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New Sufficient Conditions for Log-Balancedness, With Applications to Combinatorial Sequences

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Abstract

In this paper, we mainly study the log-balancedness of combinatorial sequences. We first give some new sufficient conditions for log-balancedness of some kinds of sequences. Then we use these results to derive the log-balancedness of a number of log-convex sequences related to derangement numbers, Domb numbers, numbers of tree-like polyhexes, numbers of walks on the cubic lattice, and so on.

1 Introduction

A sequence of positive real numbers $\{z_n\}_{n\geq 0}$ is said to be *log-convex* (or *log-concave*) if $z_n^2 \leq z_{n-1}z_{n+1}$ (or $z_n^2 \geq z_{n-1}z_{n+1}$) for each $n \geq 1$. A log-convex sequence $\{z_n\}_{n\geq 0}$ is said to be *log-balanced* if $\{\frac{z_n}{n!}\}_{n\geq 0}$ is log-concave. See Došlić [4] for more details about log-balanced sequences. It is well known that $\{z_n\}_{n\geq 0}$ is log-convex (or log-concave) if and only if its quotient sequence $\{\frac{z_{n+1}}{z_n}\}_{n\geq 0}$ is nondecreasing (or nonincreasing) and a log-convex sequence $\{z_n\}_{n\geq 0}$ is log-balanced if and only if $\frac{(n+1)z_n}{z_{n-1}} \geq \frac{nz_{n+1}}{z_n}$ for each $n \geq 1$. It is clear that the quotient sequence of a log-balanced sequence does not grow too quickly.

In combinatorics, log-convexity and log-concavity are not only instrumental in obtaining the growth rate of a combinatorial sequence, but also important sources of inequalities. Log-convexity and log-concavity have applications in many fields such as quantum physics, white noise theory, probability, economics and mathematical biology. See, for instance [1, 2, 5, 6, 7, 12, 14, 16]. Since log-balancedness is related to log-convexity and log-concavity, it can help us to find new inequalities. Hence, the log-balancedness of various sequences deserves to be studied.

In this paper, we are interested in the log-balancedness of some combinatorial sequences. In fact, there are many log-balanced sequences in combinatorics and number theory. Došlić [4] presented some sufficient conditions for the log-balancedness of sequences satisfying three-term linear recurrences. As consequences, a number of sequences such as the Motzkin numbers, the Fine numbers, the Franel numbers of orders 3 and 4, the Apéry numbers, the large and little Schröder numbers, and the central Delannoy numbers, are log-balanced (see Došlić [4]). Recently, Zhao [20] gave a sufficient condition for the log-balancedness of the product of a log-balanced sequence and a log-concave sequence and she also proved that the binomial transformation preserves the log-balancedness. Zhao [21, 20] showed that the sequences of the exponential numbers and the Catalan-Larcombe-French numbers are respectively log-balanced. Zhang and Zhao [18] gave some sufficient conditions for the log-balancedness of combinatorial sequences. In addition, for a log-balanced sequence $\{z_n\}_{n\geq 0}$, Zhang and Zhao [18] proved that $\{\sqrt{z_n}\}_{n\geq 0}$ is still log-balanced.

This paper is devoted to the study of log-balancedness of some combinatorial sequences and is organized as follows. In Section 2, we give some new sufficient conditions for log-balancedness. In Section 3, using these new results, we investigate the log-balancedness of a series of log-convex sequences.

2 Sufficient conditions for log-balancedness

Zhang and Zhao [18] proved that the sequence of the arithmetic square root of a log-balanced sequence is still log-balanced. For a log-convex sequence $\{z_n\}_{n\geq 0}$, here we prove that $\{\sqrt[r]{z_n}\}_{n\geq 0}$ is log-balanced under some conditions, where r is a fixed positive real number.

Theorem 1. *Let $\{z_n\}_{n\geq 0}$ be a log-convex sequence and r be a fixed positive real number. For $n\geq 0$, let $x_n = \frac{z_{n+1}}{z_n}$. If there exists a nonnegative integer N_r such that*

$$(n+2)^r x_n - (n+1)^r x_{n+1} \geq 0, \quad n \geq N_r,$$

the sequence $\{\sqrt[r]{z_n}\}_{n\geq N_r}$ is log-balanced.

Proof. Since the sequence $\{z_n\}_{n\geq 0}$ is log-convex, $\{\sqrt[r]{z_n}\}_{n\geq 0}$ is also log-convex. In order to prove the log-balancedness of $\{\sqrt[r]{z_n}\}_{n\geq N_r}$, it is sufficient to show that the sequence $\{\frac{\sqrt[r]{z_n}}{n!}\}_{n\geq N_r}$ is log-concave. In fact, it is clear that $\{\frac{\sqrt[r]{z_n}}{n!}\}_{n\geq N_r}$ is log-concave if and only if $\frac{\sqrt[r]{x_n}}{n+1} \geq \frac{\sqrt[r]{x_{n+1}}}{n+2}$ for every $n \geq N_r$. It follows from $(n+2)^r x_n - (n+1)^r x_{n+1} \geq 0$ that $\frac{\sqrt[r]{x_n}}{n+1} \geq \frac{\sqrt[r]{x_{n+1}}}{n+2}$. Hence the sequence $\{\frac{\sqrt[r]{z_n}}{n!}\}_{n\geq N_r}$ is log-concave. Therefore, $\{\sqrt[r]{z_n}\}_{n\geq N_r}$ is log-balanced. $\square$

Theorem 2. *Suppose that a and b are positive real numbers with $b < a$ and $\{z_n\}_{n\geq 0}$ is a log-convex sequence. If the sequence $\{z_n^a\}_{n\geq 0}$ is log-balanced, then so is the sequence $\{z_n^b\}_{n\geq 0}$.*

Proof. Since the sequence $\{z_n^a\}_{n\geq 0}$ is log-balanced, we have

$$\frac{n}{n+1} z_{n-1}^a z_{n+1}^a \leq z_n^{2a} \leq z_{n-1}^a z_{n+1}^a.$$

Then we derive

$$\left(\frac{n}{n+1}\right)^{\frac{1}{a}} z_{n-1} z_{n+1} \leq z_n^2 \leq z_{n-1} z_{n+1},$$

$$\left(\frac{n}{n+1}\right)^{\frac{b}{a}} z_{n-1}^b z_{n+1}^b \leq z_n^{2b} \leq z_{n-1}^b z_{n+1}^b.$$

