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# A Combinatorial Proof for the Generating Function of Powers of a Second-Order Recurrence Sequence

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**Abstract**

In this paper, we derive a formula for the generating function of powers of a second-order linear recurrence sequence, with initial conditions 0 and 1. As an example, we find the generating function of the powers of the nonnegative integers. We also find new formulas for computing Eulerian polynomials.

## 1 Introduction

Second-order linear recurrence sequences have been studied for hundreds of years. One of the earlier historical summaries was done by Dickson [1, Vol. 1, Chapter 17, pp. 393–411].

In the present paper, we use the notation $W_{n;(a,b;p,q)}$ to define the second-order linear recurrence sequence,

$$W_{n+2;(a,b;p,q)} = pW_{n+1;(a,b;p,q)} + qW_{n;(a,b;p,q)},$$

having initial conditions

$$W_{0;(a,b;p,q)} = a,\ W_{1;(a,b;p,q)} = b.$$

**Example 1.** The nonnegative integers are represented by $W_{n;(0,1;2,-1)}$.

Domino tiling methods have been used to find combinatorial identities and recurrence relations of certain sequences. Sellers [6] proved that the number of domino tilings of the graph $W_4 \times P_{n-1}$ equals $F_n P_n$ (sequence A001582 in Sloane's *On-Line Encyclopedia of Integer Sequences* [7]), which is the product of the $n$-th Fibonacci number and the $n$-th Pell number. Katz and Stenson [3] considered the number of tilings of a $(2 \times n)$-board A030186.

Let $F_k(x)$ be the generating function for the $k$-th powers of the Fibonacci numbers, defined by

$$F_k(x) = \sum_{n=0}^{\infty} F_n^k x^n,\ \ k \in \mathbb{N}^+.$$

For $1 \leq k \leq 10$, $F_k(x)$ can be found respectively from A000045, A007598, A056570, A056571, A056572, A056573, A056574, A056585, A056586, and A056587 in Sloane's *On-Line Encyclopedia of Integer Sequences* [7] (OEIS).

In 1962, Riordan [5] gave the following recurrence relation,

$$(1 - L_k(x) + (-1)^k x^2)F_k(x) = 1 + kx \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \frac{A_{kj}}{j} F_{k-2j}\left((-1)^j x\right),$$

where $L_k$ is the $k$-th Lucas number A000032. In 1999, Dujella [2] showed the doubly-indexed sequence $A_{kj}$, has generating function given by,

$$(1 - x - x^2)^{-j} = \sum_{k=2j}^{\infty} A_{kj} x^{k-2j},\ \ j \geq 0.$$

In 2003, Stănică [8] obtained a closed form for generating function of the non-degenerate second-order recurrence relation,

$$U_{n+2} = aU_{n+1} + bU_n,\ \ a,\ b,\ U_0, U_1 \in \mathbb{Z},$$

such that $\delta = a^2 + 4b \neq 0$. Let $\alpha = \frac{1}{2}(a + \sqrt{a^2+4b})$, $\beta = \frac{1}{2}(a - \sqrt{a^2+4b})$ and $A = \frac{U_1 - U_0\beta}{\alpha-\beta}$, $B = \frac{U_1 - U_0\alpha}{\alpha-\beta}$, $V_n = \alpha^n + \beta^n$ with initial conditions $V_0 = 2$, $V_1 = a$, and let $U_k(x) = \sum_{i=0}^{\infty} U_i^k x^i$.

If $k$ is odd, then

$$U_k(x) = \sum_{j=0}^{\frac{k-1}{2}} (-AB)^j \binom{k}{j} \frac{A^{k-2j} - B^{k-2j} + (-b)^j((B\alpha)^{k-2j} - (A\beta)^{k-2j})x}{1 - (-b)^j V_{k-2j} x - b^k x^2}.$$

If $k$ is even, then

$$U_k(x) = \sum_{j=0}^{\frac{k}{2}-1} (-AB)^j \binom{k}{j} \frac{B^{k-2j} + A^{k-2j} - (-b)^j((B\alpha)^{k-2j} + (A\beta)^{k-2j})x}{1 - (-b)^j V_{k-2j} x + b^k x^2} + \binom{k}{\frac{k}{2}} \frac{(-AB)^{\frac{k}{2}}}{1 - (-b)^{\frac{k}{2}} x}.$$

In 2004, Mansour [4] obtained a formula for the generating functions of powers of second-order recurrence sequence in terms of the determinants of certain matrices, given by

$$W_{n;(a,b;p,q)}(x) = \frac{\det(\delta_k)}{\det(\Delta_k)},$$

where

$$\begin{aligned}
\Delta_k &= \begin{bmatrix} A_{1\times 1} & C_{1\times(k-1)} \\ B_{(k-1)\times 1} & D_{(k-1)\times(k-1)} \end{bmatrix} \\
\delta_k &= \begin{bmatrix} E_{1\times 1} & C_{1\times(k-1)} \\ F_{(k-1)\times 1} & D_{(k-1)\times(k-1)} \end{bmatrix} \\
A_{1\times 1} &= \begin{bmatrix} 1 - p^k x - q^k x^2 \end{bmatrix} \\
E_{1\times 1} &= \begin{bmatrix} a^k + g_k x \end{bmatrix}, \\
B_{(k-1)\times 1} &= \begin{bmatrix} -p^{k-1}x & -p^{k-2}x & \cdots & -p^1 x \end{bmatrix}^T \\
F_{(k-1)\times 1} &= \begin{bmatrix} g_{k-1}x & g_{k-2}x & \cdots & g_1 x \end{bmatrix}^T, \\
C_{1\times(k-1)} &= \begin{bmatrix} -xp^{k-1}q^1\binom{k}{1} & -xp^{k-2}q^2\binom{k}{2} & \cdots & -xp^1q^{k-1}\binom{k}{k-1} \end{bmatrix}, \\
D_{(k-1)\times(k-1)} &= \begin{bmatrix} 1 - xp^{k-2}q^1\binom{k-1}{1} & -xp^{k-3}q^2\binom{k-1}{2} & \cdots & -xq^{k-1}\binom{k-1}{k-1} \\ -xp^{k-3}q^1\binom{k-2}{1} & 1 - xp^{k-4}q^2\binom{k-2}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -xq^1\binom{1}{1} & 0 & \cdots & 1 \end{bmatrix},
\end{aligned}$$

with $g_j = (b^j - a^j p^j)a^{k-j}$, $j = 1, 2, \ldots, k$.

