Journal of Integer Sequences, Vol. 7 (2004),
Article 04.3.4

Some Results on Summability of Random Variables

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Abstract

A convolution summability method introduced as an extension of the random-walk method generalizes the classical Euler, Borel, Taylor and Meyer-König type matrix methods. This corresponds to the distribution of sums of independent and identically distributed integer-valued random variables. In this paper, we discuss the strong regularity concept of Lorentz applied to the convolution method of summability. Later, we obtain the summability functions and absolute summability functions of this method.

1 Introduction

The methods that sum all almost convergent sequences are called strongly regular. We will show in section 2 that the matrix transformation corresponding to the regular convolution method generated by an independent and identically distributed sequence of aperiodic non-negative integer-valued random variables with finite third moment and positive variance is strongly regular.

Summability functions [7] in some sense determine the strength of the regularity of method for bounded sequences. It may also be used to show that Tauberian conditions of a certain kind may not be improved. Under the existence of first three moments, in section 3, it is shown that $\Omega(n) = o(\sqrt{n})$ are the summability functions for the convolution methods, thus extending some of previously known results for other methods such as Borel and Euler. The optimality of class of summability functions is also ascertained to show that all functions of the form, $\Omega(n) = o(\sqrt{n})$ (are only these functions) are summability functions for the $C(p, q)$ methods with some moment conditions. We conclude this paper with a discussion of absolute summability functions for this method.

The discussion will now revolve around the following types of summability methods [11] and [13]. This is a larger class of summability methods that includes random-walk method and many others.

Definition 1 Let $\{p_k\}_{k\geq 0}$ and $\{q_k\}_{k\geq 0}$ be two sequences of nonnegative numbers with $\sum_{k=0}^{\infty} p_k = 1$ and $\sum_{k=0}^{\infty} q_k = 1$. Define a summability matrix, $C = [C_{n,k}]$, whose entries are given by $C_{0,k} = q_k$ and $C_{n+1,k} := (C_{n,\cdot} * p)_k = \sum_{j=0}^{k} p_j C_{n,k-j}$ for $n, k \geq 0$. The matrix C is called a convolution summability matrix.

A useful probabilistic interpretation of C is the following. Let $Y, X_1, X_2, \ldots$ be a sequence of independent non-negative integer valued random variables such that Y has probability function q and the $X_i's$ are identically distributed with probability function p. Let $S_0 = Y$ and $S_n = Y + X_1 + \ldots + X_n$ for $n \geq 1$. Let $\{p_j\}_{j\geq 0}$ and $\{q_j\}_{j\geq 0}$ be the probability distributions of $X_1, X_2, \ldots$ and Y respectively. The n^{th} row k^{th} column entry of the convolution summability matrix C is the probability $C_{n,k} = P(S_n = k)$. The method C is regular if and only if $P(X_1 = 0) < 1$ [6]. Some classical summability methods are examples of the method C. For instance, when $Y = 0$ and $X_1 \sim \text{Binomial}(1, 1)$, then C becomes the Euler method denoted by E_r. When $Y \sim X_1 \sim \text{Poisson}(1)$ we get the Borel matrix method. When $Y \sim \text{Geometric}(1 - r)$ and $X_1 \sim Y + 1$, then we get the Taylor method. And when $Y \sim X_1 \sim \text{Geometric}(1 - r)$ we get the Meyer-König method. We shall call C a convolution method and when $Y = 0$ with probability 1, it is called the random-walk method. The method C can be extended to non-identically distributed random variables (for example, Jakimovski family of summability methods [13]); however, it will serve our purpose adequately for the time being, as it is. The regular convolution summability matrix $\{C_{n,k\geq 0}\}$, referred to everywhere in this paper has the above construction with appropriate moment conditions and in section 4 with finite moment generating function of $\{X_i, i \geq 1\}$.

2 Strong Regularity

Given below is a definition of almost convergence of a sequence, which is as we see, a generalization of ordinary convergence.

Definition 2 A bounded sequence $\{x_i\}_{i\geq 0}$ is called almost convergent, if there is a number s such that

$$\lim_{\ell\to\infty} \frac{x_n + x_{n+1} + \ldots + x_{n+\ell-1}}{\ell} = s \text{ holds uniformly in } n.$$

We denote s by Lim x_n.

Example 1 For a complex z on the boundary of the unit circle $\text{Lim} z^n = 0$ holds everywhere except for $z = +1$, as follows from

$$\frac{1}{\ell}(z^n + z^{n+1} + \cdots + z^{n+\ell-1}) = \frac{z^n}{\ell}\Big(\frac{1 - z^\ell}{1 - z}\Big).$$

We now use the following theorem of Lorentz [7]. For more details on these concepts, see [7].

Theorem 1 *(Lorentz [7]) In order that regular matrix method (transformation) $A = \{a_{n,k}\}$ sum all almost convergent sequences, it is necessary and sufficient that*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}| = 0.$$

In view of the above result, we now give below the definition for strong regularity of an almost convergent sequence.

Definition 3 A summability method A is called strongly regular, if for any almost convergent sequence $\{x_i\}_{i\geq 0}$ with $\mathrm{Lim} x_n = l$, we have $\lim_{n\to\infty}(Ax)_n = l$.

Lorentz [7] showed that the Cesàro method C_α of order $\alpha > 0$ and the Euler method E_r, with parameter r, are strongly regular. In an attempt to generalize these results, we will prove that the random-walk method is strongly regular for a probability function with finite third moment. Then using this result, we show that the convolution summability method is also strongly regular.