Since $0 < \frac{b}{a} < 1$ and $0 < \frac{n}{n+1} < 1$, we have $(\frac{n}{n+1})^{\frac{b}{a}} \geq \frac{n}{n+1}$ and hence

$$\frac{n}{n+1} z_{n-1}^b z_{n+1}^b \leq z_n^{2b} \leq z_{n-1}^b z_{n+1}^b.$$

It follows from the definition of log-balancedness that the sequence $\{z_n^b\}_{n\geq 0}$ is log-balanced. $\square$

In Theorem 2, if the condition "$b < a$" is replaced by "$b > a$", the conclusion is not valid in general. For example, the sequence $\{nn!\}_{n\geq 2}$ is log-balanced, but $\{(nn!)^2\}_{n\geq 2}$ is not log-balanced.

In the next section, we will use the results of Theorems 1–2 to derive log-balancedness of a series of sequences.

Theorem 3. *Let $\{z_n\}_{n\geq 0}$ be a log-concave sequence. If the sequence $\{n!z_n\}_{n\geq 0}$ is log-balanced, then so is the sequence $\{n!\sqrt{z_n}\}_{n\geq 0}$.*

Proof. Since the sequence $\{z_n\}_{n\geq 0}$ is log-concave, then so is the sequence $\{\sqrt{z_n}\}_{n\geq 0}$. In order to prove the log-balancedness of $\{n!\sqrt{z_n}\}_{n\geq 0}$, we only need to show that $\{n!\sqrt{z_n}\}_{n\geq 0}$ is log-convex. Since $\{n!z_n\}_{n\geq 0}$ is log-balanced, we get

$$\begin{aligned}
&n z_n^2 - (n+1) z_{n-1} z_{n+1} \leq 0,\\
&z_n \leq \sqrt{\frac{n+1}{n} z_{n-1} z_{n+1}} < \frac{n+1}{n}\sqrt{z_{n-1} z_{n+1}},\\
&(n!\sqrt{z_n})^2 \leq (n-1)!(n+1)!\sqrt{z_{n-1} z_{n+1}}.
\end{aligned}$$

Hence $\{n!\sqrt{z_n}\}_{n\geq 0}$ is log-convex. $\square$

3 Log-balancedness of some sequences

In this section, we discuss the log-balancedness of a number of log-convex sequences involving many combinatorial numbers.

3.1 The derangement numbers

The derangement numbers d_n (sequence A000166 in the OEIS) count the number of permutations of n elements with no fixed points. The sequence $\{d_n\}_{n\geq 0}$ satisfies the recurrence

$$d_{n+1} = n(d_n + d_{n-1}), \quad n \geq 1, \tag{1}$$

with $d_0 = 1$, $d_1 = 0$, $d_2 = 1$, $d_3 = 2$ and $d_4 = 9$; see Table 1 for some information about it. In particular, Liu and Wang [11] proved that $\{d_n\}_{n\geq 2}$ is log-convex.

n	0	1	2	3	4	5	6	7	8
d_n	1	0	1	2	9	44	265	1854	14833

Table 1: Some initial values of $\{d_n\}_{n\geq 0}$

Theorem 4. *For $r \geq 2$, the sequence $\{\sqrt[r]{d_n}\}_{n\geq 3}$ is log-balanced.*

Proof. We first prove that the sequence $\{\sqrt{d_n}\}_{n\geq 3}$ is log-balanced.

For $n \geq 0$, let $x_n = \frac{d_{n+1}}{d_n}$. We prove by induction that

$$\lambda_n \leq x_n \leq \lambda_{n+1}, \quad n \geq 3,$$

where $\lambda_n = \frac{2n+1}{2}$. It follows from (1) that

$$x_n = n + \frac{n}{x_{n-1}}, \quad n \geq 3. \tag{2}$$

It is clear that $\lambda_3 \leq x_3 \leq \lambda_4$. Assume that $\lambda_k \leq x_k \leq \lambda_{k+1}$ for $k \geq 3$. By applying (2), we get

$$x_{k+1} - \lambda_{k+1} = \frac{k+1}{x_k} - \frac{1}{2} \quad \text{and} \quad x_{k+1} - \lambda_{k+2} = \frac{k+1}{x_k} - \frac{3}{2}.$$

Due to $\frac{1}{\lambda_{k+1}} \leq \frac{1}{x_k} \leq \frac{1}{\lambda_k}$ $(k \geq 3)$, we have

$$x_{k+1} - \lambda_{k+1} \geq 0 \quad \text{and} \quad x_{k+1} - \lambda_{k+2} \leq 0.$$

Then we derive $\lambda_n \leq x_n \leq \lambda_{n+1}$ for $n \geq 3$.

By means of (2), we obtain

$$(n+2)^2 x_n - (n+1)^2 x_{n+1} = \frac{(n+2)^2 x_n^2 - (n+1)^3 x_n - (n+1)^3}{x_n}.$$

For any $x \in (-\infty, +\infty)$, define a function

$$f(x) = (n+2)^2 x^2 - (n+1)^3 x - (n+1)^3.$$

Then we have

$$f'(x) = 2(n+2)^2 x - (n+1)^3.$$

Since $f'(x) \geq 0$ for $x \geq \frac{(n+1)^3}{2(n+2)^2}$, f is increasing on $[\frac{(n+1)^3}{2(n+2)^2}, +\infty)$. We can verify that $\lambda_n > \frac{(n+1)^3}{2(n+2)^2}$. Hence, f is increasing on $[\lambda_n, +\infty)$. Note that

$$\begin{aligned} f(\lambda_n) &= (n+2)^2 \lambda_n^2 - (n+1)^3 \lambda_n - (n+1)^3 \\ &= \frac{2n^3 + 3n^2 - 2n - 2}{4} > 0 \quad (n \geq 1). \end{aligned}$$