In 2017, Zhang and Grossman [9] used a counting method to relate the number of tilings of a $(k \times n)$-board to the generating function of the $k$-th power of the Fibonacci numbers.

In the present paper, we extend the previous result to the generating function of the $k$-th power of an arbitrary second-order linear recurrence sequence having initial conditions 0 and 1.

Eulerian polynomials, denoted in the present paper by $A_k(x)$, were introduced by Euler in 1755 and can be determined from the identity

$$\sum_{n=0}^{\infty} n^k x^n = \frac{xA_k(x)}{(1-x)^{k+1}}.$$

The coefficients of Eulerian polynomials leads to the Eulerian numbers A008292. The recurrence relation for nonnegative integers denoted $a_n,\ n \geq 0$ is given by

$$a_{n+2} = 2a_{n+1} - a_n,\ a_0 = 0,\ a_1 = 1.$$

In the next section we derive an expression for powers of $W_{n;(0,1;p,q)}$ with arbitrary $p \neq 0$, $q$. Sequences associated with such generating functions are Pell, Jacobsthal and the nonnegative integers. Closed-form expressions for Eulerian polynomials are also given in Corollary 23 and Remark 26.

# 2 Closed form for $\mathcal{W}_{k;(0,1;p,q)}(x)$

The Fibonacci number $F_{n+1}$ gives the number of ways that $1 \times 2$ dominoes and squares can cover a $1 \times n$ checkerboard. We begin with a series of definitions.

**Definition 2.** Let $\mathcal{W}_{k;(0,1;p,q)}(x) = \sum_{n=0}^{\infty} W_n^k x^n$ be the generating function of $k$-th power of $W_n$ where $W_{n+2} = pW_{n+1} + qW_n$ with initial conditions $W_0 = 0$ and $W_1 = 1$ with $p, q \in \mathbb{R}$, $p \neq 0$.

In this section, we use the notation $(p, q)$ in the place of $(0, 1; p, q)$.

**Definition 3.** For $k \geq 1$ and $n \geq 1$, let $F_{k\times n;(p,q)}$ denote the *Fibonacci* $(k \times n; (p, q))$ *checkerboard*, which is a *checkerboard* (or simply, board) with height $k$ and length $n$, covered by squares and dominoes, such that the dominoes can only be placed horizontally. Let $\mathcal{F}_{k\times n;(p,q)}$ be the set of all $F_{k\times n;(p,q)}$ boards. Define a function $V\colon \mathcal{F}_{k\times n;(p,q)} \to \mathbb{R}$ (value of $F_{k\times n;(p,q)}$) by $V(F_{k\times n;(p,q)}) = p^s q^d$, where $d$ and $s$ denote the number of dominoes and squares respectively in this $F_{k\times n;(p,q)}$ board. Let us also redefine the function $V\colon 2^{\mathcal{F}_{k\times n;(p,q)}} \to \mathbb{R}$ by $V(A) = \sum_{a\in A} V(a)$. Let $w_{k\times n;(p,q)} = V(\mathcal{F}_{k\times n;(p,q)})$ be the sum of all the values of the different $F_{k\times n;(p,q)}$ boards.

Note that each $F_{k\times n;(p,q)}$ board is comprised of $k$ $F_{1\times n;(p,q)}$ boards (horizontal layers), each of which has at most $n$ components consisting of dominoes and squares.

We define $w_{k\times(-1);(p,q)} = 0$ and $w_{k\times 0;(p,q)} = 1$ for $k \in \mathbb{N}^+$.

**Definition 4.** The *Fibonacci* $(k \times n; (p, q))$ *minimal checkerboard* called an $M_{k\times n;(p,q)}$ *board*, is an $F_{k\times n;(p,q)}$ board which cannot be vertically divided into two Fibonacci checkerboards. Let $\mathcal{M}_{k\times n;(p,q)}$ be the set of all $M_{k\times n;(p,q)}$ boards. Let $m_{k\times n;(p,q)} = V(\mathcal{M}_{k\times n;(p,q)})$ be the sum of all the values of the different $M_{k\times n;(p,q)}$ boards.

Note that $\mathcal{M}_{k\times n;(p,q)} \subseteq \mathcal{F}_{k\times n;(p,q)}$.

**Example 5.** There are total of nine different $F_{2\times 3;(p,q)}$ boards shown in Figure 1. We find that $w_{2\times 3;(p,q)} = p^6 + 4p^4q + 4p^2q^2$. There are two different $M_{2\times 3;(p,q)}$ boards and $m_{2\times 3;(p,q)} = 2p^2q^2$.

**Lemma 6.** *For $n \geq 1$ and $k \geq 1$, $w_{1\times n;(p,q)} = W_{n+1;(p,q)}$ and $w_{k\times n;(p,q)} = W_{n+1;(p,q)}^k$.*

*Proof.* This lemma can be proven inductively.