Theorem 2 *Let $\xi_1, \xi_2, \xi_3, \ldots$ be an i.i.d. sequence of aperiodic nonnegative integer-valued random variables with finite third moment and positive variance. Then the matrix transformation corresponding to the random-walk method of the above sequence of random variables is strongly regular.*

Prior to the proof of this theorem, we need an important theorem due to Bikelis and Jasjunas [1], which gives the rate of convergence for the central limit theorem:

Theorem 3 *(Bikelis & Jasjunas [1]) For a sequence $\{\xi_i\}_{i\geq 1}$, i.i.d. aperiodic nonnegative integer-valued random variables with mean μ, positive variance σ^2, and finite third moment, the following holds:*

$$\sum_{j=-\infty}^{\infty} (1 + |\frac{j-n\mu}{\sigma\sqrt{n}}|^3)|P(S_n = j) - \frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(j-n\mu)^2/(n\sigma^2)\}| = O(n^{-1/2}),$$

where $S_n = \xi_1 + \cdots + \xi_n$.

Proof of theorem 2

As suggested in Theorem 1, we now consider

$$\sum_{k=0}^{\infty} |a_{n,k+1} - a_{n,k}| = \sum_{k=0}^{\infty} |P(S_n = k+1) - P(S_n = k)|, \text{ where } S_n = \sum_{i=0}^{n} \xi_i \text{ and } \xi_0 = 0.$$

If the mean of ξ_i is μ and standard deviation of ξ_i is σ, we write

$$\sum_{k=0}^{\infty} |P(S_n = k+1) - P(S_n = k)|$$

$$
\begin{aligned}
= & \sum_{k=0}^{\infty} |\{P(S_n = k+1) - \frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(k+1-n\mu)^2/(n\sigma^2)\}\} \\
& +\{\frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(k-n\mu)^2/(n\sigma^2)\} - P(S_n = k)\} \\
& +\{\frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(k+1-n\mu)^2/(n\sigma^2)\} - \frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(k-n\mu)^2/(n\sigma^2)\}\}| \\
\leq & \sum_{k=0}^{\infty} |P(S_n = k+1) - \frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(k+1-n\mu)^2/(n\sigma^2)\}| \\
& +\sum_{k=0}^{\infty} |\frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(k-n\mu)^2/(n\sigma^2)\} - P(S_n = k)| \\
& +\sum_{k=0}^{\infty} |\frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(k+1-n\mu)^2/(n\sigma^2)\} - \frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(k-n\mu)^2/(n\sigma^2)\}|.
\end{aligned}
$$

Subject to the finiteness of the third moment, considering the fact that $(1+|\frac{j-n\mu}{\sqrt{n\sigma^2}}|^3) \geq 1$ and restricting the values of j, for $j \geq 0$, we obtain from Bikelis and Jasjunas theorem 3 that

$$
\sum_{j=0}^{\infty} |P(S_n = j) - \frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(j-n\mu)^2/(n\sigma^2)\}| = O(n^{-1/2}).
$$

This implies that the first two sums of the above are in fact of $O(n^{-1/2})$.

For the last sum, we look at the telescopic series in the following form, noting that as $k \to \infty$, the terms increase until $k = n\mu$ and then decreases to 0 thereafter:

$$
\begin{aligned}
& \frac{1}{\sigma(2\pi n)^{\frac{1}{2}}}\{(\exp\{-\frac{1}{2}(1-n\mu)^2/(n\sigma^2)\} - \exp\{-\frac{1}{2}(0-n\mu)^2/(n\sigma^2)\}) \\
& +(\exp\{-\frac{1}{2}(2-n\mu)^2/(n\sigma^2)\} - \exp\{-\frac{1}{2}(1-n\mu)^2/(n\sigma^2)\}) + \cdots \\
& + \cdots + (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots) + \cdots \\
& + \cdots + (\exp\{-\frac{1}{2}(i+1-n\mu)^2/(n\sigma^2)\} - \exp\{-\frac{1}{2}(i-n\mu)^2/(n\sigma^2)\}) + \cdots \\
& + \cdots + (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots) + \cdots \\
& + \cdots + (\exp\{-\frac{1}{2}(n\mu-n\mu)^2/(n\sigma^2)\} - \exp\{-\frac{1}{2}(n\mu-1-n\mu)^2/(n\sigma^2)\})\} \\
& +\frac{1}{\sigma(2\pi n)^{\frac{1}{2}}}\{(\exp\{-\frac{1}{2}(n\mu+1-n\mu)^2/(n\sigma^2)\} - \exp\{-\frac{1}{2}(n\mu+2-n\mu)^2/(n\sigma^2)\}) \\
& +(\exp\{-\frac{1}{2}(n\mu+2-n\mu)^2/(n\sigma^2)\} - \exp\{-\frac{1}{2}(n\mu+3-n\mu)^2/(n\sigma^2)\}) + \cdots \\
& + \cdots + (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots) + \cdots \\
& + \cdots + (\exp\{-\frac{1}{2}(j-n\mu)^2/(n\sigma^2)\} - \exp\{-\frac{1}{2}(j+1-n\mu)^2/(n\sigma^2)\}) + \cdots \\
& + \cdots + (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)\};
\end{aligned}
$$

for $0 \le i \le n\mu - 1$ and $n\mu + 1 < j < \infty$.