Then we have $f(x_n) > 0$ for $n \geq 3$. This implies that $(n+2)^2 x_n - (n+1)^2 x_{n+1} \geq 0$ for $n \geq 3$. It follows from Theorem 1 that the sequence $\{\sqrt{d_n}\}_{n \geq 3}$ is log-balanced. For $r > 2$, it follows from Theorem 2 that $\{\sqrt[r]{d_n}\}_{n \geq 3}$ is log-balanced. $\square$

3.2 Numbers counting tree-like polyhexes

Let h_n denote the number of tree-like polyhexes with $n+1$ hexagons (Harary and Read [10]); it is sequence A002212 in the OEIS. It is well known that h_n is equal to the number of lattice paths, from $(0,0)$ to $(2n,0)$ with steps $(1,1)$, $(1,-1)$ and $(2,0)$, never falling below the x-axis and with no peaks at odd level. The sequence $\{h_n\}_{n \geq 0}$ satisfies the recurrence

$$(n+1)h_n = 3(2n-1)h_{n-1} - 5(n-2)h_{n-2}, \quad n \geq 2, \tag{3}$$

with $h_0 = h_1 = 1$, $h_2 = 3$ and $h_3 = 10$; see Table 2 for some information about it. In particular, Liu and Wang [11] showed that the sequence $\{h_n\}_{n \geq 0}$ is log-convex.

n	0	1	2	3	4	5	6	7
h_n	1	1	3	10	36	137	543	2219

Table 2: Some initial values of $\{h_n\}_{n \geq 0}$

Theorem 5. *For $r \geq 1$, the sequence $\{\sqrt[r]{h_n}\}_{n \geq 1}$ is log-balanced.*

Proof. In order to prove that $\{\sqrt[r]{h_n}\}_{n\geq 1}$ is log-balanced for $r \geq 1$, we only need to show that $\{h_n\}_{n\geq 1}$ is log-balanced by Theorem 2.

For $n \geq 0$, put $x_n = \frac{h_{n+1}}{h_n}$. We next prove by induction that

$$\lambda_n \leq x_n \leq \mu_n, \quad n \geq 0,$$

where $\lambda_n = \frac{10n+3}{2n+4}$ and $\mu_n = \frac{5n+4}{n+1}$. It follows from (3) that

$$x_n = \frac{3(2n+1)}{n+2} - \frac{5(n-1)}{(n+2)x_{n-1}}, \quad n \geq 1, \tag{4}$$

It is easy to find that $\lambda_0 \leq x_0 \leq \mu_0$. Assume that $\lambda_k \leq x_k \leq \mu_k$ for $k \geq 0$. By using (4), we derive

$$x_{k+1} - \lambda_{k+1} = \frac{3(2k+3)}{k+3} - \frac{10k+13}{2k+6} - \frac{5k}{(k+3)x_k}$$

and

$$x_{k+1} - \mu_{k+1} = \frac{3(2k+3)}{k+3} - \frac{5k+9}{k+2} - \frac{5k}{(k+3)x_k}.$$

Since $\frac{1}{\mu_k} \leq \frac{1}{x_k} \leq \frac{1}{\lambda_k}$ $(k \geq 0)$, we have

$$\begin{aligned} x_{k+1} - \lambda_{k+1} &\geq \frac{2k+5}{2(k+3)} - \frac{5k}{(k+3)\lambda_k} \\ &= \frac{16k+15}{2(k+3)(10k+3)} \geq 0 \end{aligned}$$

and

$$\begin{aligned} x_{k+1} - \mu_{k+1} &\leq \frac{k^2-3k-9}{(k+2)(k+3)} - \frac{5k}{(k+3)\mu_k} \\ &= \frac{-26k^2-67k-36}{(k+2)(k+3)(5k+4)} \leq 0. \end{aligned}$$

Then we have $\lambda_n \leq x_n \leq \mu_n$ for $n \geq 0$.

By means of (4), we obtain

$$(n+2)x_n - (n+1)x_{n+1} = \frac{(n+2)(n+3)x_n^2 - 3(n+1)(2n+3)x_n + 5n(n+1)}{(n+3)x_n}.$$

For any $x \in (-\infty, +\infty)$, define a function

$$f(x) = (n+2)(n+3)x^2 - 3(n+1)(2n+3)x + 5n(n+1).$$

It is clear that

$$(n+2)x_n - (n+1)x_{n+1} = \frac{f(x_n)}{(n+3)x_n}.$$

We note that f is increasing on $[\frac{3(n+1)(2n+3)}{2(n+2)(n+3)}, +\infty]$. We find that $\lambda_n > \frac{3(n+1)(2n+3)}{2(n+2)(n+3)}$ and

$$f(\lambda_n) = \frac{84n^2 - 41n - 27}{4(n+2)} > 0 \quad (n \geq 1).$$

Then we have $(n+2)x_n - (n+1)x_{n+1} > 0$ for $n \geq 1$. Hence $\{h_n\}_{n\geq 1}$ is log-balanced. □

3.3 Numbers counting walks on the cubic lattice

Consider the sequence $\{w_n\}_{n\geq 0}$ counting the number of walks on the cubic lattice with n steps, starting and finishing on the xy plane and never going below it (Guy [9]); it is sequence A005572 in the OEIS. The sequence $\{w_n\}_{n\geq 0}$ satisfies the recurrence

$$(n+2)w_n = 4(2n+1)w_{n-1} - 12(n-1)w_{n-2}, \quad n \geq 2, \tag{5}$$

where $w_0 = 1$, $w_1 = 4$ and $w_2 = 17$; see Table 3 for some information about it. In particular, Liu and Wang [11] showed that $\{w_n\}_{n\geq 0}$ is log-convex.

n	0	1	2	3	4	5	6
w_n	1	4	17	76	354	1704	8421

Table 3: Some initial values of $\{w_n\}_{n\geq 0}$

Theorem 6. *Let r be a positive real number. For $r \geq 1$, the sequence $\{\sqrt[r]{w_n}\}_{n\geq 0}$ is log-balanced. For $\frac{5}{6} < r < 1$, there exists a positive integer N_r such that $\{\sqrt[r]{w_n}\}_{n\geq N_r}$ is log-balanced.*

