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Degenerate space-time paths and the non-locality of quantum mechanics in a Clifford substructure of space-time

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Abstract

The quantized canonical space-time coordinates of a relativistic point particle are expressed in terms of the elements of a complex Clifford algebra which combines the complex properties of $SL(2.C)$ and quantum mechanics. When the quantum measurement principle is adapted to the generating space of the Clifford algebra we find that the transition probabilities for twofold degenerate paths in space-time equal the transition amplitudes for the underlying paths in Clifford space. This property is used to show that the apparent non-locality of quantum mechanics in a double slit experiment and in an EPR type of measurement is resolved when analyzed in terms of the full paths in the underlying Clifford space. We comment on the relationship of this model to the time symmetric formulation of quantum mechanics and to the Wheeler-Feynman model.

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1 Substructure of the canonical space-time coordinates

The fact that half-integer spin representations of the Lorentz group are realized in nature casts doubt on the assumption that space-time is a primary space. More specifically, as pointed out by Penrose [1], the fact that different spatial directions of a spin-one-half particle correspond to different complex linear combinations of the two quantum states suggests that there is a direct connection between the structure of space and the need for complex state vectors in quantum mechanics. Taken together, considerations like these point to the existence of a substructure of space-time which combines the complex properties of the Lorentz group and quantum mechanics. Substructures of space-time have been discussed in Schwartz and Van Nieuwenhuizen [2] and in Borchsenius [3, 4, 5].

To determine the nature of such a complex substructure of space-time we shall use the canonical quantization of a relativistic point particle as a model. We shall adopt Dirac's method in which space and time are treated on an equal footing, both being regarded as functions of a parameter-time τ. Reparametrization invariance imposes a constraint which can be used to define a Hamiltonian together with a set of canonical variables. The quantization results in a set of hermitian canonical space-time coordinates, the components of which satisfy

$$X^{\mu *}_{ab}(\tau) = X^{\mu}_{ba}(\tau) \tag{1}$$

These components transform under a Lorentz transformation in the index μ and under a unitary change of basis in Hilbert space in the indices a and b. To bring out the complex properties of the Lorentz group, we make use of the connection between a real four-vector and a second-rank hermitian spinor

$$V^{\mu} = \frac{1}{2}\sigma^{\mu}_{\ A\dot{B}} V^{A\dot{B}}, \qquad V^{A\dot{B}} = \sigma^{A\dot{B}}_{\mu} V^{\mu} \tag{2}$$

where σ_μ are the four hermitian Pauli matrices. The spinor form of the canonical space-time coordinates

$$X^{A\dot{B}}_{ab} \stackrel{\text{def}}{=} \sigma^{A\dot{B}}_{\mu} X^{\mu}_{ab} \tag{3}$$

exhibits two hermitian properties, one related to $SL(2.C)$ and the other to the unitary group in Hilbert space. To find a substructure of X corresponding to these two groups, we observe that the components (3) form a hermitian matrix in the combined indices (A, a) and (B, b)

$$\left(X^{A\dot{B}}_{ab}\right)^{*} = X^{B\dot{A}}_{ba} \tag{4}$$

As shown in the appendix, any hermitian matrix can be expressed in terms of the elements of a complex Clifford algebra according to (66). For the canonical space-time coordinates (4) this implies that there exists a complex Clifford algebra with elements C^{A}_{a} so that

$$X^{A\dot{B}}_{ab} = \{C^{A}_{a}, C^{*\dot{B}}_{b}\}, \qquad \{C^{A}_{a}, C^{B}_{b}\} = 0 \tag{5}$$

The complex linear space which generates the Clifford algebra, and to which the C's belong, we shall call Clifford space, and we shall refer to its elements as Clifford coordinates, borrowing from space-time terminology. To write (5) in abstract form we shall adopt the following notation. The components C_a^A which transform like a right-handed two-component spinor in the index A and as a ket vector in the index a shall be written as $\overset{>}{C}{}^A$ where the ket on top is used to distinguish it from a quantum operator and an ordinary eigenvector. Likewise $C_b^{*\dot{B}}$ will be written as the bra vector $\overset{<}{C}{}^{\dot{B}} = (\overset{>}{C}{}^B)^\dagger$ where $\dagger$ performs both the complex involution of the Clifford algebra and the quantum conjugation in Hilbert space. The commutator between a ket vector $\overset{>}{\chi}$ and a bra vector $\overset{<}{\psi}$ shall be defined as