If $n = 1$, then $w_{1\times 1;(p,q)} = p = W_{2;(p,q)}$. If $n = 2$, then $w_{1\times 2;(p,q)} = p^2 + q = W_{3;(p,q)}$.

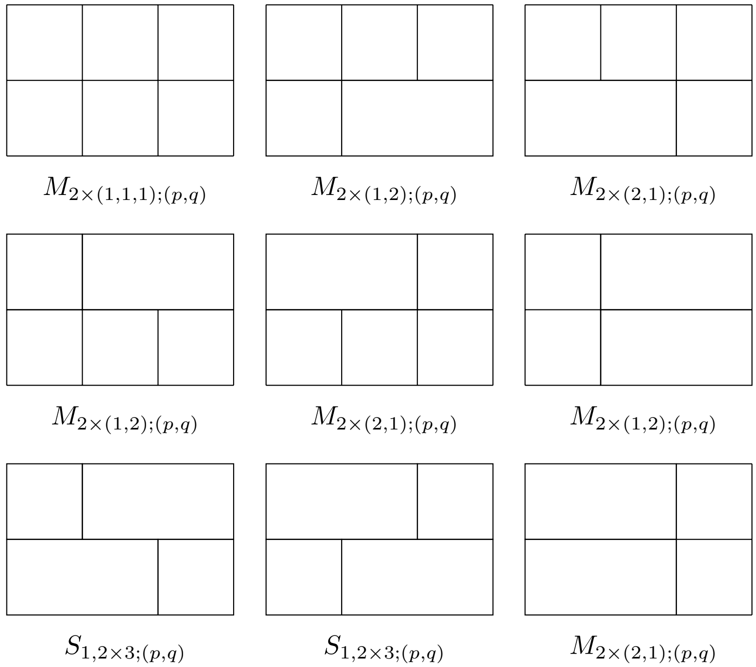



Figure 1: Nine $F_{2\times3;(p,q)}$ boards.

If $w_{1\times n;(p,q)} = W_{n+1;(p,q)}$ and $w_{1\times(n+1);(p,q)} = W_{n+2;(p,q)}$ hold, then

$$w_{1\times(n+2);(p,q)} = pw_{1\times(n+1);(p,q)} + qw_{1\times n;(p,q)} = pW_{n+2;(p,q)} + qW_{n+1;(p,q)} = W_{n+3;(p,q)}.$$

Therefore $w_{1\times n;(p,q)} = W_{n+1;(p,q)}$ for $n \geq 1$.

We know that the $F_{k\times n;(p,q)}$ board has $k$ independent layers. Also, each independent layer is an $F_{1\times n;(p,q)}$ board which implies that the sum of all the values of each layer is $w_{1\times n;(p,q)}$. Therefore, we can conclude that $w_{k\times n;(p,q)} = W^k_{n+1;(p,q)}$. $\square$

The generating function for powers of Pell numbers (A000129, A079291, A110272) is denoted $P_k(x) = \mathcal{W}_{k;(2,1)}(x)$; powers of Jacobsthal numbers (A001045, A139818), is denoted by $J_k(x) = \mathcal{W}_{k;(1,2)}(x)$; and powers of nonnegative integers (A001477, A000290, A000578, A000583, A000584, A001014, A001015, A001016, A001017, and A008454), is denoted by $\mathcal{W}_{k;(2,-1)}(x)$.

In Table 1 we listed values of $m_{4\times n;(2,-1)}$ for $1 \leq n \leq 7$. The smaller values were obtained by direct calculation, larger values were computed using Lemma 19. These values will be used to explain Example 10 and Example 12.

*Remark* 7. Any non-minimal $F_{k\times n;(p,q)}$ board can be split uniquely into at most $n$ Fibonacci minimal boards. Specifically, there exist $j$, $1 < j \leq n$, and an ordered partition $n_1 + n_2 + \cdots + n_j = n$, such that the $F_{k\times n;(p,q)}$ board can be split into $M_{k\times n_1;(p,q)}$, $M_{k\times n_2;(p,q)}$, …, $M_{k\times n_j;(p,q)}$ boards (from left to right).

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $m_{4\times n;(2,-1)}$ | 16 | $-175$ | 1760 | $-17456$ | 172832 | $-1710896$ | 16936160 |

Table 1: $m_{4\times n;(2,-1)}$ for $1 \leq n \leq 7$.

**Definition 8.** Let an $M_{k\times(n_1,n_2,\ldots,n_j);(p,q)}$ *board* be an $F_{k\times n;(p,q)}$ board with $n_1+n_2+\cdots+n_j = n$ that can be split into $M_{k\times n_1;(p,q)}$, $M_{k\times n_2;(p,q)}$, …, $M_{k\times n_j;(p,q)}$ boards. Let $\mathcal{M}_{k\times(n_1,n_2,\ldots,n_j);(p,q)}$ be the set of all different $M_{k\times(n_1,n_2,\ldots,n_j);(p,q)}$ boards. Let

$$m_{k\times(n_1,n_2,\ldots,n_j);(p,q)} = V(\mathcal{M}_{k\times(n_1,n_2,\ldots,n_j);(p,q)})$$

be the sum of all the values of the different $M_{k\times(n_1,n_2,\ldots,n_j);(p,q)}$ boards.

The value of $m_{k\times(n_1,n_2,\ldots,n_j);(p,q)}$ is quantified in Lemma 9. Note that there are two examples in Figure 1.