Since $\lim_{j\to\infty} \exp\{-\frac{1}{2}(j - n\mu)^2/(n\sigma^2)\} = 0$, the above series sums to

$$\frac{1}{\sigma(2\pi n)^{\frac{1}{2}}}\{-e^{-\frac{1}{2}n^2\mu^2/n\sigma^2} + 1 + e^{-\frac{1}{2}/n\sigma^2}\}$$

$$= \frac{1}{\sigma(2\pi n)^{\frac{1}{2}}}\{-e^{-n\mu^2/2\sigma^2} + 1 + e^{-1/2n\sigma^2}\} = O(1/\sqrt{n}).$$

This together with above leads to the fact that

$$\sum_{k=0}^{\infty} |P(S_n = k+1) - P(S_n = k)| = O(1/\sqrt{n}).$$

Now, by the Lorentz criteria, we have the strong regularity of the random-walk method.
∎

We will use the above result to prove the following generalization for the convolution summability method C defined in section 1.

Theorem 4 *Let $Y, \{X_i, i \ge 1\}$ be independent and let $\{X_i, i \ge 1\}$ be identically distributed aperiodic nonnegative integer-valued random variables with finite third moment. Let C be the convolution summability method. Then the following are equivalent.*

(i) $Var(X_1) > 0$,

(ii) C is strongly regular.

Proof.

We will first show that (i) implies (ii).

Let $\{q_j\}$ and $\{p_j\}$ be the probability weights associated with random variables Y and $\{X_i, i \ge 1\}$. The weight of the convolution summability method is

$$C_{n,k} = P(Y + S_n = k) = \sum_{j=0}^{k} q_j\{P(S_n = k - j)\}, \text{ where } S_n = \sum_{i=1}^{n} X_i.$$

Now,

$$\begin{aligned}
&\sum_{k=0}^{\infty} |C_{n,k+1} - C_{nk}| \\
= &\sum_{k=0}^{\infty} |\sum_{j=0}^{k+1} q_j P(S_n = k+1-j) - \sum_{j=0}^{k} q_j P(S_n = k - j)| \\
= &\sum_{k=0}^{\infty} |\sum_{j=0}^{k} q_j P(S_n = k+1-j) - \sum_{j=0}^{k} q_j P(S_n = k - j) + q_{k+1} P(S_n = 0)|
\end{aligned}$$

$$\leq \sum_{k=0}^{\infty}\sum_{j=0}^{k} q_j|P(S_n = k+1-j) - P(S_n = k-j)| + \sum_{k=0}^{\infty} q_{k+1}P(S_n = 0)$$
$$\leq \sum_{j=0}^{\infty} q_j \sum_{k=j}^{\infty} |P(S_n = k+1-j) - P(S_n = k-j)| + P(S_n = 0)$$

as $\sum_{k=0}^{\infty} q_k = 1$.

Change of summation index $k - j \to k$ gives

$$\sum_{k=0}^{\infty} |C_{n,k+1} - C_{nk}|$$
$$\leq \sum_{j=0}^{\infty} q_j \sum_{k=0}^{\infty} |P(S_n = k+1) - P(S_n = k)| + (p_0)^n$$
$$\leq \sum_{k=0}^{\infty} |P(S_n = k+1) - P(S_n = k)| + (p_0)^n$$

as $\sum_{k=0}^{\infty} q_k = 1$. As already seen in the previous proof, the first term is of $O(n^{-1/2})$, provided that the $\{X_i\}'s$ have finite third moment and positive variance, whereas the second term tends to 0. Since we assumed that $\mathrm{Var}(X_1) > 0$, it must be that $p_0 < 1$. Hence

$$\sum_{k=0}^{\infty} |C_{n,k+1} - C_{n,k}| \to^n 0 \qquad (n \to \infty).$$

To prove that (ii) implies (i), assume that (ii) holds and (i) fails. When $\mathrm{Var}(X_1) = 0$ there exists a nonnegative integer m such that $P(X_1 = m) = 1$. Hence,

$$C_{n,j} = P(Y + S_n = j) = P(Y = j - nm)$$
$$= \begin{cases} 0 & \text{if} \quad j < nm \\ q_{j-nm} & \text{if} \quad j \geq nm. \end{cases}$$

Therefore,

$$\sum_{j=0}^{\infty} |C_{n,j+1} - C_{n,j}| \geq \sum_{j=0}^{\infty} |q_{j+1} - q_j| \neq 0,$$

as $\sum_{j=0}^{\infty} q_j = 1$, and $q_i \geq 0$ for $i \geq 0$. This contradiction gives the result.
∎

Remark 1 It should be noted that with the condition $p_0 < 1$, Khan [6] proved the regularity of the convolution summability method. Our condition $\mathrm{Var}(X_1) > 0$ implies that $p_0 < 1$. Furthermore, it follows as a result of Theorem 4 that Taylor and Meyer-König methods are strongly regular.

3 Summability Functions

The concept of summability functions was introduced by Lorentz [7]. There are many uses of the summability functions. Summability functions, in some sense, determine the strength of the regularity of the method for bounded sequences and also may be used to show that Tauberian conditions of a certain kind cannot be improved.

Definition 4 The class $\mathcal{U}$ is the set of regular matrix methods $A = \{a_{n,k}\}$ for which

$$\lim_{n\to\infty}\{\max_{k} |a_{n,k}|\} = 0$$

is fulfilled.