$$\{\overset{>}{\chi}, \overset{<}{\psi}\}_{ab} \overset{\text{def}}{=} \{\chi_a, \psi_b\}, \qquad \{\overset{<}{\psi}, \overset{>}{\chi}\} \overset{\text{def}}{=} \{\psi_a, \chi_a\} \tag{6}$$

that is, we adopt the convention that the order of the ket and bra vectors in the first term in the commutator determines whether *both* terms are direct products or contractions. With this notation, (5) can be written in the abstract form

$$X^{A\dot{B}} = \{\overset{>}{C}{}^A, \overset{<}{C}{}^{\dot{B}}\}, \qquad \{\overset{>}{C}{}^A, \overset{>}{C}{}^B\} = 0 \tag{7}$$

X and $\overset{>}{C}$ can be expressed in terms of a complete set of eigenstates $|x_r\rangle$ and their eigenvalues

$$X^\mu = |x_r^\mu\rangle x_r^\mu \langle x_r^\mu| \tag{8}$$

$$\overset{>}{C}{}^A = |x_r^\mu\rangle c_r^A, \qquad c_r^A \overset{\text{def}}{=} \langle x_r^\mu|\, \overset{>}{C}{}^A \tag{9}$$

When these expressions are inserted into (7) we obtain

$$\{c_r^A, c_s^{*\dot{B}}\} = \delta_{rs} x_s^{A\dot{B}}, \qquad \{c_r^A, c_s^B\} = 0 \tag{10}$$

Hence the eigenvalues of X are determined by a set of mutually orthogonal elements c_r^A of the Clifford algebra. To make our discussion more transparent we shall refer to these elements as 'eigenvalues' and write the eigenstates $|x_r\rangle$ as $|c_r\rangle$. By use of (7) we obtain the expression for the expectation value of X in the state $|s\rangle$

$$\langle s|X^{A\dot{B}}|s\rangle = \langle s|\{\overset{>}{C}{}^A, \overset{<}{C}{}^{\dot{B}}\}|s\rangle = \{\bar{c}^A, \bar{c}^{*\dot{B}}\} \tag{11}$$

$$\bar{c}^A \overset{\text{def}}{=} \langle s|\, \overset{>}{C}{}^A \tag{12}$$

(12) are the Clifford coordinates corresponding to the expectation value of the space-time coordinates. Applying (9) they become

$$\bar{c}^A = \langle s|x_r\rangle c_r^A \tag{13}$$

The relationship of this equation to the expression for the expectation value of the space-time coordinates

$$\bar{x}^{\mu} = |\langle s|x_r\rangle|^2 x_r^{\mu} \tag{14}$$

can be described as a linear extraction of the quantum amplitudes as a complex substructure of the probabilities, and of the Clifford coordinates as a complex substructure of the space-time coordinates. If, conversely, we had sought a substructure of space-time which had the quantum amplitudes as a linear space of weights as in (13), we would have been led to something of the nature of the orthogonality relations (10).

In the continuum limit X has a Continuous spectrum and in the coordinate representation (9) and (10) become

$$\overset{>}{C}{}^{A} = \int |x\rangle c^A(x)\, dx \tag{15}$$

$$\{c^A(x), c^{*\dot{B}}(x')\} = x^{A\dot{B}}\delta(x - x'), \qquad \{c^A(x), c^B(x')\} = 0 \tag{16}$$

(16) generates an infinite dimensional Clifford Algebra of a type well known from the Algebra of creation and annihilation operators for a Fermi field.

The stability of Clifford space under $SL(2.C)$ implies that there are at least two values c and $-c$ of the Clifford coordinates,which correspond to the same space-time coordinates x. The well known degeneracy of $SO(1.3)$ transformations with respect to $SL(2.C)$ transformations is hereby extended to space-time itself. As we shall see in sections 3 and 4, this has the consequence that the path of a particle in Clifford space has a starting point in physical time. This is the only physical consequence of the Clifford model which differs from those of conventional quantum mechanics. To reconcile these starting times with experience in a satisfactory way, the Clifford substructure would presumably have to be applied to objects more fundamental than point particles (e.g. fields or strings) and examined in the context of a cosmological model. The viability of our model rests upon the possible success of such a program.

