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Convoluted Convolved Fibonacci Numbers

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Abstract

The convolved Fibonacci numbers $F_j^{(r)}$ are defined by $(1-x-x^2)^{-r} = \sum_{j\ge 0} F_{j+1}^{(r)} x^j$. In this note we consider some related numbers that can be expressed in terms of convolved Fibonacci numbers. These numbers appear in the numerical evaluation of a constant arising in the study of the average density of elements in a finite field having order congruent to $a \pmod d$. We derive a formula expressing these numbers in terms of ordinary Fibonacci and Lucas numbers. The non-negativity of these numbers can be inferred from Witt's dimension formula for free Lie algebras.

This note is a case study of the transform $\frac{1}{n}\sum_{d\,|\,n}\mu(d) f(z^d)^{n/d}$ (with f any formal series), which was introduced and studied in a companion paper by Moree.

1 Introduction

Let $\{F_n\}_{n=0}^{\infty} = \{0, 1, 1, 2, 3, 5, \ldots\}$ be the sequence of Fibonacci numbers and $\{L_n\}_{n=0}^{\infty} = \{2, 1, 3, 4, 7, 11, \ldots\}$ the sequence of Lucas numbers. It is well-known and easy to derive that for $|z| < (\sqrt{5}-1)/2$, we have $(1-z-z^2)^{-1} = \sum_{j=0}^{\infty} F_{j+1} z^j$. For any real number r the *convolved Fibonacci numbers* are defined by

$$\frac{1}{(1-z-z^2)^r} = \sum_{j=0}^{\infty} F_{j+1}^{(r)} z^j. \tag{1}$$

The Taylor series in (1) converges for all $z \in \mathbb{C}$ with $|z| < (\sqrt{5}-1)/2$. In the remainder of this note it is assumed that r is a positive integer. Note that

$$F_{m+1}^{(r)} = \sum_{j_1+\cdots+j_r=m} F_{j_1+1}F_{j_2+1}\cdots F_{j_r+1},$$

where the sum is over all $j_1, \ldots, j_r$ with $j_t \geq 0$ for $1 \leq t \leq r$. We also have $F_{m+1}^{(r)} = \sum_{j=0}^{m} F_{j+1} F_{m-j+1}^{(r-1)}$.

Convolved Fibonacci numbers have been studied in several papers, for some references see, e.g., Sloane [10]. The earliest reference to convolved Fibonacci numbers the author is aware of is in a book by Riordan [9], who proposed as an exercise (at p. 89) to show that

$$F_{j+1}^{(r)} = \sum_{v=0}^{r} \binom{r+j-v-1}{j-v} \binom{j-v}{v}. \tag{2}$$

However, the latter identity is false in general (it would imply $F_{j+1}^{(1)} = F_{j+1} = j$ for $j \geq 2$ for example). Using a result of Gould [1, p. 699] on Humbert polynomials (with $n = j$, $m = 2$, $x = 1/2$, $y = -1$, $p = -r$ and $C = 1$) it is easily inferred that (2) holds true with upper index of summation r replaced by $[j/2]$.

In Section 4 we give a formula expressing the convolved Fibonacci numbers in terms of Fibonacci- and Lucas numbers. Hoggatt and Bicknell-Johnson [2], using a different method, derived such a formula for $F_{j+1}^{(2)}$. However, in this note our main interest is in numbers $G_{j+1}^{(r)}$ and $H_{j+1}^{(r)}$ analogous to the convolved Fibonacci numbers, which we name *convoluted convolved Fibonacci numbers*, respectively *sign-twisted convoluted convolved Fibonacci numbers*. Given a formal series $f(z) \in \mathbb{C}[[z]]$, we define its *Witt transform* as

$$\mathcal{W}_f^{(r)}(z) = \frac{1}{r} \sum_{d\,|\,r} \mu(d) f(z^d)^{\frac{r}{d}} = \sum_{j=0}^{\infty} m_f(j,r) z^j, \tag{3}$$

where as usual μ denotes the Möbius function.