**Lemma 9.** *For $k \geq 1$ and $n \geq 1$, $n_i \in \mathbb{N}^+$ for $i \in \{1, 2, \ldots, j\}$ we have,*

$$m_{k\times(n_1,n_2,\ldots,n_j);(p,q)} = \prod_{i=1}^{j} m_{k\times n_i;(p,q)},$$

*and*

$$W^k_{n+1;(p,q)} = \sum_{j=1}^{n} \sum_{n_1+n_2+\cdots+n_j=n} \prod_{i=1}^{j} m_{k\times n_i;(p,q)}.$$

*Proof.* Since an $M_{k\times(n_1,n_2,\ldots,n_j);(p,q)}$ board is the combination of $M_{k\times n_1;(p,q)}$, $M_{k\times n_2;(p,q)}$, …, $M_{k\times n_j;(p,q)}$ boards, we have $m_{k\times(n_1,n_2,\ldots,n_j);(p,q)} = \prod_{i=1}^{j} m_{k\times n_i;(p,q)}$.

From Remark 7, each $F_{k\times n;(p,q)}$ board is either an $M_{k\times n;(p,q)}$ board or can be uniquely divided into $j$ minimal boards where $2 \leq j \leq n$. Then the set

$$\Big\{\mathcal{M}_{k\times n;(p,q)}, \bigcup_{n_1+n_2=n} \mathcal{M}_{k\times(n_1,n_2);(p,q)}, \ldots, \bigcup_{n_1+n_2+\cdots+n_n=n} \mathcal{M}_{k\times(n_1,n_2,\ldots,n_n);(p,q)}\Big\}$$

forms a partition of $\mathcal{F}_{k\times n;(p,q)}$. That is,

$$\mathcal{F}_{k\times n;(p,q)} = \mathcal{M}_{k\times n;(p,q)} \bigcup \left( \bigcup_{n_1+n_2=n} \mathcal{M}_{k\times(n_1,n_2);(p,q)} \right) \bigcup \cdots \bigcup \left( \bigcup_{n_1+n_2+\cdots+n_n=n} \mathcal{M}_{k\times(n_1,n_2,\ldots,n_n);(p,q)} \right).$$

Therefore, from Lemma 6,

$$\begin{aligned}
W^k_{n+1;(p,q)} &= w_{k\times n;(p,q)} \\
&= V(\mathcal{F}_{k\times n;(p,q)}) \\
&= V(\mathcal{M}_{k\times n;(p,q)}) + V\left(\bigcup_{n_1+n_2=n} \mathcal{M}_{k\times(n_1,n_2);(p,q)}\right) + \cdots \\
&\quad +V\left(\bigcup_{n_1+n_2+\cdots+n_n=n} \mathcal{M}_{k\times(n_1,n_2,\ldots,n_n);(p,q)}\right) \\
&= m_{k\times n;(p,q)} + \sum_{n_1+n_2=n} m_{k\times(n_1,n_2);(p,q)} + \cdots + \sum_{n_1+n_2+\cdots+n_n=n} m_{k\times(n_1,n_2,\ldots,n_n);(p,q)} \\
&= m_{k\times n;(p,q)} + \sum_{n_1+n_2=n} \prod_{i=1}^{2} m_{k\times n_i;(p,q)} + \cdots + \sum_{n_1+n_2+\cdots+n_n=n} \prod_{i=1}^{n} m_{k\times n_i;(p,q)} \\
&= \sum_{j=1}^{n} \sum_{n_1+n_2+\cdots+n_j=n} \prod_{i=1}^{j} m_{k\times n_i;(p,q)}.
\end{aligned}$$

$\square$

**Example 10.**

$$\begin{aligned}
W^4_{4;(2,-1)} &= m_{4\times 3;(2,-1)} + m_{4\times(2,1);(2,-1)} + m_{4\times(1,2);(2,-1)} + m_{4\times(1,1,1);(2,-1)} \\
&= 1760 + (-175)\cdot 16 + 16\cdot(-175) + 16\cdot 16\cdot 16 \\
&= 256 \\
&= 4^4.
\end{aligned}$$

**Lemma 11.** *For $k \in \mathbb{N}^+$, let $e_{k;(p,q)} = \sum_{n=1}^{\infty} m_{k\times n;(p,q)} x^n$. Then,*

$$\mathcal{W}_{k;(p,q)}(x) = \frac{x}{1 - e_{k;(p,q)}}.$$

*Proof.* For $k, n \in \mathbb{N}^+$, the coefficient of $x^n$ in $e_{k;(p,q)} + e^2_{k;(p,q)} + \cdots + e^n_{k;(p,q)} + \cdots$ is

$$\begin{aligned}
& m_{k\times n;(p,q)} + \sum_{\substack{n_1+n_2=n \\ n_i\in\mathbb{N}^+}} \prod_{i=1}^{2} m_{k\times n_i;(p,q)} + \sum_{\substack{n_1+n_2+n_3=n \\ n_i\in\mathbb{N}^+}} \prod_{i=1}^{3} m_{k\times n_i;(p,q)} + \cdots \\
& + \sum_{\substack{n_1+n_2+\cdots+n_n=n \\ n_i\in\mathbb{N}^+}} \prod_{i=1}^{n} m_{k\times n_i;(p,q)} \\
= & \sum_{j=1}^{n} \sum_{\substack{n_1+n_2+\cdots+n_j=n \\ n_i\in\mathbb{N}^+}} \prod_{i=1}^{j} m_{k\times n_i;(p,q)}.
\end{aligned}$$

Also, $e_{k;(p,q)} + e^2_{k;(p,q)} + \cdots + e^n_{k;(p,q)} + \cdots = \frac{1}{1-e_{k;(p,q)}} - 1$. Therefore,

$$\sum_{n=1}^{\infty}\left(\left(\sum_{j=1}^{n}\sum_{\substack{n_1+n_2+\cdots+n_j=n\\ n_i\in\mathbb{N}^+}}\prod_{i=1}^{j} m_{k\times n_i;(p,q)}\right)x^n\right) = \frac{1}{1-e_{k;(p,q)}} - 1.$$