2 Canonical equations

We consider the action:

$$\int L(c(\tau), \dot{c}(\tau))\, d\tau \tag{17}$$

Since the Lagrangian is real-valued it is natural to assume that the Clifford variables c and $\dot{c}$ occur within anticommutators. In this case the variation of L can be expressed as

$$\delta L = \{\frac{\partial L}{\partial c^A}, \delta c^A\} + c.c. + \{\frac{\partial L}{\partial \dot{c}^A}, \delta \dot{c}^A\} + c.c. \tag{18}$$

which defines the derivatives with respect to c and $\dot{c}$ up to terms which anticommute with δc. The conjugate to c is defined as

$$d^*_A = \frac{\partial L}{\partial \dot{c}^A} \tag{19}$$

If $\dot{c}$ can be eliminated in favour of d^* the Hamiltonian becomes

$$H(c,d) = \{\dot{c}^A, d^*_A\} + c.c. - L(c,\dot{c}) \tag{20}$$

with the equations of motion

$$\dot{c}^A = \frac{\partial H}{\partial d^*_A}, \qquad \dot{d}^*_A = -\frac{\partial H}{\partial c^A} \tag{21}$$

In case the action (17) has local symmetries the Hamiltonian is found by the methods of constrained dynamics.

We shall only consider Hamiltonians which can be expressed in the form

$$H(c,d) = H(x,p), \quad x^{A\dot{B}} = \{c^A, c^{*\dot{B}}\}, \quad p_{A\dot{B}} = \{d^*_A, d_{\dot{B}}\} \tag{22}$$

The system corresponding to the action (17) cannot be quantized in the usual way through Poisson brackets because c and d^* become vectors $\overset{>}{C}$ and $\overset{<}{D}$ and not operators in Hilbert space. Instead we shall determine the conditions which have to be imposed on $\overset{>}{C}$ and $\overset{<}{D}$ in order to obtain the usual canonical quantization of the system (22) with p as the momenta conjugate to x. For the Hamiltonian (22) the equations of motion (21) become

$$\dot{c}^A = \frac{\partial H}{\partial p_{A\dot{E}}} d_{\dot{E}}, \quad \dot{d}^*_A = -c^{*\dot{E}} \frac{\partial H}{\partial x^{A\dot{E}}} \tag{23}$$

The quantized form of these equations will be

$$\frac{d}{d\tau}\overset{>}{C}{}^A = -\frac{1}{2i\hbar}[H, X^{A\dot{E}}]\,\overset{>}{D}_{\dot{E}}, \qquad \frac{d}{d\tau}\overset{<}{D}_A = -\frac{1}{2i\hbar}\,\overset{<}{C}{}^{\dot{E}}\,[H, P_{A\dot{E}}] \tag{24}$$

$$X^{A\dot{B}} = \{\overset{>}{C}{}^A, \overset{<}{C}{}^{\dot{B}}\}, \qquad P_{A\dot{B}} = \{\overset{>}{D}_{\dot{B}}, \overset{<}{D}_A\} \tag{25}$$

Applying the equations of motion (24) to (25) gives

$$\begin{aligned}
\frac{d}{d\tau}X^{A\dot{B}} &= -\frac{1}{2i\hbar}[H, X^{A\dot{E}}]\{\overset{>}{D}_{\dot{E}}, \overset{<}{C}{}^{\dot{B}}\} - \frac{1}{2i\hbar}\{\overset{>}{C}{}^A, \overset{<}{D}_E\}[H, X^{\dot{B}E}] \\
\frac{d}{d\tau}P_{A\dot{B}} &= -\frac{1}{2i\hbar}[H, P_{E\dot{B}}]\{\overset{>}{C}{}^E, \overset{<}{D}_A\} - \frac{1}{2i\hbar}\{\overset{>}{D}_{\dot{B}}, \overset{<}{C}{}^{\dot{E}}\}[H, P_{A\dot{E}}]
\end{aligned} \tag{26}$$

For these equations to reduce to the usual space-time canonical equations of motion we must impose the commutation relations

$$\{\overset{>}{C}{}^A, \overset{<}{D}_B\} = \delta^A_B \mu(\tau) \tag{27}$$

where $\mu(\tau)$ is a real scalar function of τ. Then (26) becomes

$$\frac{d}{d\tau}X^{A\dot{B}} = -\frac{\mu(\tau)}{i\hbar}[H, X^{A\dot{B}}], \quad \frac{d}{d\tau}P_{A\dot{B}} = -\frac{\mu(\tau)}{i\hbar}[H, P_{A\dot{B}}] \tag{28}$$

or in reparametrized form

$$\frac{d}{d\bar{\tau}}X^{A\dot{B}} = -\frac{1}{i\hbar}[H, X^{A\dot{B}}], \quad \frac{d}{d\bar{\tau}}P_{A\dot{B}} = -\frac{1}{i\hbar}[H, P_{A\dot{B}}], \quad \frac{d\bar{\tau}}{d\tau} = \mu(\tau) \tag{29}$$

Though we obtain the standard space-time canonical equations of motion, they are subject to the (as we shall see) important restriction that the parameter $\bar{\tau}$ is only well defined for $\mu(\tau) \neq 0$.