Proof. For $n \geq 0$, let $x_n = \frac{w_{n+1}}{w_n}$. Now we prove by induction that

$$\lambda_n \leq x_n \leq \mu_n, \quad n \geq 0, \tag{6}$$

where $\lambda_n = \frac{6n+13}{n+4}$ and $\mu_n = \frac{6(n+3)}{n+4}$. It follows from (5) that

$$x_n = \frac{4(2n+3)}{n+3} - \frac{12n}{(n+3)x_{n-1}}, \quad n \geq 1, \tag{7}$$

We observe that $\lambda_k \leq x_k \leq \mu_k$ for $k = 0, 1, 2$. Assume that $\lambda_k \leq x_k \leq \mu_k$ for $k \geq 2$. By using (7), we have

$$x_{k+1} - \lambda_{k+1} = \frac{4(2k+5)}{k+4} - \frac{12(k+1)}{(k+4)x_k} - \frac{6k+19}{k+5}$$

and

$$x_{k+1}-\mu_{k+1}=\frac{4(2k+5)}{k+4}-\frac{12(k+1)}{(k+4)x_k}-\frac{6(k+4)}{k+5}.$$

Since $\frac{1}{\mu_k}\le\frac{1}{x_k}\le\frac{1}{\lambda_k}$, we derive

$$\begin{aligned}x_{k+1}-\lambda_{k+1} &> \frac{4(2k+5)}{k+4}-\frac{12(k+1)}{(k+4)\lambda_k}-\frac{6k+19}{k+5}\\ &= \frac{8k^2+17k+72}{(k+4)(k+5)(6k+13)}>0\end{aligned}$$

and

$$\begin{aligned}x_{k+1}-\mu_{k+1} &< \frac{4(2k+5)}{k+4}-\frac{12(k+1)}{(k+4)\mu_k}-\frac{6(k+4)}{k+5}\\ &= -\frac{2k^2+18k+28}{(k+3)(k+4)(k+5)}<0.\end{aligned}$$

Hence we have $\lambda_n\le x_n\le\mu_n$ for $n\ge 0$.

By applying (6), we obtain

$$\begin{aligned}(n+2)x_n-(n+1)x_{n+1} &\ge (n+2)\lambda_n-(n+1)\mu_{n+1}\\ &= \frac{n^2+7n+34}{(n+4)(n+5)}>0 \quad (n\ge 0).\end{aligned}$$

Then $\{w_n\}_{n\ge0}$ is log-balanced. For $r>1$, it follows from Theorem 2 that $\{\sqrt[r]{w_n}\}_{n\ge0}$ is log-balanced.

For $\frac{5}{6}<r<1$, by using (6), we get

$$(n+2)^r x_n-(n+1)^r x_{n+1}\ge\frac{(n+2)^r(n+5)(6n+13)-6(n+1)^r(n+4)^2}{(n+4)(n+5)}.$$

It is obvious that $(n+2)^r(n+5)(6n+13)\ge 6(n+1)^r(n+4)^2$ if and only if

$$r\ln(n+2)-r\ln(n+1)+\ln(6n^2+43n+65)-\ln(6n^2+48n+96)\ge 0.$$

We note that

$$\begin{aligned}&r\ln(n+2)-r\ln(n+1)+\ln(6n^2+43n+65)-\ln(6n^2+48n+96)\\ &=r\ln\left(1+\frac{1}{n+1}\right)-\ln\left(1+\frac{5n+31}{6n^2+43n+65}\right).\end{aligned}$$

Due to $\frac{x}{1+x}<\ln(1+x)<x$ for $x>0$, we have

$$\begin{aligned}&r\ln(n+2)-r\ln(n+1)+\ln(6n^2+43n+65)-\ln(6n^2+48n+96)\\ &>\frac{(6r-5)n^2+(43r-41)n+65r-62}{(n+2)(6n^2+43n+65)}.\end{aligned}$$

Since

$$\lim_{n\to+\infty}[(6r-5)n^2+(43r-41)n+65r-62]=+\infty,$$

there exists a positive integer N_r such that $(6r-5)n^2+(43r-41)n+65r-62>0$ for $n\geq N_r$. Then the sequence $\{\sqrt[r]{w_n}\}_{n\geq N_r}$ is log-balanced for $\frac{5}{6}<r<1$. $\square$

3.4 Numbers counting a class of arrays

For an integer $r\geq 0$, let $Q(n,r)$ denote the number of arrays (or matrices) of integers $a_{i,j}\geq 0$ $(1\leq i,j\leq n)$ such that

$$\sum_{i=1}^{n}a_{i,j}=\sum_{j=1}^{n}a_{i,j}=r$$

holds for all i and j. Consider the sequence $\{A_n\}_{n\geq 0}$, where $A_n=Q(n,2)$. The sequence $\{A_n\}_{n\geq 0}$ satisfies the recurrence

$$A_{n+1}=(n+1)^2A_n-n\binom{n+1}{2}A_{n-1},\quad n\geq 1, \tag{8}$$

where $A_0=A_1=1$ and $A_2=3$; see Table 4 for some information about it. It is sequence A000681 in the OEIS. In particular, Zhao [19] proved that the sequence $\{A_n\}_{n\geq 1}$ is log-convex (it is clear that $\{A_n\}_{n\geq 0}$ is also log-convex). See [3] for more properties of $\{A_n\}_{n\geq 0}$.

n	0	1	2	3	4	5	6
A_n	1	1	3	21	282	6210	202410

Table 4: Some initial values of $\{A_n\}_{n\geq 0}$

Theorem 7. *For $r\geq 5$, the sequence $\{\sqrt[r]{A_n}\}_{n\geq 0}$ is log-balanced.*

