaces G^σ, $\sigma<3/2+\nu$. (cf. [4] for the study of nonanalytic formal power series linearizing formally one dimensional maps $\Phi:(\mathbb{C},0)\to(\mathbb{C},0)$), while for the simultaneous Rüssmann type condition (4.14), in view of (4.16), the formal solution does not belong to any formal Gevrey spaces G^σ, $\sigma>1$.

5 Small divisors and linearization in $\mathbb{C}^4$

We would like to partition $\mathbb{Z}_+^4$ into Siegel and Poincaré sets where the homological equation decouples. Recall that we consider maps with linear part defined by (1.14) with $1=|\lambda_1|\geq|\lambda_2|>0$, $1>|\mu|>0$ and $0<|\varepsilon|\ll 1$. Set $\lambda=(\lambda_1,\lambda_2)$.

We give a representation formula for the remainder in the homological equation in $\mathbb{C}^4$. We estimate $\mathcal{R}^\varepsilon(\alpha)$ in the presence of the off-diagonal terms from the Jordan block for $M=|\alpha|\geq 2$, $\alpha_3\geq 1$. We have

$$\begin{aligned}
\mathcal{R}^\varepsilon_M(v) &= \sum_{|\alpha|=M} v_\alpha \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}x_1^{\alpha_1}x_2^{\alpha_2}\mu^{\alpha_4}x_4^{\alpha_4}[(\mu x_3+\varepsilon x_4)^{\alpha_3}-(\mu x_3)^{\alpha_3}] && (5.1)\\
&= \sum_{|\alpha|=M} v_\alpha \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}x_1^{\alpha_1}x_2^{\alpha_2}\mu^{\alpha_4}x_4^{\alpha_4}\sum_{s=0}^{\alpha_3-1}\binom{\alpha_3}{s}\mu^s\varepsilon^{\alpha_3-s}x_3^s x_4^{\alpha_3-s} && (5.2)\\
&= \sum_{\alpha_1+\alpha_2\leq M-1}\ \sum_{s=0}^{M-\alpha_1-\alpha_2-1} V^M_{\alpha_1,\alpha_2,s}(\varepsilon)\ x_1^{\alpha_1}x_2^{\alpha_2}x_3^s x_4^{M-\alpha_1-s} && (5.3)
\end{aligned}$$

where $\nu = \alpha_3 - s$ and

$$V^M_{\alpha_1,\alpha_2,s}(\varepsilon) = \sum_{\nu=1}^{M-\alpha_1-\alpha_2-s} v_{\alpha_1,\alpha-2,s+\nu,M-\alpha_1-\alpha_2-s-\nu}\binom{s+\nu}{s}\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\mu^{M-\alpha_2-\nu}\varepsilon^{\nu}. \quad (5.4)$$

Set $\ell_1 = \alpha_1$, $\ell_2 = \alpha_2$, $k = M - \ell_1 - \ell_2$, $s = k - r$, and $\mathcal{R}^{\varepsilon}_{\ell_1,\ell_2,k-r,r}[v] := V^M_{\ell_1,\ell_2,k-r}(\varepsilon)$. As in section 3, rewriting (5.4) and changing the index of summation $\nu \to k - s - \nu$ we get

$$\begin{aligned}\mathcal{R}^{\varepsilon}_{\ell_1,\ell_2,k-r,r}[v] &= \lambda_1^{\ell_1}\lambda_2^{\ell_2}\sum_{\nu=0}^{r-1} v_{\ell_1,\ell_2,k-\nu,\nu}\binom{k-\nu}{k-r}\mu^{k-r+\nu}\varepsilon^{r-\nu}\\ &= \lambda_1^{\ell_1}\lambda_2^{\ell_2}\mu^k {R'_{\ell_1,\ell_2,k-r,r}}^{\varepsilon}[v] + \varepsilon^r\lambda_1^{\ell_1}\lambda_2^{\ell_2}\mu^{k-r}\binom{k}{k-r}v_{\ell_1,\ell_2,k,0}. \quad (5.5)\end{aligned}$$