Every regular convolution summability method satisfies this property, so they form a subset of $\mathcal{U}$. For,

$$C_{n,k} = \sum_{j=0}^{k} q_j\{P(\sum_{i=1}^{n} X_i = k-j)\},$$

where $\{C_{n,k}\}$ are the convolution summability weights under consideration. Now with $\mu = EX_1$ and $0 < \sigma^2 = \mathrm{Var}(X_1) < \infty$,

$$C_{n,k} = \sum_{j=0}^{k} q_j[(\frac{1}{2\,\pi n\sigma^2})^{1/2}\exp\{-\frac{1}{2}(k-j-n\mu)^2/n\sigma^2\} + o(1/\sqrt{n})],$$

uniformly in $k-j$. Since $\sum_{j=0}^{\infty} q_j = 1$ and $q_j \geq 0,\ \forall\ j \geq 0$, we have

$$\max_{k} |C_{n,k}| = O(\frac{1}{\sqrt{n}}) \to^{n} 0.$$

The methods of the class $\mathcal{U}$ are characterized by the fact that they all possess summability functions. We now give the precise definition of the summability function.

Definition 5 Given a matrix $A = \{a_{n,k}\}$, a nonnegative sequence $\Omega(n)$ that increases to ∞ is called a summability function for A if $s_n \to 0\ (A)$ holds whenever $s_n = O(1)$ and $A(n,s) = \sum_{\nu\leq n,\ s_\nu\neq 0} 1 \leq \Omega(n)$. The sequence $A(n,s)$ is sometimes called a counting function of the sequence $\{s_n\}$.

Theorem 5 *(Lorentz [7]) The condition* $\lim_{m\to\infty}\{\max_n |a_{m,n}|\} = 0$ *is necessary as well as sufficient for the existence of an integer-valued function* $\Omega(n)$ *that increases to* ∞*, such that every bounded sequence* $x = \{x_n\}$ *for which the indices* n_ν*, with* $x_{n_\nu} \neq 0$ *have a counting function* $A(n,x) \leq \Omega(n)$ *is* $A-$*summable to zero.*

The following theorem gives sufficient conditions under which the existence of summability functions can be determined.

Theorem 6 *(Lorentz [7]) Let $A = \{a_{n,k}\}$ be a regular matrix summability method. If the integer-valued function of k, $f(k)$, is such that $0 < f(k) \uparrow \infty$ and if*

$$\sum_{k=0}^{\infty} f(k)|a_{n,k} - a_{n,k+1}| = O(1),$$

then every nonnegative sequence $\Omega(n)$ that increases to ∞, $\Omega(n) = o(f(n))$ is a summability function for A.

We make use of the above theorem to show that $\Omega(k) = o(\sqrt{k})$ is a summability function for the convolution summability methods generated by a sequence of aperiodic nonnegative integer-valued random variables with finite third moment. The optimality of the summability function so obtained is also ascertained. In this connection, we shall show that all functions $\Omega(n) = o(\sqrt{n})$ and only these functions are summability functions for the methods $\{C_{n,k}\}_{n,k\geq 1}$. Lorentz [7] showed that all functions of the form $\Omega(n) = o(\sqrt{n})$ and only these functions are summability functions for the Euler method E_α with $\alpha > 0$. We use this fact in the proof of the following theorem.

There are cases where Theorem 6 does not give all summability functions. For an example, the Nölund method N_p, with $p_n = 1/(n+1)$; Theorem 6 gives that $\Omega(n) = o(\log(n))$ are its summability functions. However, it is known that any function $\Omega(n) = O(n^{\epsilon_n})$ with $0 < \epsilon_n \to 0$ is a summability function for N_p (p. 61, [12]).

The following theorem provides summability functions for a convolution summability method.

Theorem 7 *Let $Y, \{X_i, i \geq 1\}$ be independent with $E(Y) = \mu_Y < \infty$, and $\{X_i, i \geq 1\}$ be identically distributed aperiodic nonnegative integer-valued random variables with finite third moment and positive variance. Then for the matrix transformation corresponding to the regular convolution summability method $C = \{C_{n,k}\}$, any function $0 < \Omega(n) \uparrow \infty$ of the form $\Omega(n) = o(\sqrt{n})$ gives a summability function of C. Furthermore, $0 < \Omega(n) = o(\sqrt{n})$ with $\Omega(n) \uparrow \infty$ are the only functions which are summability functions over the class of regular convolution methods under consideration.*

Proof.

The weight of the convolution summability method is given by

$$C_{n,k} = P(Y + S_n = k) = \sum_{j=0}^{k} q_j\{P(S_n = k-j)\} \text{ where } S_n = \sum_{i=1}^{n} X_i.$$

Now consider,

$$\begin{aligned}
&\sum_{k=0}^{\infty} \sqrt{k}|C_{n,k+1} - C_{nk}| \\
= \; &\sum_{k=0}^{\infty} \sqrt{k}|\sum_{j=0}^{k+1} q_j P(S_n = k+1-j) - \sum_{j=0}^{k} q_j P(S_n = k-j)|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \sqrt{k} \left| \sum_{j=0}^{k} q_j P(S_n = k+1-j) - \sum_{j=0}^{k} q_j P(S_n = k-j) + q_{k+1} P(S_n = 0) \right| \\
&\le \sum_{k=0}^{\infty} \sqrt{k} \sum_{j=0}^{k} q_j |P(S_n = k+1-j) - P(S_n = k-j)| + \sum_{k=0}^{\infty} \sqrt{k} q_{k+1} P(S_n = 0) \\
&\le \sum_{j=0}^{\infty} q_j \sum_{k=j}^{\infty} \sqrt{k} |P(S_n = k+1-j) - P(S_n = k-j)| + P(S_n = 0) \sum_{k=0}^{\infty} \sqrt{k} q_{k+1}.
\end{aligned}
$$