Proof. For $n\geq 0$, set $x_n=\frac{A_{n+1}}{A_n}$. We next prove by induction that

$$\lambda_n\leq x_n\leq\lambda_{n+1},\quad n\geq 0, \tag{9}$$

where $\lambda_n=n^2$. It follows from (8) that

$$x_n=(n+1)^2-\frac{n^2(n+1)}{2x_{n-1}},\quad n\geq 1, \tag{10}$$

It is clear that $\lambda_k \le x_k \le \lambda_{k+1}$ for $k = 0, 1, 2$. Assume that $\lambda_k \le x_k \le \lambda_{k+1}$ for $k \ge 2$. By applying (10), we get

$$\begin{aligned} x_{k+1} - \lambda_{k+1} &= 2k + 3 - \frac{(k+1)^2(k+2)}{2x_k}, \\ x_{k+1} - \lambda_{k+2} &= -\frac{(k+1)^2(k+2)}{2x_k}. \end{aligned}$$

Due to $\frac{1}{\lambda_{k+1}} \le \frac{1}{x_k} \le \frac{1}{\lambda_k}$ $(k \ge 2)$, we have

$$\begin{aligned} x_{k+1} - \lambda_{k+1} &\ge 2k + 3 - \frac{(k+1)^2(k+2)}{2\lambda_k} \\ &= \frac{3k^3 + 2k^2 - 5k - 2}{2k^2} \ge 0 \quad (k \ge 2) \end{aligned}$$

and

$$\begin{aligned} x_{k+1} - \lambda_{k+2} &\le -\frac{(k+1)^2(k+2)}{2\lambda_{k+1}} \\ &= -\frac{k+2}{2} \le 0. \end{aligned}$$

Then we derive $\lambda_n \le x_n \le \lambda_{n+1}$ for $n \ge 0$.

By using (9), we have

$$\begin{aligned} (n+2)^5 x_n - (n+1)^5 x_{n+1} &\ge (n+2)^5 \lambda_n - (n+1)^5 \lambda_{n+2} \\ &= (n+2)^2(n^4 + 2n^3 - 2n^2 - 5n - 1) > 0 \quad (n \ge 2). \end{aligned}$$

On the other hand, we note that $(k+2)^5 x_k - (k+1)^5 x_{k+1}$ for $k = 0, 1$. Then $(n+2)^5 x_n - (n+1)^5 x_{n+1} > 0$ holds for $n \ge 0$. It follows from Theorem 1 that the sequence $\{\sqrt[5]{A_n}\}_{n\ge 0}$ is log-balanced. For $r > 5$, it follows from Theorem 2 that $\{\sqrt[r]{A_n}\}_{n\ge 0}$ is log-balanced. □

3.5 Numbers satisfying a three-term recurrence

Let t_n counting the number of integer sequences $(f_j, \dots, f_2, f_1, 1, 1, g_1, g_2, \dots, g_k)$ with $j + k + 2 = n$ in which every f_i is the sum of one or more contiguous terms immediately to its right, and g_i is likewise the sum of one or more contiguous terms immediately to its left; see Odlyzko [13]. Fishburn et al. [8] proved that the sequence $\{t_n\}_{n\ge 1}$ satisfies the recurrence

$$t_{n+1} = 2nt_n - (n-1)^2 t_{n-1}, \quad n \ge 2, \tag{11}$$

where $t_1 = t_2 = 1$ and $t_3 = 3$; see Table 5 for some information about it. It is sequence A005189 in the OEIS. Zhao [19] showed that the sequence $\{t_n\}_{n\ge 1}$ is log-convex.

Theorem 8. *For $r \ge 3$, the sequence $\{\sqrt[r]{t_n}\}_{n\ge 3}$ is log-balanced.*

n	1	2	3	4	5	6	7
t_n	1	1	3	14	85	626	5387

Table 5: Some initial values of $\{t_n\}_{n\geq 0}$

Proof. For $n \geq 0$, put $x_n = \frac{t_{n+1}}{t_n}$. We first prove by induction that

$$\lambda_k \leq x_k \leq \mu_k, \quad k \geq 3, \tag{12}$$

where $\lambda_k = k + \sqrt{k} - \frac{1}{4}$ and $\mu_k = k + 1 + \sqrt{k+1}$. It follows from (11) that

$$x_n = 2n - \frac{(n-1)^2}{x_{n-1}}, \quad n \geq 2, \tag{13}$$

It is clear that $\lambda_k \leq x_k \leq \mu_k$ for $k = 3, 4$. Assume that $\lambda_k \leq x_k \leq \mu_k$ for $k \geq 4$. By using (13), we have

$$x_{k+1} - \lambda_{k+1} = k - \sqrt{k+1} + \frac{5}{4} - \frac{k^2}{x_k} \quad \text{and} \quad x_{k+1} - \mu_{k+1} = k - \sqrt{k+2} - \frac{k^2}{x_k}.$$

Since $\frac{1}{\mu_k} \leq \frac{1}{x_k} \leq \frac{1}{\lambda_k}$ for $k \geq 4$, we get

$$\begin{aligned} x_{k+1} - \lambda_{k+1} &\geq k + \frac{5}{4} - \sqrt{k+1} - \frac{k^2}{\lambda_k} \\ &= \frac{1}{\lambda_k}\left(k\sqrt{k} + k + \frac{5\sqrt{k}}{4} + \frac{\sqrt{k+1}}{4} - \frac{5}{16} - k\sqrt{k+1} - \sqrt{k(k+1)}\right) \\ &> \frac{1}{\lambda_k}\left(\sqrt{k} + k - \sqrt{k(k+1)} - \frac{5}{16}\right) \\ &> \frac{k+1-\sqrt{k(k+1)}}{\lambda_k} > 0 \end{aligned}$$

and

$$\begin{aligned} x_{k+1} - \mu_{k+1} &\leq k - \sqrt{k+2} - \frac{k^2}{\mu_k} \\ &= \frac{k + k\sqrt{k+1} - (k+1)\sqrt{k+2} - \sqrt{(k+1)(k+2)}}{k+1+\sqrt{k+1}} \\ &\leq -\frac{\sqrt{k+2}}{k+1+\sqrt{k+1}} \leq 0. \end{aligned}$$

Then we derive $\lambda_k \leq x_k \leq \mu_k$ for $k \geq 3$.