For every integer $r \geq 1$ we put

$$G_{j+1}^{(r)} = m_f(j,r) \text{ with } f = \frac{1}{1-z-z^2}.$$

Similarly we put

$$H_{j+1}^{(r)} = (-1)^r m_f(j,r) \text{ with } f = \frac{-1}{1-z-z^2}. \tag{4}$$

On comparing (4) with (1) one sees that

$$G_{j+1}^{(r)} = \frac{1}{r} \sum_{d\,|\,\gcd(r,j)} \mu(d) F_{\frac{j}{d}+1}^{(\frac{r}{d})} \text{ and } H_{j+1}^{(r)} = \frac{(-1)^r}{r} \sum_{d\,|\,\gcd(r,j)} \mu(d) (-1)^{\frac{r}{d}} F_{\frac{j}{d}+1}^{(\frac{r}{d})}. \tag{5}$$

In Tables 1, 2 and 3 below some values of convolved, convoluted convolved, respectively sign-twisted convoluted convolved Fibonacci numbers are provided. The purpose of this paper is to investigate the properties of these numbers. The next section gives a motivation for studying these numbers.

2 Evaluation of a constant

Let g be an integer and p a prime not dividing g. Then by $\mathrm{ord}_p(g)$ we denote the smallest positive integer k such that $g^k \equiv 1 \pmod{p}$. Let $d \geq 1$ be an integer. It can be shown that the set of primes p for which $\mathrm{ord}_p(g)$ is divisible by d has a density and this density can be explicitly computed. It is easy to see that the primes p for which $\mathrm{ord}_p(g)$ is even are the primes that divide some term of the sequence $\{g^r+1\}_{r=0}^{\infty}$. A related, but much less studied, question is whether given integers a and d the set of primes p for which $\mathrm{ord}_p(g) \equiv a \pmod d$ has a density. Presently this more difficult problem can only be resolved under assumption of the Generalized Riemann Hypothesis, see, e.g., Moree [6]. In the explicit evaluation of this density and also that of its average value (where one averages over g) the following constant appears:

$$B_\chi = \prod_p \left(1 + \frac{[\chi(p)-1]p}{[p^2-\chi(p)](p-1)}\right),$$

where χ is a Dirichlet character and the product is over all primes p (see Moree [5]). Recall that the Dirichlet L-series for χ^k, $L(s,\chi^k)$, is defined, for $\mathrm{Re}(s) > 1$, by $\sum_{n=1}^{\infty} \chi^k(n)n^{-s}$. It satisfies the Euler product

$$L(s,\chi^k) = \prod_p \frac{1}{1-\chi^k(p)p^{-s}}.$$

We similarly define $L(s,-\chi^k) = \prod_p (1+\chi^k(p)p^{-s})^{-1}$. The Artin constant, which appears in many problems involving the multiplicative order, is defined by

$$A = \prod_p \left(1 - \frac{1}{p(p-1)}\right) = 0.3739558136\ldots$$

The following result, which follows from Theorem 2 (with $f(z) = -z^3/(1-z-z^2)$) and some convergence arguments, expresses the constant B_χ in terms of Dirichlet L-series. Since Dirichlet L-series in integer values are easily evaluated with very high decimal precision this result allows one to evaluate B_χ with high decimal precision.

Theorem 1 1) *We have*

$$B_\chi = A\frac{L(2,\chi)}{L(3,-\chi)} \prod_{r=1}^{\infty} \prod_{j=3r+1}^{\infty} L(j,(-\chi)^r)^{-e(j,r)},$$

where $e(j,r) = G^{(r)}_{j-3r+1}$.
2) *We have*

$$B_\chi = A\frac{L(2,\chi)L(3,\chi)}{L(6,\chi^2)} \prod_{r=1}^{\infty} \prod_{j=3r+1}^{\infty} L(j,\chi^r)^{f(j,r)},$$

where $f(j,r) = (-1)^{r-1}H^{(r)}_{j-3r+1}$.

Proof. Moree [5] proved part 2, and a variation of his proof yields part 1. □

In the next section it is deduced (Proposition 1) that the numbers $e(j,r)$ appearing in the former double product are actually positive integers and that the $f(j,r)$ are non-zero integers that satisfy $\mathrm{sgn}(f(j,r)) = (-1)^{r-1}$. The proof makes use of properties of the Witt transform that was introduced by Moree [7].

3 Some properties of the Witt transform

We recall some of the properties of the Witt transform (as defined by (3)) and deduce consequences for the (sign-twisted) convoluted convolved Fibonacci numbers.

Theorem 2 [5]. *Suppose that $f(z) \in \mathbb{Z}[[z]]$. Then, as formal power series in y and z, we have*

$$1 - yf(z) = \prod_{j=0}^{\infty}\prod_{r=1}^{\infty}(1 - z^j y^r)^{m_f(j,r)}.$$

Moreover, the numbers $m_f(j,r)$ are integers. If

$$1 - yf(z) = \prod_{j=0}^{\infty}\prod_{r=1}^{\infty}(1 - z^j y^r)^{n(j,r)},$$

for some numbers $n(j,r)$, then $n(j,r) = m_f(j,r)$.