Making the change of summation index $k-j \to k$ in the first sum and $k+1 \to k$ in the second sum, we obtain

$$
= \sum_{j=0}^{\infty} q_j \sum_{k=0}^{\infty} \sqrt{k+j} |P(S_n = k+1) - P(S_n = k)| + P(S_n = 0) \sum_{k=1}^{\infty} \sqrt{k-1} q_k.
$$

Since the sequence $\{q\}$ has a finite first moment, say μ_Y, it follows that

$$
\sum_{k=0}^{\infty} \sqrt{k} |C_{n,k+1} - C_{nk}| \le \sum_{j=0}^{\infty} q_j \sum_{k=0}^{\infty} (\sqrt{k} + \sqrt{j}) |P(S_n = k+1) - P(S_n = k)| + \mu_Y (p_0)^n.
$$

Note that the last term tends to 0, since $\mathrm{Var}(X_1) > 0$, which implies that $p_0 < 1$.

$$
\sum_{k=0}^{\infty} \sqrt{k} |C_{n,k+1} - C_{nk}|
$$

$$
\le \mu_Y \sum_{k=0}^{\infty} |P(S_n = k+1) - P(S_n = k)| + \sum_{k=0}^{\infty} \sqrt{k} |P(S_n = k+1) - P(S_n = k)| + o(1).
$$

The first sum also tends to 0, since $\mathrm{Var}(X_1) > 0$ is a necessary and sufficient condition for the strong regularity of the convolution summability method. Now what remains is to show that

$$
\sum_{k=0}^{\infty} \sqrt{k} |P(S_n = k+1) - P(S_n = k)| = O(1).
$$

We begin with the following.

$$
\begin{aligned}
\sqrt{k} &= \sqrt{(k - n\mu) + n\mu} \le \sqrt{|k - n\mu| + n\mu} \le \sqrt{|k - n\mu|} + (n\mu)^{1/2} \\
&\le \left| \frac{k - n\mu}{\sigma\sqrt{n}} \right|^{1/2} (\sigma\sqrt{n})^{1/2} + (n\mu)^{1/2}.
\end{aligned}
$$

With the assumption of the finiteness of the third moment of the sequence $\{p_j\}$ of i.i.d. random variables Theorem 3 of Bikelis and Jasjunas [1] gives

$$
\sum_{j=-\infty}^{\infty} \left(1 + \left| \frac{j - n\mu}{\sigma\sqrt{n}} \right|^3\right) \left| P(S_n = j) - \frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(j - n\mu)^2/(n\sigma^2)\} \right| = O(n^{-1/2}),
$$

where $S_n = X_1 + X_2 + \cdots + X_n$. Then

$$\begin{aligned}
& \sum_{k=0}^{\infty} \sqrt{k}|P(S_n = k+1) - P(S_n = k)| \\
\le\ & (\sigma\sqrt{n})^{1/2} \sum_{k=0}^{\infty} |\frac{k - n\mu}{\sigma\sqrt{n}}|^{1/2} |P(S_n = k+1) - P(S_n = k)| \\
& +(n\mu)^{1/2} \sum_{k=0}^{\infty} |P(S_n = k+1) - P(S_n = k)| = \sum_1 + \sum_2 \text{ say.}
\end{aligned}$$

We have already shown in the proof of Theorem 2 that $\sum_2 = O(1)$.

For $\sum_1$, we will proceed as follows:

$$\begin{aligned}
\sum_1 \ \le\ & (\sigma\sqrt{n})^{1/2} \sum_{k=0}^{\infty} |\frac{k - n\mu}{\sigma\sqrt{n}}|^{1/2} |P(S_n = k+1) \\
& - \frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(k+1-n\mu)^2/(n\sigma^2)\}| \\
& +(\sigma\sqrt{n})^{1/2} \sum_{k=0}^{\infty} |\frac{k - n\mu}{\sigma\sqrt{n}}|^{1/2} |\frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \exp\{-\frac{1}{2}(k-n\mu)^2/(n\sigma^2)\} - P(S_n = k)| \\
& +(\sigma\sqrt{n})^{1/2} \sum_{k=0}^{\infty} |\frac{k - n\mu}{\sigma\sqrt{n}}|^{1/2} \frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} |\exp\{-\frac{1}{2}(k+1-n\mu)^2/(n\sigma^2)\} \\
& - \exp\{-\frac{1}{2}(k-n\mu)^2/(n\sigma^2)\}|.
\end{aligned}$$

Note that

$$\left(1 + |\frac{k - n\mu}{\sigma\sqrt{n}}|^3\right) > |\frac{k - n\mu}{\sigma\sqrt{n}}|^{1/2} \text{ for all } k \ge 0 \text{ and all } n \ge 0.$$