We note that in order to define the decomposition we need to study the multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}^n_+(2)$, with $\beta = \alpha_3 + \alpha_4$, such that at least one of the following three quantities becomes small:

$$|\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\mu^{\beta} - \lambda_1|, \quad |\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\mu^{\beta} - \lambda_2|, \quad |\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\mu^{\beta} - \mu|.$$

Proposition 5.1 *Let $|\lambda_1| = |\lambda_2| = 1$. Then there is an $\mathcal{L}$-partition $S \cup P$ defined by*

$$S = \{\alpha = (\alpha_S, \alpha_J) \in \mathbb{Z}^4_+(2), \alpha_S = (\alpha_1, \alpha_2),\ \alpha_J = (\alpha_3, \alpha_4),\ |\alpha_J| \le 1\}, \quad (5.6)$$

$P = \mathbb{Z}^4_+(2) \setminus S$.

Proof: We note that if $|\beta| \ge 2$ we have

$$\min\{|\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\mu^{\beta} - \lambda_1|, |\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\mu^{\beta} - \lambda_2|, |\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\mu^{\beta} - \mu|\} \ge |\mu|(1 - |\mu|) \quad (5.7)$$

for all $\alpha_1, \alpha_2 \in \mathbb{Z}$. If $\beta = 1$ we have

$$\min\{|\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\mu - \lambda_1|, |\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\mu - \lambda_2|\} \ge 1 - |\mu|, \qquad (\alpha_1, \alpha_2) \in \mathbb{Z}^2_+(1) \quad (5.8)$$

while $|\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\mu - \mu| = |\mu||\lambda_1^{\alpha_1}\lambda_2^{\alpha_2} - 1|$ may become small. Finally for $\beta = 0$

$$|\lambda_1^{\alpha_1}\lambda_2^{\alpha_2} - \mu| \ge 1 - |\mu|, \qquad (\alpha_1, \alpha_2) \in \mathbb{Z}^2_+(2) \quad (5.9)$$

while $|\lambda_1^{\alpha_1}\lambda_2^{\alpha_2} - \lambda_1|$ $|\lambda_1^{\alpha_1}\lambda_2^{\alpha_2} - \lambda_2|$ may become small. The proof is complete. □

Before dealing with the part ii) of Theorem 1 we introduce a notion which will clarify why inhomogeneous conditions occur in the near-resonance case.

Definition 5.2 *We say that $(\lambda_1, \lambda_2, \mu) \in \mathbb{C}^3$, where $|\lambda_j| \le 1$ for $j = 1, 2$ and $|\mu| < 1$, is uniformly nonresonant if: i) $(\lambda_1, \lambda_2, \mu)$ is a nonresonant; ii) $|\lambda_k|^j \ne |\mu|$ and $|\lambda_k| \ne |\mu|^j$ for every $j \in \mathbb{N}$, $k = 1, 2$.*

We note that in Proposition 5.1 $(\lambda_1, \lambda_2, \mu)$ is uniformly nonresonant. Moreover, by comparison with the definition of near-resonance we see that $(\lambda_1, \lambda_2, \mu)$ satisfying $|\lambda_1| = 1$, $0 < |\lambda_2|, |\mu| < 1$ is either uniformly nonresonant or near-resonant. In particular, fixing μ, we obtain a geometrical interpretation of the near-resonance. Namely, it occurs iff λ_2 lies on one of the circles $\{z \in \mathbb{C} \mid |z| = |\mu|^j\}$, $j \in \mathbb{N}$ or μ lies on one of the circles $\{z \in \mathbb{C} \mid |z| = |\lambda_2|^j\}$, $j \in \mathbb{N}$.