For certain choices of f identities as above arise in the theory of Lie algebras, see, e.g., Kang and Kim [3]. In this theory they go by the name of denominator identities.

Theorem 3 [7]. *Let $r \in \mathbb{Z}_{\geq 1}$ and $f(z) \in \mathbb{Z}[[z]]$. Write $f(z) = \sum_j a_j z^j$.*
1) *We have*

$$(-1)^r \mathcal{W}_{-f}^{(r)}(z) = \begin{cases} \mathcal{W}_f^{(r)}(z) + \mathcal{W}_f^{(r/2)}(z^2), & \text{if } r \equiv 2 \ (\text{mod } 4); \\ \mathcal{W}_f^{(r)}(z), & \text{otherwise.} \end{cases}$$

2) *If $f(z) \in \mathbb{Z}[[z]]$, then so is $\mathcal{W}_f^{(r)}(z)$.*
3) *If $f(z) \in \mathbb{Z}_{\geq 0}[[z]]$, then so are $\mathcal{W}_f^{(r)}(z)$ and $(-1)^r \mathcal{W}_{-f}^{(r)}(z)$.*
Suppose that $\{a_j\}_{j=0}^{\infty}$ is a non-decreasing sequence with $a_1 \geq 1$.
4) *Then $m_f(j,r) \geq 1$ and $(-1)^r m_{-f}(j,r) \geq 1$ for $j \geq 1$.*
5) *The sequences $\{m_f(j,r)\}_{j=0}^{\infty}$ and $\{(-1)^r m_{-f}(j,r)\}_{j=0}^{\infty}$ are both non-decreasing.*

In Moree [7] several further properties regarding monotonicity in both the j and r direction are established that apply to both $G_j^{(r)}$ and $H_j^{(r)}$. It turns out that slightly stronger results in this direction for these sequences can be established on using Theorem 8 below.

3.1 Consequences for $G_j^{(r)}$ and $H_j^{(r)}$

Since clearly $F_{j+1}^{(r)} \in \mathbb{Z}$ we infer from (5) that $rG_{j+1}^{(r)}, rH_{j+1}^{(r)} \in \mathbb{Z}$. More is true, however:

Proposition 1 *Let $j, r \geq 1$ be integers. Then*
1) *$G_j^{(r)}$ and $H_j^{(r)}$ are non-negative integers.*
2) *When $j \geq 2$, then $G_j^{(r)} \geq 1$ and $H_j^{(r)} \geq 1$.*
3) *We have*

$$H_j^{(r)} = \begin{cases} G_j^{(r)} + G_{\frac{j+1}{2}}^{(r/2)}, & \text{if } r \equiv 2 \ (\text{mod } 4) \text{ and } j \text{ is odd;} \\ G_j^{(r)}, & \text{otherwise.} \end{cases}$$

4) *The sequences $\{G_j^{(r)}\}_{j=1}^{\infty}$ and $\{H_j^{(r)}\}_{j=1}^{\infty}$ are non-decreasing.*

The proof easily follows from Theorem 3.

4 Convolved Fibonacci numbers reconsidered

We show that the convolved Fibonacci numbers can be expressed in terms of Fibonacci and Lucas numbers.

Theorem 4 *Let $j \geq 0$ and $r \geq 1$. We have*

$$F_{j+1}^{(r)} = \sum_{\substack{k=0 \\ r+k\equiv 0 \pmod 2}}^{r-1} \binom{r+k-1}{k}\binom{r-k+j-1}{j}\frac{L_{r-k+j}}{5^{(k+r)/2}}+$$

$$\sum_{\substack{k=0 \\ r+k\equiv 1 \pmod 2}}^{r-1} \binom{r+k-1}{k}\binom{r-k+j-1}{j}\frac{F_{r-k+j}}{5^{(k+r-1)/2}}.$$