This shows that the first two sums are of order $O(1)$ as we expected. Now we consider the last sum:

$$\begin{aligned}
(\sigma\sqrt{n})^{1/2} \frac{1}{\sigma(2\pi n)^{\frac{1}{2}}} \sum_{k=0}^{\infty} |\frac{k - n\mu}{\sigma\sqrt{n}}|^{1/2} |\exp\{-\frac{1}{2}(k+1-n\mu)^2/(n\sigma^2)\} \\
- \exp\{-\frac{1}{2}(k-n\mu)^2/(n\sigma^2)\}|.
\end{aligned}$$

Let $t_{n,k} = \frac{k-n\mu}{\sigma\sqrt{n}}$ and let $\Delta t_{n,k} = t_{n,k+1} - t_{n,k} = \frac{1}{\sigma\sqrt{n}}$. The last sum is

$$\begin{aligned}
& \frac{\sqrt{\sigma\sqrt{n}}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{1}{\sigma\sqrt{n}} |t_{n,k}|^{1/2} \left| e^{-\frac{1}{2}t_{n,k+1}^2} - e^{-\frac{1}{2}t_{n,k}^2} \right| \\
= \ & \frac{\sqrt{\sigma\sqrt{n}}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \left(t_{n,k+1} - t_{n,k}\right) |t_{n,k}|^{1/2} \left| e^{-\frac{1}{2}\left(t_{n,k} + \frac{1}{\sigma\sqrt{n}}\right)^2} - e^{-\frac{1}{2}t_{n,k}^2} \right|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{\sigma\sqrt{n}}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \Big(\Delta t_{n,k}\Big) |t_{n,k}|^{1/2} \Big| e^{-\frac{1}{2}t_{n,k}^2} \Big(e^{-\frac{1}{2}\Big(2t_{n,k}\Delta t_{n,k} + \Big(\Delta t_{n,k}\Big)^2\Big)} - 1 \Big) \Big| \\
&= \sqrt{\sigma\sqrt{n}} \sum_{k=0}^{\infty} \Big(\Delta t_{n,k}\Big) |t_{n,k}|^{1/2} \phi\Big(t_{n,k}\Big) \Big| e^{t_{n,k}\Delta t_{n,k}} e^{-\frac{1}{2}\Big(\Delta t_{n,k}\Big)^2} - 1 \Big| \\
&= \sqrt{\sigma\sqrt{n}} \sum_{k=0}^{\infty} \Big(\Delta t_{n,k}\Big) |t_{n,k}|^{1/2} \phi\Big(t_{n,k}\Big) \Big| \sum_{j=0}^{\infty} \frac{\Big(-t_{n,k}\Delta t_{n,k} - \frac{\Big(\Delta t_{n,k}\Big)^2}{2}\Big)^j}{j!} - 1 \Big| \\
&\leq \sqrt{\sigma\sqrt{n}} \sum_{k=0}^{\infty} \Big(\Delta t_{n,k}\Big) |t_{n,k}|^{1/2} \phi\Big(t_{n,k}\Big) \sum_{j=0}^{\infty} \Big(\Delta t_{n,k}\Big) \frac{\Big(|t_{n,k}| + 1\Big)^j}{j!} \\
&\leq \frac{\sqrt{\sigma\sqrt{n}}}{\sigma\sqrt{n}} \sum_{k=0}^{\infty} \Big(\Delta t_{n,k}\Big) |t_{n,k}|^{1/2} \phi\Big(t_{n,k}\Big) e^{1+|t_{n,k}|} \\
&\sim \frac{1}{\sqrt{\sigma\sqrt{n}}} \int_{-\infty}^{\infty} |t|^{1/2} e^{|t|+1} \phi(t)\,dt \\
&= O\Big(\frac{1}{n^{1/4}}\Big).
\end{aligned}
$$

This now concludes the proof of the first half of the theorem.

Since E_α with $\alpha > 0$ are members of the convolution method and as proved in Lorentz [7], $\Omega(n) = o(\sqrt{n})$ are the only summability functions of the method E_α, one cannot enlarge the class of summability functions over the space of convolution methods under consideration. This concludes the sharpness of the result.

∎

Remark 2 The summability functions for the following methods were obtained by Lorentz [7], of which $\Omega(n) = o(\sqrt{n})$ for the Euler E_p method agrees with the above theorem.

1. For the $(C,1)$ method; $\Omega(n) = o(n)$. As the methods (C,α) $(\alpha > 0)$ and the Abel method $\mathcal{A}$ are equivalent to the $(C,1)$ method for bounded sequences, they also have the same summability functions.

2. For the Euler E_p method; $\Omega(n) = o(\sqrt{n})$.

Let $R_{n,j}$ be the weight of the random-walk method. As usual, by writing μ, σ^2 for the mean and variance of the sequence of i.i.d. random variables with finite third moment, we obtain

$$\max_j |R_{n,j}| = O\Big(\frac{1}{n^{1/2}}\Big) \text{ as } n \to \infty$$

as follows from

$$R_{n,j} = P(X_1 + X_2 + \ldots + X_n = j) = \frac{1}{\sigma(2\pi n)^{1/2}} \exp\{\frac{-1}{2}(j - n\mu)^2/n\sigma^2\} + o(1/\sqrt{n})$$

uniformly in j [2]. Hence, the set of all random-walk methods with finite third moment is contained in the class $\mathcal{U}$ of the matrix method. The following corollary can be easily drawn from the above theorem.