Normally the compatibility of the commutation relations with the equations of motion is ensured by the Poisson brackets. This also applies in the present case to the space-time commutation relations

$$[X^\mu, X^\nu] = 0, \quad [P_\mu, P_\nu] = 0, \quad [X^\mu, P_\nu] = i\hbar\delta^\mu_\nu \tag{30}$$

which are compatible with the equations of motion (28). These equations, however, assume the validity of the Clifford commutation relations (27) which are not related to any poisson brackets. We shall prove the compatibility of these commutation relations with the equations of motion in the classical case where they reduce to

$$\{c_A, d^*_B\} = -\epsilon_{AB}\,\mu(\tau) \tag{31}$$

Since all skewsymmetric second rank spinors are proportional to ϵ_{AB}, (31) is equivalent to the vanishing of the symmetric part of the commutator

$$\{c_{(A}, d^*_{B)}\} = 0 \tag{32}$$

The equations of motion (21) subject to the constraint (32) can be obtained from

$$\int \{\dot{c}^A, d^*_A\} + c.c. - H(c,d) + \lambda^{AB}(\tau)\{c_{(A}, d^*_{B)}\} + c.c.\, d\tau, \quad \lambda^{AB} = \lambda^{BA} \tag{33}$$

by independent variation of c and d where λ^{AB} are six Lagrange multipliers. A local $SL(2.C)$ transformation

$$c_A = S_A^{\ E}(\tau)\bar{c}_E, \quad d^*_A = S_A^{\ E}(\tau)\bar{d}^*_E \tag{34}$$

turns (33) into

$$\int \{\dot{\bar{c}}^A, \bar{d}^*_A\} + c.c. - H(\bar{c}, \bar{d}) + (\dot{S}^{EA}S_E^{\ B} + \bar{\lambda}^{AB})\{\bar{c}_{(A}, \bar{d}^*_{B)}\} + c.c.\, d\tau \tag{35}$$

The last two terms in (35) can be made to vanish if

$$\dot{S}^{EA}(\tau)S_E^{\ B}(\tau) = -\bar{\lambda}^{AB}(\tau) \tag{36}$$

Taking λ to be small, the infinitesimal $SL(2.C)$ transformation

$$S^{AB}(\tau) = \epsilon^{AB} + \kappa^{AB}(\tau), \quad \kappa^{AB} = \kappa^{BA} \tag{37}$$

turns (36) into

$$\dot{\kappa}^{AB} = -\lambda^{AB} \tag{38}$$

which can always be solved for $\kappa^{AB}(\tau)$ in terms of $\lambda^{AB}(\tau)$. The constraint (32) can therefore be absorbed into a local $SL(2.C)$ transformation of the dynamical variables and will accordingly preserve the form of the equations of motion. In section 4 we shall examine a specific model of the relativistic point particle and find that also the quantum form of the Clifford commutation relations leads to a consistent result.

3 Degenerate space-time paths

When $\mu(\tau)$ in (27) has a zero, the parameter $\bar{\tau}$ of the space-time equations of motion is ill-defined. We should therefore be prepared to encounter complete solutions $C(\tau)$,$D(\tau)$ to the equations of motion (24) which generate incomplete solutions $X(\tau)$,$P(\tau)$ to the space-time equations of motion. To understand what happens, let us assume that $\mu(0) = 0$. Then for $\tau = 0$ the commutation relations (27) reduce to

$$\{\overset{>}{C}{}^A(0), \overset{<}{D}_B(0)\} = 0 \tag{39}$$

Let us expand $C(\tau)$

$$\overset{>}{C}(\tau) = \overset{>}{C}(0) + \cdots + \frac{1}{n!}\overset{>}{C}{}^{(n)}(0)\tau^n + \cdots \tag{40}$$

The higher order derivatives $C^{(n)}(\tau)$ are obtained by differentiating the equations of motion (24) and reinserting the expressions for $\dot{X}$ and $\dot{P}$ obtained from (26). Because of the commutation relations (39) this can only result in coefficients which contain terms of the form

$$F^A_B(X(0),P(0))\,\overset{>}{C}{}^B(0) \quad \text{or} \quad G^{A\dot{B}}(X(0),P(0))\,\overset{>}{D}_{\dot{B}}(0) \tag{41}$$