In particular, $5F_{j+1}^{(2)} = (j+1)L_{j+2} + 2F_{j+1}$ *and*

$$50F_{j+1}^{(3)} = 5(j+1)(j+2)F_{j+3} + 6(j+1)L_{j+2} + 12F_{j+1}.$$

Proof. Suppose that $\alpha, \beta \in \mathbb{C}$ with $\alpha\beta \neq 0$ and $\alpha \neq \beta$. Then it is not difficult to show that we have the following partial fraction decomposition:

$$(1-\alpha z)^{-r}(1-\beta z)^{-r} =$$

$$\sum_{k=0}^{r-1}\binom{-r}{k}\frac{\alpha^r\beta^k}{(\alpha-\beta)^{r+k}}(1-\alpha z)^{k-r} + \sum_{k=0}^{r-1}\binom{-r}{k}\frac{\beta^r\alpha^k}{(\beta-\alpha)^{r+k}}(1-\beta z)^{k-r},$$

where $\binom{t}{k} = 1$ if $k=0$ and $\binom{t}{k} = t(t-1)\cdots(t-k+1)/k!$ otherwise (with t any real number). Using the Taylor expansion (with t a real number)

$$(1-z)^t = \sum_{j=0}^{\infty}(-1)^j\binom{t}{j}z^j,$$

we infer that $(1-\alpha z)^{-r}(1-\beta z)^{-r} = \sum_{j=0}^{\infty}\gamma(j)z^j$, where

$$\gamma(j) = \sum_{k=0}^{r-1}\binom{-r}{k}\frac{\alpha^r\beta^k}{(\alpha-\beta)^{r+k}}(-1)^j\binom{k-r}{j}\alpha^j+$$

$$\sum_{k=0}^{r-1}\binom{-r}{k}\frac{\beta^r\alpha^k}{(\beta-\alpha)^{r+k}}(-1)^j\binom{k-r}{j}\beta^j.$$

Note that $1-z-z^2 = (1-\alpha z)(1-\beta z)$ with $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. On substituting these values of α and β and using that $\alpha - \beta = \sqrt{5}$, $\alpha\beta = -1$, $L_n = \alpha^n + \beta^n$ and $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, we find that

$$F_{j+1}^{(r)} = \sum_{\substack{k=0 \\ r+k\equiv 0 \pmod 2}}^{r-1} (-1)^k\binom{-r}{k}(-1)^j\binom{k-r}{j}\frac{L_{r-k+j}}{5^{(k+r)/2}}+$$

$$\sum_{\substack{k=0\\ r+k\equiv 1 \pmod 2}}^{r-1} (-1)^k \binom{-r}{k} (-1)^j \binom{k-r}{j} \frac{F_{r-k+j}}{5^{(k+r-1)/2}}.$$

On noting that $(-1)^k\binom{-r}{k} = \binom{r+k-1}{k}$ and $(-1)^j\binom{k-r}{j} = \binom{r-k+j-1}{j}$, the proof is completed. $\square$

Let $r \geq 1$ be fixed. From the latter theorem one easily deduces the asymptotic behaviour of $F_{j+1}^{(r)}$ considered as a function of j.

Proposition 2 *Let $r \geq 2$ be fixed. Let $[x]$ denote the integer part of x. Let $\alpha = (1+\sqrt{5})/2$. We have $F_{j+1}^{(r)} = g(r)j^{r-1}\alpha^j + O_r(j^{r-2}\alpha^j)$, as j tends to infinity, where the implicit error term depends at most on r and $g(r) = \alpha^r 5^{-[r/2]}/(r-1)!$.*

5 The numbers $H_{j+1}^{(r)}$ for fixed r

In this and the next section we consider the numbers $H_{j+1}^{(r)}$ for fixed r, respectively for fixed j. Very similar results can of course be obtained for the convoluted convolved Fibonacci numbers $G_{j+1}^{(r)}$.

For small fixed r we can use Theorem 4 in combination with (5) to explicitly express $H_{j+1}^{(r)}$ in terms of Fibonacci- and Lucas numbers. In doing so it is convenient to work with the characteristic function χ of the integers, which is defined by $\chi(r) = 1$ if r is an integer and $\chi(r) = 0$ otherwise. We demonstrate the procedure for $r = 2$ and $r = 3$. By (5) we find $2H_{j+1}^{(2)} = F_{j+1}^{(2)} + F_{\frac{j}{2}+1}\chi(\frac{j}{2})$ and $3H_{j+1}^{(3)} = F_{j+1}^{(3)} - F_{\frac{j}{3}+1}\chi(\frac{j}{3})$. By Theorem 4 it then follows for example that

$$150H_{j+1}^{(3)} = 5(j+1)(j+2)F_{j+3} + 6(j+1)L_{j+2} + 12F_{j+1} - 50F_{\frac{j}{3}+1}\chi(\frac{j}{3}).$$

The asymptotic behaviour, for r fixed and j tending to infinity can be directly inferred from (5) and Proposition 2.