Employing Lemmas 6 and 9, we obtain,

$$\begin{aligned}
\mathcal{W}_{k;(p,q)}(x) &= \sum_{n=0}^{\infty} W^k_{n;(p,q)}x^n \\
&= W^k_{0;(p,q)} + W^k_{1;(p,q)}x + \sum_{n=2}^{\infty} W^k_{n;(p,q)}x^n \\
&= x + \sum_{n=1}^{\infty} W^k_{n+1;(p,q)}x^{n+1} \\
&= x + x\sum_{n=1}^{\infty} W^k_{n+1;(p,q)}x^{n} \\
&= x + x\sum_{n=1}^{\infty}\left(\left(\sum_{j=1}^{n}\sum_{\substack{n_1+n_2+\cdots+n_j=n\\ n_i\in\mathbb{N}^+}}\prod_{i=1}^{j} m_{k\times n_i;(p,q)}\right)x^n\right) \\
&= x + x\left(\frac{1}{1-e_{k;(p,q)}} - 1\right) \\
&= \frac{x}{1-e_{k;(p,q)}}.
\end{aligned}$$

□

**Example 12.** From Table 1, we have $e_{4;(2,-1)} = \sum_{n=1}^{\infty} m_{4\times n;(2,-1)}x^n = 16x - 175x^2 + 1760x^3 - \cdots$ and $\mathcal{W}_{4;(2,-1)}(x) = \frac{x}{1-e_{4;(2,-1)}}$.

Now we determine the closed form of $e_{k;(p,q)}$.

**Lemma 13.** *Suppose $S_{1,n}$, $S_{2,n}$, $\ldots$, $S_{m,n}$ are real-valued sequences such that $S_{i,n+1} = \sum_{j=1}^{m} a_{i,j}S_{j,n}$ for $1 \le i \le m$ and $n \ge 0$.*

*Let* $[a_{i,j}]_{m\times m} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m} \end{bmatrix}_{m\times m}$, $[S_{j,n}]_{m\times 1} = \begin{bmatrix} S_{1,n} \\ S_{2,n} \\ \vdots \\ S_{m,n} \end{bmatrix}_{m\times 1}$, $B_{m+1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^T_{1\times m}$.

*Then* $\sum_{j=1}^{m} S_{j,n} = B_{m+1}[a_{i,j}]^{n-l}_{m\times m}[S_{j,l}]_{m\times 1}$.

*Proof.* Since

$$[a_{i,j}]_{m\times m}^{n-l}[S_{j,l}]_{m\times 1} = [a_{i,j}]_{m\times m}^{n-l-1}[a_{i,j}]_{m\times m}[S_{j,l}]_{m\times 1} = [a_{i,j}]_{m\times m}^{n-l-1}[S_{j,l+1}]_{m\times 1} = \cdots = [S_{j,n}]_{m\times 1},$$

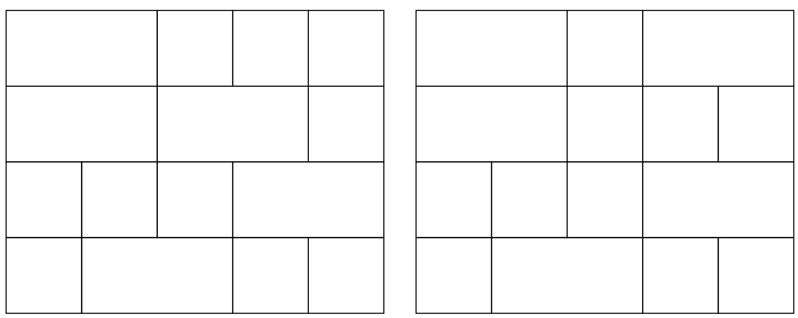
then

$$\sum_{j=1}^{m} S_{j,n} = B_{m+1}[S_{j,n}]_{m\times 1} = B_{m+1}[a_{i,j}]_{m\times m}^{n-l}[S_{j,l}]_{m\times 1}.$$

□

**Definition 14.** Let an $S_{j,k\times n;(p,q)}$ *board* ($j \in \{1,2,3,\ldots,k\}, n \geq 2$) be an $M_{k\times n;(p,q)}$ board with $j$ dominoes in last two columns. Let $\mathcal{S}_{j,k\times n;(p,q)}$ be the set of all $S_{j,k\times n;(p,q)}$ boards. Let $s_{j,k\times n;(p,q)} = V(\mathcal{S}_{j,k\times n;(p,q)})$ be the sum of all the values of the different $S_{j,k\times n;(p,q)}$ boards.

**Example 15.** In Figure 2, the board on the left is an $S_{1,4\times 5;(p,q)}$ board and the board on the right is a combination of an $S_{1,4\times 3;(p,q)}$ board and an $S_{2,4\times 2;(p,q)}$ board.



$M_{4\times 5;(p,q)}$, also $S_{1,4\times 5;(p,q)}$ $\qquad$ $M_{4\times (3,2);(p,q)}$

Figure 2: Two different boards.