We have

Proposition 5.3 *Let $(\lambda_1, \lambda_2, \mu) \in \mathbb{C}^3$ be uniformly nonresonant with $|\lambda_1| = 1, |\lambda_2| < 1, |\mu| < 1$. Then there is an $\mathcal{L}$-partition $S \cup P$ as in (5.6). Moreover, $(\lambda_1, \lambda_2, \mu, \mu)$ satisfies $(\mathcal{BR}_2)$ iff (λ_1, λ_2) does (i.e., we have a splitting).*

Proof. Set $\lambda = (\lambda_1, \lambda_2)$. If $|\mu| < |\lambda_2|$, we obtain

$$\begin{aligned} \mathcal{S}^{\lambda,\mu}_{\alpha,\beta} &:= \min\{|\lambda^\alpha \mu^\beta - \lambda_1|, |\lambda^\alpha \mu^\beta - \lambda_2|, |\lambda^\alpha \mu^\beta - \mu|\} \\ &\geq \min\{|\mu|(1 - |\mu|), |\lambda_2| - |\mu|\}, \qquad \alpha \in \mathbb{Z}^2_+, \beta \in \mathbb{Z}_+(2). \end{aligned} \tag{5.10}$$

Let now $|\mu| > |\lambda_2|$. Choose $\beta_0 \in \mathbb{Z}_+(2)$ to be the smallest integer exceeding $\ln |\lambda_2| / \ln |\mu|$ (the uniform nonresonance excludes equality). One gets readily that

$$\mathcal{S}^{\lambda,\mu}_{\alpha,\beta} \geq \min\{|\mu| - |\mu|^{\beta_0}, |\lambda_2| - |\mu|^{\beta_0}\} > 0 \tag{5.11}$$

for all $\alpha \in \mathbb{Z}^2_+$, $\beta \in \mathbb{Z}_+(\beta_0)$. If $\beta_0 \geq 3$, in view of the uniform nonresonance and the choice of β_0 we obtain

$$\mathcal{S}^{\lambda,\mu}_{\alpha,\beta} \geq |\mu|^\beta - |\lambda_2| > 0, \qquad (\alpha_1, \alpha_2) \in \mathbb{Z}^2_+, \beta = 2, \ldots, \beta_0 - 1. \tag{5.12}$$

On the other hand, if $\beta = 1$,

$$\min\{|\lambda_1^{\alpha_1} \lambda_2^{\alpha} \mu - \lambda_1|, |\lambda_1^{\alpha_1} \lambda_2^{\alpha} \mu - \lambda_2|\} \geq \min\{|\mu| - |\lambda_2|, |\lambda_2| - |\mu|\} > 0, \tag{5.13}$$

for all $(\alpha_1, \alpha_2) \in \mathbb{Z}^2_+(1)$ while the only small divisors may appear in $|\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \mu - \mu| = |\mu||\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} - 1|$. In a similar way we study the case $\beta = 0$, which concludes the proof. $\square$

Now we deal with the presence of near–resonances.

In fact, the same statements are true for a nonresonant Siegel part of any finite dimension when there is one two-dimensional Jordan block. For a Jordan block of dimension m, $S = \{(\alpha_S, \alpha_J) \mid |\alpha_J| < m\}$. To clarify the other cases we need the following:

Proposition 5.4 *Let $(\lambda_1, \lambda_2, \mu)$ be as in iii) of Theorem 1.7 (near-resonant spectrum). Then there is an $\mathcal{L}$-partition $S \cup P$, of one of the following types:*

(i) $S = \{\alpha = (\alpha_P, \alpha_S, \alpha_J) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}^2_+ \mid \alpha_P = 0, |\alpha_J| < 2\} \bigcup \{\alpha = (\alpha_P, \alpha_S, \alpha_J) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}^2_+ \mid \alpha_P = k, |\alpha_J| = 0\}$.

(ii) $\{\alpha=(\alpha_P,\alpha_S,\alpha_J)\in\mathbb{Z}_+\times\mathbb{Z}_+\times\mathbb{Z}_+^2 \mid \alpha_P=0,|\alpha_J|<2\}\bigcup\{\alpha=(\alpha_P,\alpha_S,\alpha_J)\in\mathbb{Z}_+\times\mathbb{Z}_+\times\mathbb{Z}_+^2 \mid \alpha_P=0,(\alpha_J)=(\ell-r,r)\ \mathit{for}\ r=0\ldots\ell\}$.