It follows from (12) that

$$\begin{aligned}
(n+2)^3x_n-(n+1)^3x_{n+1} &\geq (n+2)^3\left(n+\sqrt{n}-\frac{1}{4}\right)-(n+1)^3(n+2+\sqrt{n+2})\\
&= \left(\frac{3}{4}-\frac{2}{\sqrt{n}+\sqrt{n+2}}\right)n^3+\frac{3n^2-4n-8}{2}\\
&\quad +3n(2(n+2)\sqrt{n}-(n+1)\sqrt{n+2})+8\sqrt{n}-\sqrt{n+2}\\
&> 0 \quad (n\geq 3).
\end{aligned}$$

On the other hand, we can verify that $(n+2)^3x_n-(n+1)^3x_{n+1}>0$ for $1\leq n\leq 2$. Hence the sequence $\{\sqrt[3]{t_n}\}_{n\geq 1}$ is log-balanced. For $r>3$, it follows from Theorem 2 that $\{\sqrt[r]{t_n}\}_{n\geq 1}$ is log-balanced. □

3.6 Numbers counting bipermutations

For a given nonnegative integer k, a relation $\Re$ is called a k-permutation of $[n]=\{1,2,\ldots,n\}$ if all vertical sections and all horizontal sections have k elements. The k-permutation $\Re$ is called a bipermutation when $k=2$. Let $P(n,k)$ denote the number of these relations. Let $P_n=P(n,2)$. The sequence $\{P_n\}$ satisfies the recurrence

$$P_{n+1}=\binom{n+1}{2}(2P_n+nP_{n-1}),\quad n\geq 1, \tag{14}$$

where $P_0=1$ and $P_1=0$; see Table 6 for some information about it. It is sequence A001499 in the OEIS. In particular, Zhao [19] showed that the sequence $\{P_n\}_{n\geq 2}$ is log-convex.

n	0	1	2	3	4	5	6
P_n	1	0	1	6	90	2040	67950

Table 6: Some initial values of $\{P_n\}_{n\geq 0}$

Theorem 9. *For $r\geq 3$, the sequence $\{\sqrt[r]{P_n}\}_{n\geq 3}$ is log-balanced.*

Proof. For $n\geq 2$, put $x_n=\frac{P_{n+1}}{P_n}$. It follows from (14) that

$$x_k=k(k+1)+k\binom{k+1}{2}\frac{1}{x_{k-1}},\quad k\geq 3. \tag{15}$$

We prove that

$$\lambda_n\leq x_n\leq\lambda_{n+1},\quad n\geq 2, \tag{16}$$

where $\lambda_n = n(n+1)$. It is evident that $\lambda_k < x_k < \lambda_{k+1}$ for $k = 2, 3$. Assume that $\lambda_k \le x_k \le \lambda_{k+1}$ for $k \ge 3$. By applying (15), we get

$$x_{k+1} - \lambda_{k+1} = (k+1)\binom{k+2}{2}\frac{1}{x_k} > 0$$

and

$$x_{k+1} - \lambda_{k+2} = -2(k+2) + \frac{(k+1)^2(k+2)}{2x_k}.$$

Since $\frac{1}{x_k} \le \frac{1}{\lambda_k}$, we have

$$x_{k+1} - \lambda_{k+2} \le -\frac{3k^2+5k-2}{2k} < 0.$$

Then we have $\lambda_n \le x_n \le \lambda_{n+1}$ for $n \ge 2$.

It follows from (15) and (16) that

$$\begin{aligned}(n+2)^3x_n - (n+1)^3x_{n+1} &= (n+2)^3x_n - (n+1)^4(n+2) - \frac{(n+1)^5(n+2)}{2x_n} \\ &\ge (n+2)^3\lambda_n - (n+1)^4(n+2) - \frac{(n+1)^5(n+2)}{2\lambda_n} \\ &= \frac{(n+1)(n+2)(n^3-n^2-5n-1)}{2n} > 0 \quad (n \ge 3).\end{aligned}$$

We have from Theorem 1 that the sequence $\{\sqrt[3]{P_n}\}_{n\ge3}$ is log-balanced. For $r > 3$, it follows from Theorem 2 that $\{\sqrt[r]{P_n}\}_{n\ge3}$ is log-balanced. $\square$

3.7 Numbers satisfying a four-term recurrence ("minus" case)

Let G_n stand for the number of graphs on the vertex set $[n] = \{1, 2, \ldots, n\}$, whose every component is a cycle, and put $G_0 = 1$. The sequence $\{G_n\}$ satisfies the recurrence

$$G_{n+1} = (n+1)G_n - \binom{n}{2}G_{n-2}, \quad n \ge 2, \tag{17}$$

where $G_1 = 1$, $G_2 = 2$, and $G_3 = 5$; see Table 7 for some information about it. It is sequence A002135 in the OEIS. This example is Exercise 5.22 of Stanley [15], and one can find its combinatorial proof in Stanley [15, p. 121]. In addition, Došlić [7] showed that the sequence $\{G_n\}_{n\ge0}$ is log-convex.

Theorem 10. *For $r \ge 2$, the sequence $\{\sqrt[r]{G_n}\}_{n\ge0}$ is log-balanced.*

n	0	1	2	3	4	5	6
G_n	1	1	2	5	17	73	388

Table 7: Some initial values of $\{G_n\}_{n\geq 0}$

Proof. For $n \geq 0$, let $x_n = \frac{G_{n+1}}{G_n}$. We next prove by induction that

$$\lambda_n \leq x_n \leq \lambda_{n+1}, \quad n \geq 0, \tag{18}$$

where $\lambda_n = n$. It follows from (17) that

$$x_n = n + 1 - \binom{n}{2}\frac{1}{x_{n-1}x_{n-2}}, \quad n \geq 2. \tag{19}$$

Firstly, we have $\lambda_k \leq x_k \leq \lambda_{k+1}$ for $0 \leq k \leq 4$. Assume that $\lambda_k \leq x_k \leq \lambda_{k+1}$ for $k \geq 4$. By using (19), we have

$$x_{k+1} - \lambda_{k+2} = -\binom{k+1}{2}\frac{1}{x_k x_{k-1}} < 0$$

and

$$\begin{aligned} x_{k+1} - \lambda_{k+1} &= 1 - \binom{k+1}{2}\frac{1}{x_k x_{k-1}} \\ &\geq \frac{2k(k-1) - k(k+1)}{2x_k x_{k-1}} > 0. \end{aligned}$$

Then we derive $\lambda_n \leq x_n \leq \lambda_{n+1}$ for $n \geq 0$.