Proposition 3 *With the same notation and assumptions as in Proposition 2 we have $H_{j+1}^{(r)} = g(r)j^{r-1}\alpha^j/r + O_r(j^{r-2}\alpha^j)$.*

6 The numbers $H_{j+1}^{(r)}$ for fixed j

In this section we investigate the numbers $H_{j+1}^{(r)}$ for fixed j. We first investigate this question for the convolved Fibonacci numbers.

The coefficient $F_{j+1}^{(r)}$ of z^j in $(1-z-z^2)^{-r}$ is equal to the coefficient of z^j in $(1+F_2z+F_3z^3+\cdots+F_{j+1}z^j)^r$. By the multinomial theorem we then find

$$F_{j+1}^{(r)} = \sum_{\sum_{k=1}^{j} kn_k = j} \binom{r}{n_1,\cdots,n_j} F_2^{n_1}\cdots F_{j+1}^{n_j}, \tag{6}$$

where the multinomial coefficient is defined by

$$\binom{r}{m_1, \cdots, m_s} = \frac{r!}{m_1! m_2! \cdots m_s! (r - m_1 - \cdots - m_s)!}$$

and $m_k \geq 0$ for $1 \leq k \leq s$.

Example. We have

$$\begin{aligned} F_5^{(r)} &= \binom{r}{4} + 2\binom{r}{2,1} + 4\binom{r}{2} + 3\binom{r}{1,1} + 5\binom{r}{1} \\ &= \frac{7}{4}r + \frac{59}{24}r^2 + \frac{3}{4}r^3 + \frac{r^4}{24}. \end{aligned}$$

This gives an explicit description of the sequence $\{F_5^{(r)}\}_{r=1}^{\infty}$, which is sequence A006504 of Sloane's OEIS [10].

The sequence $\{\binom{r}{m_1, \cdots, m_k}\}_{r=0}^{\infty}$ is a polynomial sequence where the degree of the polynomial is $m_1 + \cdots + m_k$. It follows from this and (6) that $\{F_{j+1}^{(r)}\}_{r=0}^{\infty}$ is a polynomial sequence of degree $\max\{n_1 + \cdots + n_j | \sum_{k=1}^{j} k n_j = j\} = j$. The leading term of this polynomial in r is due to the multinomial term having $n_1 = j$ and $n_t = 0$ for $2 \leq t \leq j$. All other terms in (6) are of lower degree. We thus infer that $F_{j+1}^{(r)} = r^j/j! + O_j(r^{j-1})$, $r \to \infty$. We leave it to the reader to make this more precise by showing that the coefficient of r^{j-1} is $3/(2(j-2)!)$. If $n_1, \ldots, n_j$ satisfy $\max\{n_1 + \cdots + n_j | \sum_{k=1}^{j} k n_j = j\} = j$, then $j!/(n_1! \ldots n_j!)$ is an integral multiple of a multinomial coefficient and hence an integer. We thus infer that $j! F_{j+1}^{(r)}$ is a monic polynomial in $\mathbb{Z}[r]$ of degree j. Note that the constant term of this polynomial is zero. To sum up, we have obtained:

Theorem 5 *Let $j, r \geq 1$ be integers. There is a polynomial*

$$A(j, r) = r^j + \frac{3}{2} j(j-1) r^{j-1} + \cdots \in \mathbb{Z}[r]$$

with $A(j, 0) = 0$ such that $F_{j+1}^{(r)} = A(j, r)/j!$.

Using this result, the following regarding the sign-twisted convoluted convolved Fibonacci numbers can be established.