**Lemma 16.** *For $k \in \mathbb{N}^+$, $s_{j,k\times 2;(p,q)} = p^{2(k-j)}q^j\binom{k}{j}$ for $1 \leq j \leq k$ and*

$$m_{k\times n;(p,q)} = \begin{cases} p^k, & \text{if } n = 1; \\ q^k + \sum_{j=1}^{k-1} s_{j,k\times 2;(p,q)}, & \text{if } n = 2; \\ \sum_{j=1}^{k-1} s_{j,k\times n;(p,q)}, & \text{if } n \geq 3. \end{cases}$$

*Proof.* For $n \geq 3$, $\mathcal{S}_{1,k\times n;(p,q)}$, $\mathcal{S}_{2,k\times n;(p,q)}$, …, $\mathcal{S}_{k-1,k\times n;(p,q)}$ forms a partition of $\mathcal{M}_{k\times n;(p,q)}$. Therefore,

$$m_{k\times n;(p,q)} = V(\mathcal{M}_{k\times n;(p,q)}) = \sum_{j=1}^{k-1} V(\mathcal{S}_{j,k\times n;(p,q)}) = \sum_{j=1}^{k-1} s_{j,k\times n;(p,q)}.$$

For $n=2$ and $j=k$, there is only one $S_{k,k\times 2;(p,q)}$ board and $V(\mathcal{S}_{k,k\times 2;(p,q)})=q^k$. For $1\le j\le k-1$, there are $\binom{k}{j}$ different $S_{j,k\times 2;(p,q)}$ boards. Then $s_{j,k\times 2;(p,q)}=p^{2(k-j)}q^j\binom{k}{j}$. Therefore,

$$m_{k\times 2;(p,q)}=q^k+\sum_{j=1}^{k-1}s_{j,k\times 2;(p,q)}=q^k+\sum_{j=1}^{k-1}p^{2(k-j)}q^j\binom{k}{j}.$$

□

**Lemma 17.** *For $n\ge 2$ and $1\le j\le k-1$,*

$$s_{j,k\times(n+1);(p,q)}=\sum_{i=1}^{k-j}p^{k-2j}q^j\binom{k-i}{j}s_{i,k\times n;(p,q)}.$$

*Proof.* An $S_{j,k\times(n+1);(p,q)}$ board, $1\le j\le k-1$, can be obtained from an $S_{i,k\times n;(p,q)}$ board, $1\le i\le k-j$, by the following procedure. Suppose we start with an $S_{i,k\times n;(p,q)}$ board, $1\le i\le k-j$, then this board has $k-i$ squares in the last column. Subsequently, choose $j$ squares from this column, replace the chosen squares with dominoes. For each replacement, since each square has weight $p$ and each domino has weight $q$, we multiply $s_{i,k\times n;(p,q)}$ by $\left(\frac{q}{p}\right)^j$.

So now we obtain an $S_{j,k\times(n+1);(p,q)}$ board by filling the remaining $(k-j)$ empty positions in the $n+1$ column with squares. Thus, we need to multiply $s_{i,k\times n;(p,q)}\left(\frac{q}{p}\right)^j$ by $p^{k-j}$. Therefore,

$$s_{j,k\times(n+1);(p,q)}=\sum_{i=1}^{k-j}p^{k-2j}q^j\binom{k-i}{j}s_{i,k\times n;(p,q)}$$

for $1\le j\le k-1$. □

In Figure 3, there is an example of Lemma 17.

**Definition 18.** For $k\ge 2$, define matrices $A_{k;(p,q)}=[a_{ji}]_{(k-1)\times(k-1)}$ where $a_{ji}=p^{k-2j}q^j\binom{k-i}{j}$; $B_k=[b_{1i}]_{1\times(k-1)}$ such that $b_{1i}=1$; $C_{k;(0,1;p,q)}=[c_{j1}]_{(k-1)\times 1}$ where $c_{j1}=p^{2(k-j)}q^j\binom{k}{j}$, and $I_k$ is the $(k-1)\times(k-1)$ identity matrix.

If $k=1$, let $A_{1;(p,q)}=B_1=C_{1;(0,1;p,q)}=0$, $I_1=1$.

**Lemma 19.** *For $k\in\mathbb{N}^+$,*

$$m_{k\times n;(p,q)}=\begin{cases}q^k+B_kA^0_{k;(p,q)}C_{k;(0,1;p,q)}, & \text{if } n=2;\\ B_kA^{n-2}_{k;(p,q)}C_{k;(0,1;p,q)}, & \text{if } n\ge 3.\end{cases}$$