When $C(\tau)$,$D(\tau)$ is a solution to (24), so is $-C(-\tau)$,$D(\tau)$. Thus $C(\tau)$ must be odd under a change of sign of $C(0)$ and τ. It follows that in the expansion (40) all terms of even order must have coefficients of the first type in (41) and all terms of odd order must have coefficients of the second type. When therefore the expansion (40) is inserted into (5) to determine $X(\tau)$ we find that, because of the commutation relations (39), all anti-commutators between terms of odd order and terms of even order vanish. Accordingly $X(\tau)$ can only contain terms of even order and must therefore be an even function of τ:

$$X(-\tau) = X(\tau) \tag{42}$$

This implies that $C(\tau)$ reproduces $X(\tau)$ twice, making it twofold degenerate. Hence there exist complete paths in Clifford space which have either a beginning or an end in physical time. If we assume that it is the first possibility which applies, then the only way to avoid a contradiction with experience is to assume that the 'starting times' of the particles are of cosmological origin. The viability of the Clifford model therefore depends on the construction of a cosmological model which would presumably go beyond the framework of the quantum mechanics of point particles.

The classical paths will, like $X(\tau)$, be even functions of τ. In the quantum regime, however, paths for which $x(\tau) \neq x(-\tau)$ will also contribute to the transition amplitudes. Consequently the Clifford model will seem to be non-local from a space-time point of view. We shall interpret this non-locality in section 6.

4 The relativistic point particle

Since there exists no $SL(2.C)$ invariant hermitian second rank spinor, but only the real skewsymmetric metric ϵ_{AB}, the simplest reparametrization invariant action for a relativistic point particle which only depends on $\dot{c}$ is

$$-2^{\frac{7}{4}}\sqrt{m}\int \sqrt[4]{\{\dot{c}^A,\dot{c}^{*\dot{B}}\}\{\dot{c}_A,\dot{c}^*_{\dot{B}}\}}\, d\tau \tag{43}$$

The conjugate to c is

$$d^*_A = -2^{\frac{3}{4}}\sqrt{m}(\{\dot{c}^E,\dot{c}^{*\dot{F}}\}\{\dot{c}_E,\dot{c}^*_{\dot{F}}\})^{-\frac{3}{4}}\{\dot{c}_A,\dot{c}^*_{\dot{B}}\}\dot{c}^{*\dot{B}} \tag{44}$$

Not unexpectedly the Hamiltonian (20) vanishes because of reparametrization invariance. By use of the relation

$$V_{A\dot{F}}V^{B\dot{F}} = \delta^B_A V_\mu V^\mu \tag{45}$$

for a hermitian second rank spinor, we obtain from (44) the associated constraint

$$\frac{1}{2}\{d^*_A, d_{\dot{B}}\}\{d^{*A}, d^{\dot{B}}\} = m^2 \tag{46}$$

or

$$p_\mu p^\mu = m^2 \tag{47}$$

This is the same constraint as would have been obtained from the usual space-time Lagrangian $m\sqrt{\dot{x}^2}$, but with the important difference that p_μ is no longer a primary dynamical variable. The new Hamiltonian is proportional to the constraint:

$$H(\overset{>}{C},\overset{<}{D}) = \nu(\tau)(P_\mu P^\mu - m^2) \tag{48}$$

The gauge is fixed by choosing $\nu(\tau) = \frac{1}{2m}$. By use of the space-time commutation relations the equations of motion (24) become

$$\frac{d}{d\tau}\overset{>}{C}{}^A = \frac{1}{2m}P^{A\dot{E}}\overset{>}{D}_{\dot{E}}, \qquad \frac{d}{d\tau}\overset{<}{D}_A = 0 \tag{49}$$

with the solution

$$\overset{>}{C}{}^{A}(\tau) = \overset{>}{C}{}^{A}(0) + \frac{1}{2m} P^{A\dot{E}}(0)\, \overset{>}{D}_{\dot{E}}(0)\tau, \quad \overset{<}{D}_{A}(\tau) = \overset{<}{D}_{A}(0) \tag{50}$$

¿From (50) we obtain

$$\{\overset{>}{C}{}^{A}(\tau), \overset{<}{D}_{B}(\tau)\} = \{\overset{>}{C}{}^{A}(0), \overset{<}{D}_{B}(0)\} + \frac{1}{2m} P_{B\dot{E}}(0) P^{A\dot{E}}(0)\tau \tag{51}$$