Theorem 6 *Let $\chi(r) = 1$ if r is an integer and $\chi(r) = 0$ otherwise. We have*

$$H_1^{(r)} = \begin{cases} 1, & \text{if } r \leq 2; \\ 0, & \text{otherwise,} \end{cases}$$

furthermore $H_2^{(r)} = 1$. We have

$$2H_3^{(r)} = 3 + r - (-1)^{r/2} \chi(\frac{r}{2}) \text{ and } 6H_4^{(r)} = 8 + 9r + r^2 - 2\chi(\frac{r}{3}).$$

Also we have

$$24H_5^{(r)} = 42 + 59r + 18r^2 + r^3 - (18+3r)(-1)^{\frac{r}{2}}\chi(\frac{r}{2}) \text{ and}$$

$$120H_6^{(r)} = 264 + 450r + 215r^2 + 30r^3 + r^4 - 24\chi(\frac{r}{5}).$$

In general we have

$$H_{j+1}^{(r)} = \sum_{d\,|\,j,\ 2\nmid d} \mu(d)\chi(\frac{r}{d})\frac{A(\frac{j}{d},\frac{r}{d})}{r(j/d)!} + \sum_{d\,|\,j,\ 2\,|\,d} \mu(d)(-1)^{r/2}\chi(\frac{r}{d})\frac{A(\frac{j}{d},\frac{r}{d})}{r(j/d)!}.$$

Let $j \geq 3$ be fixed. As r tends to infinity we have

$$H_{j+1}^{(r)} = \frac{r^{j-1}}{j!} + \frac{3r^{j-2}}{2(j-2)!} + O_j(r^{j-2}).$$

Proof. Using that $\sum_{d\,|\,n}\mu(d) = 0$ if $n > 1$, it is easy to check that

$$H_1^{(r)} = \frac{(-1)^r}{r}\sum_{d\,|\,r}\mu(r)(-1)^{r/d-1} = \begin{cases} 1, & \text{if } r \leq 2; \\ 0, & \text{if } r > 2. \end{cases}$$

The remaining assertions can be all derived from (5), (6) and Theorem 5. □

7 Monotonicity

Inspection of the tables below suggests monotonicity properties of $F_j^{(r)}, G_j^{(r)}$ and $H_j^{(r)}$ to hold true.

Proposition 4
1) *Let $j \geq 2$. Then $\{F_j^{(r)}\}_{r=1}^{\infty}$ is a strictly increasing sequence.*
2) *Let $r \geq 2$. Then $\{F_j^{(r)}\}_{j=1}^{\infty}$ is a strictly increasing sequence.*

The proof of this is easy. For the proof of part 2 one can make use of the following simple observation.

Lemma 1 *Let $f(z) = \sum_j a(j)z^j \in \mathbb{R}[[z]]$ be a formal series. Then $f(z)$ is said to have k-nondecreasing coefficients if $a(k) > 0$ and $a(k) \leq a(k+1) \leq a(k+2) < \ldots$. If $a(k) > 0$ and $a(k) < a(k+1) < a(k+2) < \ldots$, then f is said to have k-increasing coefficients.*
If f, g are k-increasing, respectively l-nondecreasing, then fg is $(k+l)$-increasing.
If f is k-increasing and g is l-nondecreasing, then $f+g$ is $\max(k,l)$-increasing.
If f is k-increasing, then $\sum_{j\geq 1} b(j)f^j$ with $b(j) \geq 0$ and $b(1) > 0$ is k-increasing.

We conclude this paper by establishing the following result:

Theorem 7
1) *Let $j \geq 4$. Then $\{G_j^{(r)}\}_{r=1}^{\infty}$ is a strictly increasing sequence.*
2) *Let $r \geq 1$. Then $\{G_j^{(r)}\}_{j=2}^{\infty}$ is a strictly increasing sequence.*
3) *Let $j \geq 4$. Then $\{H_j^{(r)}\}_{r=1}^{\infty}$ is a strictly increasing sequence.*
4) *Let $r \geq 1$. Then $\{H_j^{(r)}\}_{j=2}^{\infty}$ is a strictly increasing sequence.*

The proof rests on expressing the entries of the above sequences in terms of certain quantities occurring in the theory of free Lie algebras and circular words (Theorem 8) and then invoke results on the monotonicity of these quantities to establish the result.

7.1 Circular words and Witt's dimension formula

We will make use of an easy result on cyclic words. A word $a_1 \cdots a_n$ is called *circular* or *cyclic* if a_1 is regarded as following a_n, where $a_1 a_2 \cdots a_n$, $a_2 \cdots a_n a_1$ and all other cyclic shifts (rotations) of $a_1 a_2 \cdots a_n$ are regarded as the same word. A circular word of length n may conceivably be given by repeating a segment of d letters n/d times, with d a divisor of n. Then one says the word is of *period* d. Each word belongs to an unique smallest period: the *minimal period.*