Corollary 1 *Let $\{X_i, i \geq 1\}$ be independent and identically distributed aperiodic nonnegative integer-valued random variables with finite third moment and positive variance. Then for the matrix transformation corresponding to the regular random-walk method, $\{R_{n,j}\}$; any function $0 < \Omega(n) \uparrow \infty$ of the form $\Omega(n) = o(\sqrt{n})$ gives the possible summability functions. Furthermore, $0 < \Omega(n) = o(\sqrt{n})$ with $\Omega(n) \uparrow \infty$ are the only functions which are summability functions over the class of regular random-walk methods.*

4 Absolute Summability Functions

Definition of the summability functions has been improved by introducing the concept of absolute summability functions.

Definition 6 Let $\Omega(n)$ be a non-decreasing positive function which tends to $+\infty$ with n. We say that $\Omega(n)$ is an absolute summability function of a summability matrix $A = \{a_{n,k}\}_{n,k\geq 0}$, if any bounded sequence $\{f(k),\ k \geq 0\}$ for which $f(k) = 0$ except for a subsequence $\{n_\nu\}$ with the counting function $A(n, f) \leq \Omega(n)$ is absolutely A-summable, that is, $\sum_{n=0}^{\infty} |\sigma_n - \sigma_{n-1}| < +\infty$ for any such sequence, where $\sigma_n = \sum_{k=0}^{\infty} a_{n,k} f(k)$ for $n \geq 0$.

Theorem 7 of Lorentz [9] addresses question of the existence of absolute summability functions.

Theorem 8 *(Lorentz [9]) The method of summation A generated by the matrix $A = \{a_{n,k}\}_{n,k\geq 0}$ for which $\sum_{k=0}^{\infty} |a_{0,k}| < +\infty$ has absolute summability functions if and only if the variation of the $k-$th column $V_k = var_n\ a_{n,k}$ defined by $\sum_{n=0}^{\infty} |a_{n+1,k} - a_{n,k}|$ converges to 0 for $k \to \infty$.*

As we will show below, a regular convolution summability method that has been considered in the preceding sections has this structure. Hence, according to theorem 7 of Lorentz [9], a regular convolution summability method under consideration has absolute summability functions. Since the moment generating function (mgf) may exist for some real arguments but not all, we simply insist that the characteristic function is to be entire (analytic) in the results to follow [10]. For the probabilistic relevance of the mgf condition, see [3].

Theorem 9 *Let $Y, \{X_i, i \geq 1\}$ be independent with $E(Y) = \mu_Y < \infty$, and $\{X_i, i \geq 1\}$ be identically distributed aperiodic nonnegative integer-valued random variables with characteristic function is analytic. The matrix transformation corresponding to the regular convolution summability method $C = \{C_{n,k}\}_{n,k\geq 0}$ has absolute summability functions.*

Proof.

First, we verify that

$$\sum_{k=0}^{\infty} |C_{0,k}| = \sum_{k=0}^{\infty} P(Y = k) = \sum_{k=0}^{\infty} q_k = 1 < \infty.$$

This means that the method already satisfies the condition of hypotheses.
Let $S_n = X_1 + X_2 + \ldots + X_n$ and $S_0 = 0$. We now consider,

$$\begin{aligned}
& \sum_{n=0}^{\infty} |C_{n+1,k} - C_{n,k}| \\
= & \sum_{n=0}^{\infty} |P(Y + S_{n+1} = k) - P(Y + S_n = k)| \\
= & \sum_{n=0}^{\infty} |\sum_{j=0}^{k+1} q_j P(S_n = k+1-j) - \sum_{j=0}^{k} q_j P(S_n = k-j)| \\
\leq & \sum_{n=0}^{\infty} |\sum_{j=0}^{k} |P(S_n = k+1-j) - P(S_n = k-j)| + \sum_{n=0}^{\infty} q_{k+1} P(S_n = 0) \\
\leq & \sum_{j=0}^{k} q_j \sum_{n=0}^{\infty} |P(S_n = k+1-j) - P(S_n = k-j)| + q_{k+1} \sum_{n=0}^{\infty} (p_0)^n \\
= & \sum_{j=0}^{k} q_j \sum_{n=0}^{\infty} |P(S_n = k+1-j) - P(S_n = k-j)| + q_{k+1} \Big(\frac{1}{1-p_0}\Big).
\end{aligned}$$

The last term on the right is $o_k(1)$ as the method is regular ($p_0 < 1$,) and $\sum_{k=0}^{\infty} q_k = 1$. The convergence of first (other) sum is evident from Theorem 4 of Kesten [5]. Now, we show that this sum is in fact $o_k(1)$, where k denote that order notation has taken as $k \to \infty$. For,

$$I_k = \sum_{j=0}^{k} q_j \sum_{n=0}^{\infty} |P(S_n = k+1-j) - P(S_n = k-j)| + o_k(1),$$

using the Chung-Erdõs inequality cited in page 706 of Kesten [5], of the form: If for some integer $k \geq 0, a,\ P(S_k = a)P(S_{k+m} = a + j) > 0$ holds, then for every $\epsilon > 0$ there exists a $\delta > 0$ such that sufficiently large n,

$$\begin{aligned}
P(S_n = i_n) & \leq (1+\epsilon) P(S_{n+m} = i_n + j) + e^{-\delta n} \text{ and} \\
P(S_{n+m} = i_n + j) & \leq (1+\epsilon) P(S_n = i_n) + e^{-\delta n}.
\end{aligned}$$