Proof: In the first case the Siegel partition contains a new direction that is subject to a new inhomogeneous Diophantine restriction. Let $\nu=\frac{\lambda_2^t}{\mu}$. We need $\tilde{\omega}(\nu,\lambda_1)=|\nu\lambda_1^s-\lambda_1|>e^{\frac{s}{\psi(s)}}$ in addition to the original Diophantine condition on λ_1 to show convergence. In the second case a similar condition can be defined. Note that for (ii) there is also interaction between the Jordan structure, the remainder terms $\mathcal{R}$ and this new condition. Since there is a bounded number of dependencies among the Siegel type coefficients, the accumulated effect of the Siegel estimates can be controlled. More precisely, if the near-resonance is of order t and the small divisors are bounded by $\Gamma(\delta,1)$, then after the accumulation of small divisors from the remainder term the worst case is bounded by $\Gamma(\delta,t)$. $\square$

6 Convergence estimates

The solution of the homological equation can be used to define (inductively) a sequence of formal transformations which in the limit conjugate Φ with its linear part. We have already estimated the solutions to the homological equation in the presence of nontrivial Jordan blocks in $\Phi'(0)$, and this is the main difficulty. In this section we indicate estimates which show that the limit transformation converges following the standard KAM type approach (e.g., see [21], [13], [8], [7]). If $g=O(2^k)$ and v satisfies the homological equation $\mathcal{L}v=g$ then writing $u=Id+v$ and $f=\Lambda+g$ one computes that the nonlinear part of $f_+=u^{-1}\circ f\circ u$ is of order 2^{k+1}. We can perform a domain shrinking KAM argument on a sequence of polydisks of radii T_k where $T_{k+1}=T_ke^{-\delta_k}$ for a suitably chosen positive, decreasing, summable sequence δ_k. Recall (2.14)

$$\Gamma(\delta,q)=\sum_{n\geq 2}e^{\frac{qn}{\psi(n)}}e^{-n\delta}.$$

For a multiplication operator $\mathcal{A}$ of constant type whose multipliers, a_α, satisfy

$$|a_\alpha|\leq e^{\frac{|\alpha|}{\psi(|\alpha|)}},$$

one can show that

$$\|\mathcal{A}u\|_{Te^{-\delta}}\leq\sum e^{\frac{|\alpha|}{\psi(|\alpha|)}}\|u\|_T T^{-|\alpha|}(Te^{-\delta})^{|\alpha|}\leq\Gamma(\delta,1)\|u\|_T.$$

In our situation we have instead $\mathcal{L}_\alpha=\Omega(\alpha)-\epsilon N+\mathcal{R}_\alpha^\varepsilon$ where $\mathcal{R}^\varepsilon$ captures the interaction terms caused by the Jordan block. On the slice $|\alpha|=M$ we estimate $u_\alpha=(\Omega(\alpha)-\epsilon N)^{-1}g_\alpha+(\Omega(\alpha)-\epsilon N)^{-1}\mathcal{R}_\alpha^\varepsilon$.

Let m denote the algebraic multiplicity of the Jordan block. The degenerate case (algebraic multiplicity exceeding the geometric multiplicity) slightly improves the estimates. One checks that on the Siegel part

$$\|\mathcal{L}_S^{-1}u\|_{Te^{-\delta}} \leq c(\Gamma(\delta,2))\|u\|_T$$

holds for the case $n=3$ and that for $n \geq 4$, $\|\mathcal{L}_S^{-1}u\|_{Te^{-\delta}} \leq c(\Gamma(\delta,t))\|u\|_T$ where $t=m$ is the algebraic multiplicity or (if there is a near-resonance of the form $|\mu^k| = |\lambda|$) $t = \max\{m,k\}$.

Let $\Phi_0 = \Phi, u_0 = Id$, u_n be the solution of the n^{th} homological equation and inductively define $\Phi_{n+1} = u_{n+1}^{-1} \circ \Phi \circ u_{n+1}$.