Consider circular words of length n on an alphabet $x_1, \ldots, x_r$ consisting of r letters. The total number of ordinary words such that x_i occurs n_i times equals

$$\frac{n!}{n_1! \cdots n_r!},$$

where $n_1 + \cdots + n_r = n$. Let $M(n_1, \ldots, n_r)$ denote the number of circular words of length $n_1 + \cdots + n_r = n$ and minimal period n such that the letter x_i appears exactly n_i times. This leads to the formula

$$\frac{n!}{n_1! \cdots n_r!} = \sum_{d \,|\, \gcd(n_1, \ldots, n_r)} \frac{n}{d} M(\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_r}{d}). \tag{7}$$

whence it follows by Möbius inversion that

$$M(n_1, \ldots, n_r) = \frac{1}{n} \sum_{d \,|\, \gcd(n_1, \ldots, n_r)} \mu(d) \frac{\frac{n}{d}!}{\frac{n_1}{d}! \cdots \frac{n_r}{d}!}. \tag{8}$$

Note that $M(n_1, \ldots, n_r)$ is totally symmetric in the variables $n_1, \ldots, n_r$. The numbers $M(n_1, \ldots, n_r)$ also occur in a classical result in Lie theory, namely Witt's formula for the homogeneous subspaces of a finitely generated free Lie algebra L: if H is the subspace of L generated by all homogeneous elements of multidegree $(n_1, \ldots, n_r)$, then $\dim(H) = M(n_1, \ldots, n_r)$, where $n = n_1 + \cdots + n_r$.

In Theorem 8 a variation of $M(n_1, \ldots, n_r)$ appears.

Lemma 2 [7]. *Let r be a positive integer and let $n_1, \ldots, n_r$ be non-negative integers and put $n = n_1 + \cdots + n_r$. Let*

$$V_1(n_1, \ldots, n_r) = \frac{(-1)^{n_1}}{n} \sum_{d \,|\, \gcd(n_1, \ldots, n_r)} \mu(d)(-1)^{\frac{n_1}{d}} \frac{\frac{n}{d}!}{\frac{n_1}{d}! \cdots \frac{n_r}{d}!}.$$

Then

$$V_1(n_1,\ldots,n_r)=\begin{cases} M(n_1,\ldots,n_r)+M(\frac{n_1}{2},\ldots,\frac{n_r}{2}), & \text{if } n_1\equiv 2 \pmod 4 \text{ and } 2\,|\,\gcd(n_1,\ldots,n_r);\\ M(n_1,\ldots,n_r) & \text{otherwise.}\end{cases}$$

The numbers $V_1(n_1,\ldots,n_r)$ can also be interpreted as dimensions (in the context of free Lie superalgebras), see, e.g., Petrogradsky [8].

The numbers M and V_1 enjoy certain monotonicity properties.

Lemma 3 [7]. *Let $r\ge 1$ and $n_1,\ldots,n_r$ be non-negative numbers.*
1) The sequence $\{M(m,n_1,\ldots,n_r)\}_{m=0}^{\infty}$ is non-decreasing if $n_1+\cdots+n_r\ge 1$ and strictly increasing if $n_1+\cdots+n_r\ge 3$.
2) The sequence $\{V_1(m,n_1,\ldots,n_r\}_{m=0}^{\infty}$ is non-decreasing if $n_1+\cdots+n_r\ge 1$ and strictly increasing if $n_1+\cdots+n_r\ge 3$.

Using (7) one infers (on taking the logarithm of either side and expanding it as a formal series) that

$$1-z_1-\cdots-z_r=\prod_{n_1,\ldots,n_r=0}^{\infty}(1-z_1^{n_1}\cdots z_r^{n_r})^{M(n_1,\cdots,n_r)},\tag{9}$$

where $(n_1,\ldots,n_r)=(0,\ldots,0)$ is excluded in the product. From the latter identity it follows that

$$1+z_1-z_2-\cdots-z_r=$$
$$\prod_{\substack{n_1,\ldots,n_r=0\\ 2\,|\,n_1}}^{\infty}(1-z_1^{n_1}\cdots z_r^{n_r})^{M(n_1,\ldots,n_r)}\prod_{\substack{n_1,\ldots,n_r=0\\ 2\nmid n_1}}^{\infty}\left(\frac{1-z_1^{2n_1}\cdots z_r^{2n_r}}{1-z_1^{n_1}\cdots z_r^{n_r}}\right)^{M(n_1,\ldots,n_r)},$$

whence, by Lemma 2,

$$1+z_1-z_2-\cdots-z_r=\prod_{n_1,\ldots,n_r=0}^{\infty}(1-z_1^{n_1}\cdots z_r^{n_r})^{(-1)^{n_1}V_1(n_1,\ldots,n_r)}.\tag{10}$$