Thus, for $\delta_1 \neq \delta_2$, we have I_k

$$\begin{aligned}
\leq & \sum_{j=0}^{k} q_j \sum_{n=0}^{\infty} |(1+\epsilon_1) P(S_{n+k-j} = 0) + e^{-\delta_1(n+k-j)} - (1+\epsilon_2) P(S_{n+k-j} = 0) - e^{-\delta_2(n+k-j)}| + o_k(1) \\
\leq & \sum_{j=0}^{k} q_j \Big\{ |\epsilon_1 - \epsilon_2| \sum_{n=0}^{\infty} (p_0)^{n+k-j} + \Big| \sum_{n=0}^{\infty} (e^{-\delta_1(n+k-j)} - \sum_{n=0}^{\infty} (e^{-\delta_2(n+k-j)} \Big| \Big\} + o_k(1) \\
\leq & |\epsilon_1 - \epsilon_2| (p_0)^k \sum_{j=0}^{k} q_j p_0^{-j} \Big(\frac{1}{1-p_0}\Big) + \sum_{j=0}^{k} q_j e^{\max(\delta_1,\delta_2) j} \Big| \sum_{n=0}^{\infty} \Big\{ e^{-\delta_1 n} - e^{-\delta_2 n} \Big\} \Big| \Big(e^{-\min(\delta_1,\delta_2)} \Big)^k + o_k(1)
\end{aligned}$$

$$\leq \quad |\epsilon_1 - \epsilon_2|(p_0)^k \sum_{j=0}^{k} q_j p_0^{-j} \left(\frac{1}{1-p_0}\right) + \sum_{j=0}^{k} q_j e^{\max(\delta_1,\delta_2)j} \left| \frac{1}{1-e^{-\delta_1}} - \frac{1}{1-e^{-\delta_2}} \right| \left(e^{-\min(\delta_1,\delta_2)}\right)^k + o_k(1).$$

This gives $I_k = o_k(1)$ with characteristic function of $\{X_i, i \geq 1\}$ is analytic.
∎

In analogy to Theorem 9 of Lorentz [9], we prove the following generalization for the convolution summability method; which now includes Taylor and Meyer-König summability methods and many others. We will use most of the preliminary facts from Lorentz [9] and avoid further discussions as we proceed.

Theorem 10 *A function $\Omega(n)$ is an absolute summability function of the regular convolution summability method generated by a sequence of $Y, \{X_i, i \geq 1\}$ of independent with $E(Y) = \mu_Y < \infty$, and $\{X_i, i \geq 1\}$ of identically distributed aperiodic non negative integer-valued random variables with characteristic function is analytic, if and only if*

$$\sum_{n=1}^{\infty} n^{-3/2} \Omega(n) < +\infty.$$

Proof.
The sufficiency of the theorem is proved as follows: Let $\{C_{n,k}\}_{n,k\geq 1}$ be the convolution summability matrix corresponding to the given sequence of random variables. From Theorem 9 the variation of the $k-$th column is

$$V_k = \mathrm{var}_n\ C_{n,k} = \sum_{n=0}^{\infty} |C_{n+1,k} - C_{n,k}| = \alpha(p_0)^k + O\left(e^{-\beta k}\right) + \gamma q_{k+1}$$

for some positive constants α, β, and γ as follows from the proof of Theorem 9. If $\{n_\nu\}$ is a sequence of integers with the counting function $\omega(n) \leq \Omega(n)$, we have $\sum_{\nu=1}^{\infty} k_\nu^{-\beta} < \infty$ by Lemma 2 (p. 247 of Lorentz [9]), and noting that $e^{-\beta k_\nu} < k_\nu{}^{-\beta}$, we see that

$$\sum_{\nu=1}^{\infty} \mathrm{var}_n C_{n,k_\nu} = \sum_{\nu=1}^{\infty} \left\{ \alpha(p_0)^{k_\nu} + O\left(e^{-\beta k_\nu}\right) + \gamma q_{k_\nu+1} \right\} < \infty.$$

The later is the necessary and sufficient condition for $\Omega(n)$ to be an absolute summability function of the given matrix method (Theorem 6 of Lorentz [9]).

For the necessity, using the fact that Euler method E_t, $0 < t < 1$ and the Borel method B are members of the convolution summability methods, it suffices to proceed in the following manner. Suppose the series $\sum_{n=1}^{\infty} n^{-\frac{3}{2}} \Omega(n)$ be divergent. Then taking the Euler method E_t, $0 < t < 1$ or the Borel method B, we have that the series is convergent for either of these methods. This contradiction concludes the assertion.
∎

Most of the discussions and remarks appeared in [7], [8], [9], and [12] now follow without further proofs and hold for the random-walk method and all members of the convolution summability method.

5 Acknowledgments

This paper stems from the author's Ph.D. dissertation [4] at Kent State University under Professor M. Kazim Khan. The author wishes to thank him for proposing this topic for further exploration with some pertinent insights to the problem in hand. Other readers have helped to improve this paper to a greater extent. Thanks are also due to the referee for his(her) valuable comments, suggestions and submitting some writing errors on earlier versions of this manuscript. They were immensely helpful to improve this paper.

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2000 *Mathematics Subject Classification*: Primary 40A05; Secondary 40C05, 42B08, 43A99 .

Keywords: absolute summability functions, almost convergence, convolution summability method, random-walk method, regularity, summability functions, strong regularity.

Received September 11 2003. revised version received August 18 2004. Published in *Journal of Integer Sequences*, September 30 2004.

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