Applying (45) and the quantum form of (47) to (51) it becomes

$$\{\overset{>}{C}{}^{A}(\tau), \overset{<}{D}_{B}(\tau)\} = \{\overset{>}{C}{}^{A}(0), \overset{<}{D}_{B}(0)\} + \delta^{A}_{B} \frac{m}{2}\tau \tag{52}$$

Accordingly, the Clifford commutation relations (27) are preserved in time by the equations of motion, and with the choice $\tau = 0$ for the zero-point of $\mu(\tau)$ we obtain

$$\mu(\tau) = \frac{m}{2}\tau, \quad \bar{\tau} = \frac{m}{4}\tau^2 \tag{53}$$

The corresponding space-time solution is

$$X^{\mu}(\bar{\tau}) = X^{\mu}(0) + \frac{1}{m} P^{\mu}(0)\bar{\tau}, \quad P_{\mu}(\bar{\tau}) = P_{\mu}(0) \tag{54}$$

In accordance with the general result in section 3, the complete solutions to the Clifford equations of motion (49) are double coverings of the incomplete solutions $X(\bar{\tau}), \bar{\tau} \geq 0$ to the space-time equations of motion.

5 Measurement principle

The measurement principle in quantum mechanics says that the (abstract) state vector is constant in time as long as no measurement is being performed. After a measurement has been performed the state vector is replaced by the eigenvector of the measured quantity for subsequent times ('state vector reduction'). This measurement principle applies equally well to Dirac's parameter-time formalism when 'time' is taken to be a parameter-time with the same direction as our $\bar{\tau}$ in the foregoing. Recognizing the primary character of Clifford space, we shall instead assume that the reduction of the state vector takes place in the positive direction of τ itself which therefore comes to represent the true direction of causality:

> Measurement Principle. *The state vector of the particle is constant in parameter-time τ as long as no measurement is being performed. When the particle is measured to be in the eigenstate $|x_P\rangle$ of X the state vector is replaced by $|c_P\rangle = |x_P\rangle$ for parameter-times $\tau > \tau_P$ where c_P is an 'eigenvalue' of $\overset{>}{C}(\tau_P)$ and c_P and $\overset{>}{C}(\tau_P)$ satisfy $\{c_P, c_P^*\} = x_P$ and $\{\overset{>}{C}(\tau_P), \overset{<}{C}(\tau_P)\} = X$ respectively*

Using a convenient terminology we shall say that the Clifford position of the particle has been measured to be c_P at $\tau = \tau_P$. The measurement principle respects the fact that since the interaction-Hamiltonians used for measuring space-time positions depend only on C *through* X, the state vector reduction in Clifford space should also be defined trough X and its eigenvalues. In the following we shall examine the consequences of this principle.

Let the space-time position of the particle have been measured to be x_Q. From (42) and (10) it follows that $C(\tau)$ satisfies the criteria in the measurement principle at two parameter-times $\tau = \pm\tau_Q$. Let us call the corresponding 'eigenvalues' for c_{Q+} and c_{Q-}. Hence the state vector will be $|c_{Q-}\rangle$ and $|c_{Q+}\rangle$ (both equal to $|x_Q\rangle$) right after $\tau = -\tau_Q$ and $\tau = \tau_Q$ respectively. If no measurement is being performed between $\tau = -\tau_Q$ and $\tau = \tau_Q$ the particle will arrive at $\tau = \tau_Q$ in the state $|c_{Q-}\rangle$. Since after $\tau = \tau_Q$ the state is $|c_{Q+}\rangle$, the transition amplitude is $\langle c_{Q-}|c_{Q+}\rangle = \langle x_Q|x_Q\rangle = 1$. Therefore the measurement principle is self-consistent as long as no measurement is being performed between $\tau = -\tau_Q$ and $\tau = \tau_Q$. Let us now assume that such a measurement *is* being performed, resulting in the space-time position x_P corresponding to the Clifford positions $c_{P\pm}$ at $\tau = \pm\tau_P$ respectively, where $\tau_P < \tau_Q$. The transition amplitude for the particle to pass from c_{Q-} through c_{P-} and c_{P+} to c_{Q+} is

$$\langle c_{Q-}|c_{P-}\rangle\langle c_{P+}|c_{Q+}\rangle = |\langle x_P|x_Q\rangle|^2 \tag{55}$$

and therefore equals the transition probability for the particle to move from x_P to x_Q. We conclude that the space-time transition probabilities arise as transition amplitudes for the complete paths in Clifford space.