By applying the arguments from [8], [13], or [21] one can show that the nonlinear part of Φ satisfies

$$\|\Phi_{n+1} - \Lambda^{\varepsilon}\| \leq c\Gamma(\delta,2t)\epsilon_n^2 \tag{6.1}$$

where $\epsilon_n = \|\Phi_n' - \Lambda^{\varepsilon}\|$ and c is constant. Thus for all $n = 0, 1, \ldots$ we obtain the recursion relation for ϵ_n,

$$\epsilon_{n+1} \leq \frac{c}{\delta_n}\Gamma(\delta_n, 2t)\epsilon_n^2.$$

With ϵ_0 small enough these estimates which are quadratic in the previous error term lead to superexponential convergence of the right hand sides and convergence of the conjugacy.

7 Concluding remarks and open problems

It seems natural to study commuting holomorphic maps with linear parts admitting nontrivial Jordan blocks (for recent results on simultaneous normal forms of commuting maps (respectively, vector fields) see [9], [13], [16], [18] (respectively [3], [25], [29])) and the references therein.

We have to investigate the simultaneous Jordan normal forms. The problem turns out to be fairly complicated. Unlike the case of commuting semisimple matrices (e.g., see [11], [13], [25]) where we can always find a nonsingular matrix which conjugates simultaneously the given matrices to diagonal ones, here complicated centralizers occur (see F.R. Gantmacher [12]). Here is an example, which explains the impossibility to reduce simultaneously to Jordan normal forms.

Example 7.1 *Let*

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & \varepsilon \\ 0 & 0 & \lambda \end{pmatrix} \quad B = \begin{pmatrix} p & 0 & q \\ r & s & t \\ 0 & 0 & s \end{pmatrix}, \tag{7.1}$$

where $\lambda, \varepsilon, p, q, r, s, t \in \mathbb{C} \setminus 0$. Then A and B commute but it is impossible, in general, to reduce B to the Jordan block structure of A, preserving that of A. We point out that the problem of finding conditions guaranteeing simultaneous reduction of commuting matrices to Jordan normal forms is related to questions in the Lie group theory (e.g., see Chapter IV, [6], and the references therein).

A more detailed study of commuting maps admitting nontrivial Jordan will be carried out in a work in progress.

Acknowledgements. Support from the CIME Italy, National Science Foundation, University of Cagliari, Italy, and University of North Texas made direct contact possible and enabled this work to be carried out. The authors appreciate the encouragement of Profs. Rafael de la Llave and Ricardo Pérez-Marco. We also thank Prof. Michael Herman for illuminating discussions regarding Bruno and related arithmetic conditions. The excellent working conditions at the University of Missouri, Columbia facilitated the completion of our paper. In particular we thank Prof. Stamatis Dostoglou for the invitation and for his assistance.

References

[1] V. I. Arnold: Geometrical Methods in the Theory of Ordinary Differential Equations. Springer, New York - Heidelberg - Berlin, 1983

[2] A.D. Bruno: The analytic form of differential equations, *Tr. Mosk. Mat. O-va* **25**, 119-262 (1971) and **26**, 199-239 (1972) (in Russian); see also Trans. Mosc. Math. Soc. **25**, 131-288 (1971) and **26**, 199-239 (1972)

[3] A.D. Bruno and S. Walcher: Symmetries and convergence of normalizing transformations, J. Math. Anal. Appl. **183**, 571–576 (1994)

[4] T. Carletti and S. Marmi: Linearization of analytic and non-analytic germs of diffeomorphisms of $(C, 0)$, Bull. Soc. Math. France, **128**, 69–85 (2000)

[5] K.T. Chen: Diffeomorphisms: $C^\infty-$realizations of formal properties, Amer. J. Math. **87**, 140–157 (1965)

[6] G. Cicogna and G. Gaeta: Symmetry and perturbation theory in nonlinear dynamics. Lecture Notes in Physics. New Series m: Monographs, **57**. Springer–Verlag, Berlin, 1999

[7] R. De La Llave: A tutorial on KAM theory. to appear in Proc. of 1999 Summer Research Institute on Smooth Ergodic Theory and Applications, July 26–Augsut 13, 1999, Univ. of Washington, Seattle.