It follows from (17) and (18) that

$$\begin{aligned} (n+2)^2 x_n - (n+1)^2 x_{n+1} &= (n+2)^2 x_n - (n+1)^2(n+2) + \frac{(n+1)^2\binom{n+1}{2}}{x_n x_{n-1}} \\ &\geq (n+2)^2 \lambda_n - (n+1)^2(n+2) + \frac{(n+1)^2\binom{n+1}{2}}{\lambda_{n+1}\lambda_n} \\ &= \frac{n^2-3}{2} > 0 \quad (n \geq 2). \end{aligned}$$

On the other hand, we note that $(k+2)^2 x_k - (k+1)^2 x_{k+1} > 0$ for $k = 0, 1$. Then $(n+2)^2 x_n - (n+1)^2 x_{n+1} > 0$ holds for $n \geq 0$. We have from Theorem 1 that the sequence $\{\sqrt{G_n}\}_{n\geq 0}$ is log-balanced. For $r > 2$, it follows from Theorem 2 that $\{\sqrt[r]{G_n}\}_{n\geq 0}$ is log-balanced. $\square$

3.8 Numbers satisfying a four-term recurrence ("plus" case)

Let be given a set of Δ of n straight lines in the plane, $\delta_1, \delta_2, \ldots, \delta_n$, lying in general position (no two among them are parallel, and no three among are concurrent). Let P be the set of their points of intersection, $|P| = \binom{n}{2}$. We call any set of n points from P such that any three different points are not collinear, a *cloud*. Let $\mathscr{G}(\Delta)$ stand for the set of clouds of Δ and $g_n = |\mathscr{G}(\Delta)|$. The sequence $\{g_n\}_{n\geq 0}$ satisfies the recurrence

$$g_{n+1} = ng_n + \binom{n}{2} g_{n-2}, \quad n \geq 2, \tag{20}$$

where $g_0 = 1$, $g_1 = g_2 = 0$, $g_3 = 1$, $g_4 = 3$ and $g_5 = 12$; see Table 8 for some information about it. It is sequence A001205 in the OEIS. In particular, Zhao [19] proved that the sequence $\{g_n\}_{n\geq 3}$ is log-convex. For more properties of $\{g_n\}_{n\geq 0}$, see Comtet [3].

n	0	1	2	3	4	5	6	7	8
g_n	1	0	0	1	3	12	70	465	3507

Table 8: Some initial values of $\{g_n\}_{n\geq 0}$

Theorem 11. *For $r \geq 2$, the sequence $\{\sqrt[r]{g_n}\}_{n\geq 5}$ is log-balanced.*

Proof. For $n \geq 3$, let $x_n = \frac{g_{n+1}}{g_n}$. It follows from (20) that

$$x_n = n + \binom{n}{2} \frac{1}{x_{n-1}x_{n-2}}, \quad n \geq 5. \tag{21}$$

Now we show that

$$\lambda_n \leq x_n \leq \lambda_{n+1}, \quad n \geq 3, \tag{22}$$

where $\lambda_n = n$. We can verify that $\lambda_k \leq x_k \leq \lambda_{k+1}$ for $3 \leq k \leq 5$. Assume that $\lambda_k \leq x_k \leq \lambda_{k+1}$ for $k \geq 5$. By applying (21), we get

$$x_{k+1} - \lambda_{k+1} = \binom{k+1}{2} \frac{1}{x_k x_{k-1}} > 0$$

and

$$\begin{aligned} x_{k+1} - \lambda_{k+2} &= \frac{k(k+1)}{2x_k x_{k-1}} - 1 \\ &\leq -\frac{k(k-3)}{2x_k x_{k-1}} < 0. \end{aligned}$$

Then we have $\lambda_n \le x_n \le \lambda_{n+1}$ for $n \ge 3$.

It follows from (21) and (22) that

$$\begin{aligned}(n+2)^2x_n - (n+1)^2x_{n+1} &= (n+2)^2x_n - (n+1)^3 - \frac{(n+1)^2\binom{n+1}{2}}{x_nx_{n-1}} \\ &\ge (n+2)^2\lambda_n - (n+1)^3 - \frac{(n+1)^2\binom{n+1}{2}}{\lambda_n\lambda_{n-1}} \\ &= \frac{n^3-3n^2-7n+1}{2(n-1)} > 0 \quad (n \ge 5).\end{aligned}$$

We have from Theorem 1 that the sequence $\{\sqrt{g_n}\}_{n\ge5}$ is log-balanced. For $r > 2$, it follows from Theorem 2 that $\{\sqrt[r]{g_n}\}_{n\ge5}$ is log-balanced. □

3.9 Numbers counting permutation with ordered orbits

Consider the sequence $\{T_n\}_{n\ge2}$ defined by

$$T_{n+1} = (n-1)T_n + \frac{n!}{2}, \quad n \ge 2, \tag{23}$$

where $T_2 = 1$; see Table 9 for some information about it. The value of T_n is related to the number of permutations with ordered orbits. In particular, Zhao [19] proved that the sequence $\{T_n\}_{n\ge2}$ is log-convex. It is sequence A006595 in the OEIS. For more properties of $\{T_n\}_{n\ge2}$, see Comtet [3].

n	2	3	4	5	6	7
T_n	1	2	7	33	192	1320

Table 9: Some initial values of $\{T_n\}_{n\ge0}$

Theorem 12. *For $r \ge 2$, the sequence $\{\sqrt[r]{T_n}\}_{n\ge2}$ is log-balanced.*