Note that viewed from space-time it appears as if there are two amplitudes, one moving forward in time from x_P to x_Q and the other moving backwards in time from x_Q to x_P. This resembles the situation in the time symmetric formulation of quantum mechanics by Aharonov and Vaidman [6], Costa de Beauregard [7], and Werbos [8]. In the present model the two state vectors of time symmetric quantum mechanics are recognized to be one and the same, propagating along a path which covers the space-time path twice. The use of parameter-time in our model is necessitated by the secondary character of physical time, but it has the added advantage of ensuring manifest Lorentz invariance.

The present model should also be compared to the so-called 'double space-time interpretation of quantum mechanics' Bialynicki-Birula [9], inspired by Schwinger's time loop integrated amplitudes. The main problem in this interpretation is how to join the two space-time sheets at infinity to allow a particle to travel along a single path on the two sheets.

The choice of taking the causal direction of state vector reduction to be in the positive direction of parameter-time τ rather than of 'affine time' $\bar{\tau}$ strongly suggests that the same should apply to the direction of propagation of classical fields. The following heuristic observation shows that this is not necessarily inconsistent with experience. Let the union of all possible particle trajectories for $\tau \leq 0$ and for $\tau \geq 0$ form regions Ω_- and Ω_+ of Clifford space which correspond to the same space-time region. For the field to propagate in the positive direction of τ we should choose the advanced field on Ω_- and the retarded field on Ω_+. The contribution to the electrodynamic action in the proper-time interval $[\bar{\tau}_1;\bar{\tau}_2]$ of a test-particle with charge e traversing this region is

$$\frac{1}{2}\int_{-\tau_2}^{-\tau_1} m\sqrt{\dot{x}^2} + A_{adv}e(-\dot{x})\,d\tau + \frac{1}{2}\int_{\tau_1}^{\tau_2} m\sqrt{\dot{x}^2} + A_{ret}e\dot{x}\,d\tau$$
$$= \int_{\bar{\tau}_1}^{\bar{\tau}_2} m\sqrt{\dot{x}^2} + (\frac{1}{2}A_{adv} + \frac{1}{2}A_{ret})e\dot{x}\,d\bar{\tau} \qquad (56)$$

The test-particle will therefore detect the effective field to be the time symmetric half-advanced plus half-retarded field. Assuming complete absorption and no self-interaction Wheeler and Feynman [10] have shown that this time symmetric field leads to the conventional rules of electrodynamics.

6 Interpretation of non-locality

Consider a particle which travels in space-time from a point P to a point Q and is forced to travel trough two alternative points S_1 and S_2. This corresponds to the double slit experiment with the two slits being opened at given times. As follows from our foregoing discussion the particle can follow four alternative sets of paths in Clifford space corresponding to the the four sequences of positions in Clifford space ordered according to parameter time:

$$c_{Q-},\ c_{S_i-},\ c_{P-},\ c_{P+},\ c_{S_j+},\ c_{Q+},\ i,j=1,2 \qquad (57)$$

The amplitude for the particle to travel from c_{Q-} to c_{Q+} is the sum of the amplitudes for all four different sets of paths

$$\sum_{i,j=1}^{2} \langle c_{Q-}|c_{S_i-}\rangle\langle c_{S_i-}|c_{P-}\rangle\langle c_{P+}|c_{S_j+}\rangle\langle c_{S_j+}|c_{Q+}\rangle$$
$$= |\langle x_P|x_{S_1}\rangle\langle x_{S_1}|x_Q\rangle + \langle x_P|x_{S_2}\rangle\langle x_{S_2}|x_Q\rangle|^2 \qquad (58)$$

which is the well known probability for the particle to travel from P to Q. The customary interpretation of this transition probability is that there are two alternative paths and that the transition probability is the sum of the probabilities for each path plus two interference terms which seem to signal a non-local influence of one path on the other. From (58) we see that there are really four different sets of paths and that the two interference terms are the amplitudes for the two sets of paths where the particle goes through each slit at opposite parameter times. The apparent non-locality can be entirely attributed to the twofold degeneracy of the space-time paths.

If we measure the position of the particle at one of the slits, say S_1, then according to our measurement principle the particle has to travel through both

c_{S_1-} and c_{S_1+} or neither of them, and this excludes the two sets of paths where the particle passes through both slits. This removes the interference terms in accordance with the space-time view of quantum mechanics. This analysis is readily extended to a many-slit experiment by observing that all interference terms arise from pairs of slits.