[8] D. DeLatte: On normal forms in Hamiltonian dynamics, a new approach to some convergence questions. Ergod. Th. & Dynam. Sys. **15**, 49-66 (1995)

[9] D. DeLatte: Diophantine conditions for the linearization of commuting holomorphic functions. Discr. & Cont. Dynam. Sys. **3**, 317-332 (1997)

[10] D. Dickinson, T. Gramchev and M. Yoshino: Perturbations of vector fields on tori: resonant normal forms and Diophantine phenomena. *Mathematical Physics Preprint Archive mp_arc* $00-11$, Jan 10, 00 (2000), to appear in Proc. Edinb. Math. Soc.

[11] J. Ecalle and B. Vallet: Correction and linearization of resonant vector fields and diffeomorphisms. Math. Z. **229**, 249-318 (1998)

[12] F.R. Gantmacher: The theory of matrices, Vols. 1, 2. Chelsea Publishing Co., New York, 1959

[13] T. Gramchev and M. Yoshino: Rapidly convergent iteration method for simultaneous normal forms of commuting maps. Math. Z., **231**, 745–770 (1999)

[14] M. Herman: Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of $\mathbb{C}^n$ near a fixed point. Proc. of the VIIIth International Congress on Mathematical Physics, Marseille, 1986, World Scientific Publ., Singapore, 138-184 (1987)

[15] Y. Il'yashenko: Divergence of series reducing an analytic differential equation to linear form at a singular point. Funct. Anal. and Appl. **13**, 227-229 (1979)

[16] B. Kra: The conjugating map for commuting groups of circle diffeomorphisms. Israel Jour. Math. **100**, 303-316 (1996)

[17] J. Levesley: A general inhomogeneous Jarník–Besicovitch theorem. J. Number Theory, **71**, 65–80 (1998)

[18] J. Moser: On commuting circle mappings and simultaneous Diophantine approximations. Math. Z. **205**, 105-121 (1990)

[19] R. Pérez Marco: Non linearizable holomorphic dynamics having an uncountable number of symmetries. Inv. Math. **119**, 67-127 (1995)

[20] J. Pöschel: Small divisors with spatial structures in infinite dimensional hamiltonian systems. Comm. in Math. Physics **127**, 351-393 (1990)

[21] H. Rüssmann: On the one-dimensional Schrödinger equation with a periodic potential. Ann. NY Acad. Sci. **357**, 90 - 107 (1980)

[22] W. Schmidt: Diophantine approximation. *Lect. Notes in Mathematics*, **785**, Springer Verlag, Berlin - Heidelberg - New York 1980

[23] C. L. Siegel: Iteration of analytic functions. Ann. Math. **43**, 607-614 (1942)

[24] S. Sternberg: The structure of local homeomorphisms II, III, Amer. J. Math **80**, 623-632 and **81**, 578-604 (1958)

[25] L. Stolovitch: Singular complete integrability. Publ. Math. I.H.E.S., **91**, 134-210 (2000)

[26] J.-C. Yoccoz: Linéarisation des germes de difféomorphismes holomorphes de $(\mathbb{C}, 0)$. C. R. Acad. Sci. Paris Sér. I Math., **306**, 55–58 (1988)

[27] J.-C. Yoccoz: Recent development in dynamics, Proc. of the International Congress of Mathematicians, Zürich, Switzerland, August 1994, Birkhäuser, 246-265 (1995)

[28] J.-C. Yoccoz: A remark on Siegel's theorem for nondiagonalizable linear part, manuscript, 1978, see also Théorème de Siegel, nombres de Bruno e polynômes quadratic, Astérisque **231**, 3-88 (1995)

[29] M. Yoshino: Simultaneous normal forms of commuting maps and vector fields. A. Degasperis, G. Gaeta eds., Symmetry and perturbation theory SPT 98, Rome 16–22 December 1998, World Scientific, Singapore, 287–294 (1999)

[30] E. Zehnder: A simple proof of a generalization of a theorem by C. L. Siegel. Lect. Notes in Math. **597**, 855-866 (1977)