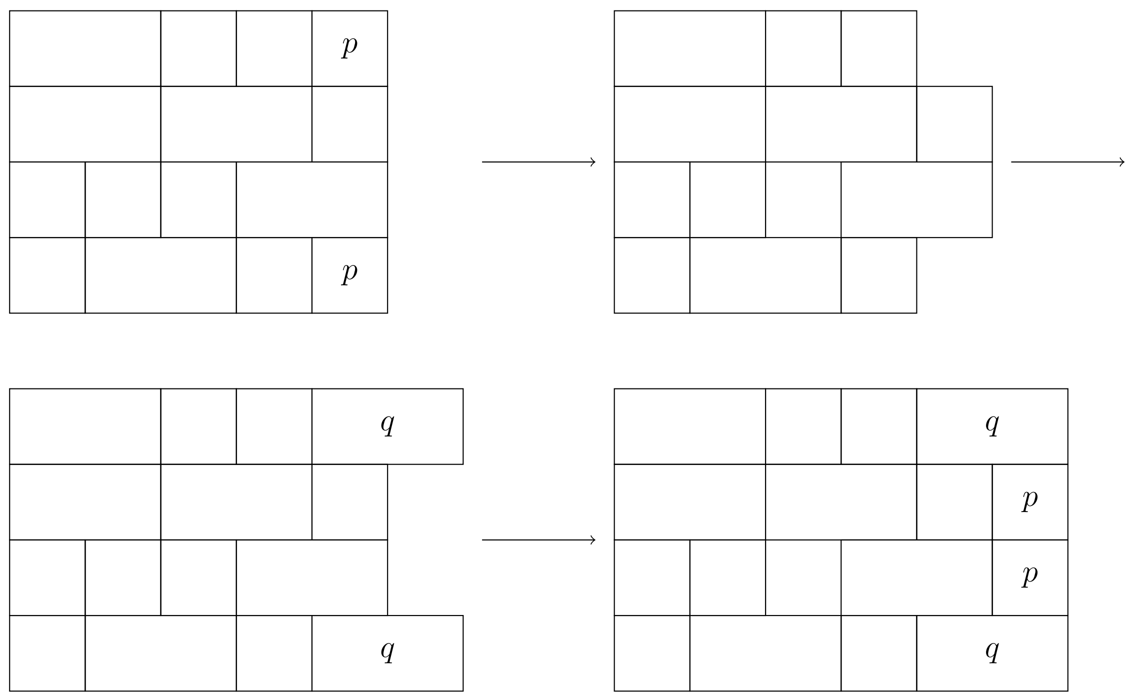



Figure 3: How to transform $S_{1,4\times5;(p,q)}$ to $S_{2,4\times6;(p,q)}$.

*Proof.* Employing Lemmas 13, 17, and 16, with $k \in \mathbb{N}^+$ and $n \geq 3$, we have,

$$\begin{aligned}
m_{k\times n;(p,q)} &= \sum_{j=1}^{k-1} s_{j,k\times n;(p,q)} \\
&= B_k A_{k;(p,q)}^{n-2} \begin{bmatrix} s_{1,k\times 2;(p,q)} \\ s_{2,k\times 2;(p,q)} \\ \vdots \\ s_{k-1,k\times 2;(p,q)} \end{bmatrix}_{(k-1)\times 1} \\
&= B_k A_{k;(p,q)}^{n-2} \begin{bmatrix} p^{2(k-1)} q^1 \binom{k}{1} \\ p^{2(k-2)} q^2 \binom{k}{2} \\ \vdots \\ p^2 q^{k-1} \binom{k}{k-1} \end{bmatrix}_{(k-1)\times 1} \\
&= B_k A_{k;(p,q)}^{n-2} C_{k;(0,1;p,q)}.
\end{aligned}$$

For $n = 2$, $m_{k\times 2;(p,q)} = q^k + \sum_{j=1}^{k-1} s_{j,k\times 2;(p,q)} = q^k + B_k A_{k;(p,q)}^0 C_{k;(0,1;p,q)}$. $\square$

Table 2 contains $s_{j,4\times n;(2,-1)}$ for $2 \leq n \leq 7$, with $j = 1, 2, 3$. We have that,
$s_{1,4\times(n+1);(2,-1)} = (-12) \cdot s_{1,4\times n;(2,-1)} + (-8) \cdot s_{2,4\times n;(2,-1)} + (-4) \cdot s_{3,4\times n;(2,-1)}$,

$s_{2,4\times(n+1);(2,-1)} = 3\cdot s_{1,4\times n;(2,-1)} + 1\cdot s_{2,4\times n;(2,-1)} + 0\cdot s_{3,4\times n;(2,-1)}$, and
$s_{3,4\times(n+1);(2,-1)} = (-\frac{1}{4})\cdot s_{1,4\times n;(2,-1)} + 0\cdot s_{2,4\times n;(2,-1)} + 0\cdot s_{3,4\times n;(2,-1)}$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $m_{4\times n;(2,-1)}$ | 16 | $-175$ | 1760 | $-17456$ | 172832 | $-1710896$ | 16936160 |
| $s_{1,4\times n;(2,-1)}$ | | $-256$ | 2368 | $-23296$ | 230464 | $-2281216$ | 22581568 |
| $s_{2,4\times n;(2,-1)}$ | | 96 | $-672$ | 6432 | $-63456$ | 627936 | $-6215712$ |
| $s_{3,4\times n;(2,-1)}$ | | $-16$ | 64 | $-592$ | 5824 | $-57616$ | 570304 |

Table 2: $s_{j,4\times n;(2,-1)}$, $j=1,2,3$, for $2\le n\le 7$.

**Example 20.** From Lemma 19, and for $n\ge 3$,

$$\begin{aligned} m_{4\times n;(2,-1)} &= s_{1,4\times n;(2,-1)} + s_{2,4\times n;(2,-1)} + s_{3,4\times n;(2,-1)} \\ &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -12 & -8 & -4 \\ 3 & 1 & 0 \\ -\frac{1}{4} & 0 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} -256 \\ 96 \\ -16 \end{bmatrix}. \end{aligned}$$

**Theorem 21.** *For $k\in\mathbb{N}^+$,*

$$\mathcal{W}_{k;(p,q)}(x) = \frac{x}{1-p^k x - q^k x^2 - x^2 B_k (I_k - xA_{k;(p,q)})^{-1} C_{k;(0,1;p,q)}}.$$

*Proof.* For $k\ge 2$, according to Lemmas 11, 16, and 19 we obtain,

$$\begin{aligned} e_{k;(p,q)} &= \sum_{n=1}^{\infty} m_{k\times n;(p,q)} x^n \\ &= p^k x + q^k x^2 + \sum_{n=2}^{\infty} B_k A_{k;(p,q)}^{n-2} C_{k;(0,1;p,q)} x^n \\ &= p^k x + q^k x^2 + x^2 \sum_{n=0}^{\infty} B_k A_{k;(p,q)}^{n} C_{k;(0,1;p,q)} x^n \\ &= p^k x + q^k x^2 + x^2 B_k \left( \sum_{n=0}^{\infty} (xA_{k;(p,q)})^n \right) C_{k;(0,1;p,q)} \\ &= p^k x + q^k x^2 + x^2 B_k \left( I_k - xA_{k;(p,q)} \right)^{-1} C_{k;(0,1;p,q)}. \end{aligned}$$