As the second example of non-locality we shall consider an EPR type of measurement. Consider a composite system (PQ) consisting of two spin $\frac{1}{2}$ particles P and Q. First the position and the total spin of the composite system is measured to be $x_{(PQ)}$ and 0. After this measurement P and Q become separated by a spacelike distance and their position and spin along some axis are measured to be x_P and $\frac{1}{2}$ and x_Q and $-\frac{1}{2}$ respectively. The last two measurements appear to be correlated despite the spacelike separation of P and Q, giving thereby the impression of 'action at a distance'. However, according to our measurement principle, the position measurements, and together with them the spin measurements, each correspond to two measurements in Clifford space at opposite values of τ . If the measurements of $x_{(PQ)}$, x_P and x_Q correspond to parameter-times $\tau = \pm\tau_{(PQ)}$, $\tau = \pm\tau_P$ and $\tau = \pm\tau_Q$ respectively, then the sequence of events for negative τ can be described as follows. First at parameter-times $\tau = -\tau_P$ and $\tau = -\tau_Q$ the spins along some axis of P and Q are measured to be $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. At the later parameter-time $\tau = -\tau_{PQ} > -\tau_P, -\tau_Q$, P and Q merge into a composite system (PQ) and the total spin is measured to be 0 . We would not object to this last sequence of events because it suggests no correlation between the spacelike separated states of P and Q. Rather, it suggests an obvious correlation between the states of P and Q on the one hand and the state of the composite system (PQ) on the other, which invokes no need for 'action at a distance'. Nevertheless these two sequences of events, corresponding to opposite values of τ, together form a single series of events in the causal direction of state vector reduction in Clifford space and are both the result of the same space-time measurements on a degenerate space-time path. They are therefore on an equal footing and we conclude that it has no absolute meaning to say whether the spin-measurements on P and Q are correlated or independent. Accordingly, the apparent manifestation of 'action at a distance' loses its significance.

A Appendix. Clifford algebras and Hermitian quadratic forms

A real Clifford algebra arises naturally as the 'square root' of a real quadratic form Q on a linear space V:

$$v^2 = Q(v),\ v \in V \tag{59}$$

Q can have any signature (N_+, N_0, N_-) . In case Q is degenerate $(N_0 \neq 0)$, the algebra contains Grassmann elements. When v is expanded on an orthogonal basis e_i of V it follows that (59) is satisfied if

$$\frac{1}{2}\{e_i, e_j\} = \delta_{ij} Q(e_i) \tag{60}$$

The basis e_i generates the Clifford algebra. Consider now a quadratic form Q with signature $(2N_+, 2N_0, 2N_-)$. We can rearrange the generators e_i into two sets a_i and b_i, $i = 1, \ldots, N$ which when normalized satisfy

$$\frac{1}{2}\{a_i, a_j\} = \frac{1}{2}\{b_i, b_j\} = \delta_{ij} Q(a_i), \qquad \{a_i, b_j\} = 0 \tag{61}$$

a_i and b_i can be used as 'real' and 'imaginary' parts to define the complex quantities

$$f_j = a_j + i\, b_j \tag{62}$$

where i is the imaginary unit. The elements f_j are seen to satisfy the commutation relations

$$\frac{1}{4}\{f_i, f_j^*\} = \delta_{ij} Q(a_i), \qquad \{f_i, f_j\} = 0 \tag{63}$$

where $*$ is any complex involution induced by a self-involution in the real algebra. The algebra generated by f_i is a complex Clifford algebra.

For any hermitian quadratic form H on a linear space V there exists a complex Clifford algebra generated by V which satisfies

$$\frac{1}{2}\{v, v^*\} = H(v),\ v \in V, \qquad v^2 = 0,\ v \in V \tag{64}$$

The proof follows by expanding v on an orthogonal basis f_i of V. (64) is seen to be satisfied if

$$\frac{1}{2}\{f_i, f_j^*\} = \delta_{ij} H(f_i), \qquad \{f_i, f_j\} = 0 \tag{65}$$

which is recognized to be the generating algebra of a complex Clifford algebra. Expressed in matrix language, (64) implies that any hermitian matrix H_{ij} can be expressed in terms of elements v_i of a complex Clifford algebra:

$$H_{ij} = \{v_i, v_j^*\}, \qquad \{v_i, v_j\} = 0 \tag{66}$$

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