Theorem 8 *Let $r\ge 1$ and $j\ge 0$. We have*

$$G_{j+1}^{(r)}=\sum_{k=0}^{[j/2]}M(r,k,j-2k)\text{ and }H_{j+1}^{(r)}=\sum_{k=0}^{[j/2]}V_1(r,k,j-2k).$$

Proof. By Theorem 2 and the definition of $G_j^{(r)}$ we infer that

$$1-\frac{y}{1-z-z^2}=\prod_{j=0}^{\infty}\prod_{r=1}^{\infty}(1-z^jy^r)^{G_{j+1}^{(r)}}.$$

The left hand side of the latter equality equals $(1-z-z^2-y)/(1-z-z^2)$. On invoking (9) with $z_1=z$, $z_2=z^2$ and $z_3=y$ the claim regarding $G_j^{(r)}$ follows from the uniqueness assertion in Theorem 2.

The proof of the identity for $H_{j+1}^{(r)}$ is similar, but makes use of identity (10) instead of (9). $\square$

Proof of Theorem 7. 1) For $j \geq 5$, $k+j-2k \geq j-[j/2] \geq 3$ and hence each of the terms $M(r,k,j-2k)$ with $0 \leq k \leq [j/2]$ is strictly increasing in r by Lemma 3. For $3 \leq j \leq 4$, by Lemma 3 again, all terms $M(r,k,j-2k)$ with $0 \leq k \leq [j/2]$ are non-decreasing in r and at least one of them is strictly increasing. The result now follow by Theorem 8.
2) In the proof of part 1 replace the letter 'M' by 'V_1'.
3) For $r=1$ we have $G_{j+!}^{(1)} = F_{j+1}$ and the result is obvious. For $r=2$ each of the terms $M(r,k,j-2k)$ with $0 \leq k \leq [j/2]$ is non-decreasing in j. For $j \geq 2$ one of these is strictly increasing. Since in addition $G_2^{(2)} < G_3^{(2)}$ the result follows for $r=2$. For $r \geq 3$ each of the terms $M(r,k,j-2k)$ with $0 \leq k \leq [j/2]$ is strictly increasing in j. The result now follows by Theorem 8.
4) In the proof of part 3 replace the letter 'G' by 'H and 'M' by 'V_1'. $\square$

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8 Tables

Table 1: Convolved Fibonacci numbers $F_j^{(r)}$

$r\backslash j$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	2	3	5	8	13	21	34	55	89
2	1	2	5	10	20	38	71	130	235	420	744
3	1	3	9	22	51	111	233	474	942	1836	3522
4	1	4	14	40	105	256	594	1324	2860	6020	12402
5	1	5	20	65	190	511	1295	3130	7285	16435	36122

Table 2: Convoluted convolved Fibonacci numbers $G_j^{(r)}$

$r\backslash j$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	2	3	5	8	13	21	34	55	89
2	0	1	2	5	9	19	34	65	115	210	368
3	0	1	3	7	17	37	77	158	314	611	1174
4	0	1	3	10	25	64	146	331	710	1505	3091
5	0	1	4	13	38	102	259	626	1457	3287	7224
6	0	1	4	16	51	154	418	1098	2726	6570	15308
7	0	1	5	20	70	222	654	1817	4815	12265	30217
8	0	1	5	24	89	309	967	2871	8043	21659	56123
9	0	1	6	28	115	418	1396	4367	12925	36542	99385
10	0	1	6	33	141	552	1946	6435	20001	59345	168760

Table 3: Sign twisted convoluted convolved Fibonacci numbers $H_j^{(r)}$

$r\backslash j$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	2	3	5	8	13	21	34	55	89
2	1	1	3	5	11	19	37	65	120	210	376
3	0	1	3	7	17	37	77	158	314	611	1174
4	0	1	3	10	25	64	146	331	710	1505	3091
5	0	1	4	13	38	102	259	626	1457	3287	7224
6	0	1	5	16	54	154	425	1098	2743	6570	15345
7	0	1	5	20	70	222	654	1817	4815	12265	30217
8	0	1	5	24	89	309	967	2871	8043	21659	56123
9	0	1	6	28	115	418	1396	4367	12925	36542	99385
10	0	1	7	33	145	552	1959	6435	20039	59345	168862

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