Thus,

$$\mathcal{W}_{k;(p,q)}(x) = \frac{x}{1-e_{k;(p,q)}} = \frac{x}{1-p^k x - q^k x^2 - x^2 B_k (I_k - xA_{k;(p,q)})^{-1} C_{k;(0,1;p,q)}}.$$

Since $(B_1(I_1 - xA_{1;(p,q)})^{-1}C_{1;(0,1;p,q)} = 0$, the theorem is also true for $k=1$. $\square$

**Example 22.** Let $k=4$, $p=2$ and $q=-1$. Then

$$(I_4 - xA_{4;(2,-1)})^{-1} = \frac{1}{1+11x+11x^2+x^3}\begin{bmatrix} 1-x & -8x & 4x(1-x) \\ 3x & 1+12x-x^2 & -12x^2 \\ \frac{x(x-1)}{4} & 2x^2 & 1+11x+12x^2 \end{bmatrix},$$

$$B_4(I_4 - xA_{4;(2,-1)})^{-1}C_{4;(0,1,2,-1)} = \frac{-16(2x^2+11x+11)}{1+11x+11x^2+x^3},$$

therefore,

$$\mathcal{W}_{4;(2,-1)}(x) = \frac{x}{1-16x-x^2-x^2\frac{-16(2x^2+11x+11)}{1+11x+11x^2+x^3}} = \frac{x(1+11x+11x^2+x^3)}{(1-x)^5}.$$

The polynomial $1+11x+11x^2+x^3$ is the Eulerian polynomial $A_4(x)$.

**Corollary 23.** $\det(I_k - xA_{k;(2,-1)}) = A_k(x)$.

*Proof.* By Theorem 21,

$$\begin{aligned}
\mathcal{W}_{k;(p,q)}(x) &= \frac{x}{1-p^k x - q^k x^2 - x^2 B_k (I_k - xA_{k;(p,q)})^{-1} C_{k;(0,1;p,q)}} \\
&= \frac{x}{1-p^k x - q^k x^2 - x^2 B_k \frac{\operatorname{adj}(I_k - xA_{k;(p,q)})}{\det(I_k - xA_{k;(p,q)})} C_{k;(0,1;p,q)}} \\
&= \frac{x\det(I_k - xA_{k;(p,q)})}{(1-p^k x - q^k x^2)\det(I_k - xA_{k;(p,q)}) - x^2 B_k \operatorname{adj}(I_k - xA_{k;(p,q)})C_{k;(0,1;p,q)}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{xA_k(x)}{(1-x)^{k+1}} &= \sum_{n=0}^{\infty} n^k x^n \\
&= \mathcal{W}_{k;(2,-1)}(x) \\
&= \frac{x\det(I_k - xA_{k;(2,-1)})}{(1-2^k x - (-1)^k x^2)\det(I_k - xA_{k;(2,-1)}) - x^2 B_k \operatorname{adj}(I_k - xA_{k;(2,-1)})C_{k;(0,1;2,-1)}}.
\end{aligned}$$

Since $A_k(1) = k! \neq 0$, $(1-x) \nmid A_k(x)$ and $\deg\left(\det(I_k - xA_{k;(2,-1)})\right) \leq k-1$, then $\deg\left(\det(I_k - xA_{k;(2,-1)})\right) = k-1$. Thus, $\det(I_k - xA_{k;(2,-1)}) = A_k(x)$. $\square$

**Example 24.** If $k=6$,

$$\begin{aligned}
\det(I_6 - xA_{6;(2,-1)}) &= \det \begin{bmatrix} 1+80x & 64x & 48x & 32x & 16x \\ -40x & 1-24x & -12x & -4x & 0 \\ 10x & 4x & 1+x & 0 & 0 \\ \frac{-5x}{4} & \frac{-x}{4} & 0 & 1 & 0 \\ \frac{x}{16} & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= 1+57x+302x^2+302x^3+57x^4+x^5 \\
&= A_6(x).
\end{aligned}$$

**Corollary 25.** *Let $D_{k;(p,q)} = [d_{ji}]_{(k-1)\times(k-1)}$ be the diagonal matrix with $d_{jj} = p^{2j-k}q^{-j}$ and $d_{ji} = 0$ if $j \neq i$. Then*

$$\det(D_{k;(p,q)} - xA_{k;(1,1)}) = \det(I_k - xA_{k;(p,q)}).$$

*Proof.* Let $\lambda_{ji} = 1$ if $j = i$, otherwise $\lambda_{ji} = 0$. Then

$$\begin{aligned}
\det(I_k - xA_{k;(p,q)}) &= \det \left[\lambda_{ji} - p^{k-2j}q^j \binom{k-i}{j} x\right]_{(k-1)\times(k-1)} \\
&= \det \left[p^{2j-k}q^{-j}\lambda_{ji} - \binom{k-i}{j} x\right]_{(k-1)\times(k-1)} \\
&= \det \left[d_{ji} - \binom{k-i}{j} x\right]_{(k-1)\times(k-1)} \\
&= \det(D_{k;(p,q)} - xA_{k;(1,1)}).
\end{aligned}$$

□

*Remark* 26. From Corollaries 23 and 25, we have

$$\det(D_{k;(p,q)} - xA_{k;(1,1)}) = A_k(x).$$

*Remark* 27. From Corollary 23, Remark 26, and $A_k(1) = k!$, we have

$$\det(I_k - A_{k;(2,-1)}) = \det(D_{k;(2,-1)} - A_{k;(1,1)}) = k!.$$

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