Proof. For $n \ge 2$, let $x_n = \frac{T_{n+1}}{T_n}$. It is easy to verify that

$$T_{n+1} = (2n-1)T_n - (n-2)nT_{n-1}, \quad n \ge 3. \tag{24}$$

It follows from (24) that

$$x_n = 2n-1 - \frac{(n-2)n}{x_{n-1}}, \quad n \ge 3. \tag{25}$$

Now we prove by induction that

$$\lambda_n \le x_n \le \lambda_{n+1}, \quad n \ge 2,$$

where $\lambda_n = n$. It is not difficult to verify that $\lambda_k \le x_k \le \lambda_{k+1}$ for $2 \le k \le 4$. Assume that $\lambda_k \le x_k \le \lambda_{k+1}$ for $k \ge 4$. Using (25), we have

$$\begin{aligned} x_{k+1} - \lambda_{k+1} &= k - \frac{(k+1)(k-1)}{x_k} \\ &\ge k - \frac{(k+1)(k-1)}{k} > 0 \end{aligned}$$

and

$$x_{k+1} - \lambda_{k+2} \le k - 1 - \frac{(k+1)(k-1)}{\lambda_{k+1}} = 0.$$

Then we derive that $\lambda_n \le x_n \le \lambda_{n+1}$ for $n \ge 2$. By means of (26), we obtain

$$(n+2)^2 x_n - (n+1)^2 x_{n+1} = \frac{(n^2+4n+4)x_n^2 - (2n^3+5n^2+4n+1)x_n + n^4 + 2n^3 - 2n - 1}{x_n}$$

For any $x \in (-\infty, +\infty)$, define a function

$$f(x) = (n^2+4n+4)x^2 - (2n^3+5n^2+4n+1)x + n^4 + 2n^3 - 2n - 1.$$

We can prove that f is increasing on $[\sigma_n, +\infty)$, where $\sigma_n = \frac{2n^3+5n^2+4n+1}{2(n^2+4n+4)}$, f is increasing on $[\sigma_n, +\infty)$. We can verify that $\lambda_n > \sigma_n$. Hence f is increasing on $[\lambda_n, +\infty)$. We note that

$$\begin{aligned} f(\lambda_n) &= (n^2+4n+4)\lambda_n^2 - (2n^3+5n^2+4n+1)\lambda_n + n^4 + 2n^3 - 2n - 1 \\ &= n^3 - 3n - 1 \\ &> 0 \quad (n \ge 2). \end{aligned}$$

Then we have $f(x_n) > 0$ for $n \ge 2$. This implies that $(n+2)^2 x_n - (n+1)^2 x_{n+1} \ge 0$ for $n \ge 2$. It follows from Theorem 1 that the sequence $\{\sqrt{T_n}\}_{n\ge2}$ is log-balanced. For $r > 2$, it follows from Theorem 2 that $\{\sqrt[r]{T_n}\}_{n\ge2}$ is log-balanced. □

3.10 The Domb numbers

Let $\{D_n\}_{n\ge0}$ be the sequence of the Domb numbers. The value of D_n is the number of $2n$-step polygons on the diamond lattice. The sequence $\{D_n\}_{n\ge0}$ satisfies the recurrence

$$n^3 D_n = 2(2n-1)(5n^2-5n+2)D_{n-1} - 64(n-1)^3 D_{n-2}, \quad n \ge 2, \tag{26}$$

where $D_0 = 1$ and $D_1 = 4$; see Table 10 for some information about it. It is sequence A002895 in the OEIS. In particular, Wang and Zhu [17] proved that the sequence $\{D_n\}_{n\ge0}$ is log-convex.

Theorem 13. *For $r \ge 2$, the sequence $\{\sqrt[r]{D_n}\}_{n\ge1}$ is log-balanced.*

n	0	1	2	3	4	5	6	7
D_n	1	4	28	256	2716	31504	387136	4951552

Table 10: Some initial values of $\{D_n\}_{n\geq 0}$

Proof. For $n \geq 0$, let $x_n = \frac{D_{n+1}}{D_n}$. It follows from (26) that

$$x_n = \frac{2(2n+1)(5n^2+5n+2)}{(n+1)^3} - \frac{64n^3}{(n+1)^3 x_{n-1}}, \quad n \geq 1. \tag{27}$$

We first show that

$$\lambda_n \leq x_n \leq \mu_n, \quad n \geq 1, \tag{28}$$

where $\lambda_n = \frac{16(n-1)}{n+1}$ and $\mu_n = \frac{16n}{n+1}$. It is obvious that $\lambda_k < x_k < \mu_k$ for $1 \leq k \leq 3$. Assume that $\lambda_k \leq x_k \leq \mu_k$ for $k \geq 3$. By means of (27), we have

$$\begin{aligned} x_{k+1} - \lambda_{k+1} &\geq \frac{2(2k+3)(5k^2+15k+12)}{(k+2)^3} - \frac{16k}{k+2} - \frac{64(k+1)^3}{(k+2)^3\lambda_k} \\ &= \frac{2(3k^3+12k^2-9k-38)}{(k-1)(k+2)^3} > 0 \quad (k \geq 3) \end{aligned}$$

and

$$\begin{aligned} x_{k+1} - \mu_{k+1} &\leq \frac{2(2k+3)(5k^2+15k+12)}{(k+2)^3} - \frac{16(k+1)}{k+2} - \frac{64(k+1)^3}{(k+2)^3\mu_k} \\ &= -\frac{2(3k^3+7k^2+4k+2)}{k(k+2)^3} < 0. \end{aligned}$$

Then we have $\lambda_n \leq x_n \leq \mu_n$ for $n \geq 1$.

It follows from (28) that

$$\begin{aligned} (n+2)^2 x_n - (n+1)^2 x_{n+1} &\geq \frac{16[(n-1)(n+2)^3 - (n+1)^4]}{(n+1)(n+2)} \\ &= \frac{16(n^3-8n-9)}{(n+1)(n+2)} \\ &> 0 \quad (n \geq 4). \end{aligned}$$

On the other hand, we observe that $(n+2)^2 x_n - (n+1)^2 x_{n+1} > 0$ for $0 \leq n \leq 3$. We have from Theorem 1 that the sequence $\{\sqrt{D_n}\}_{n\geq 0}$ is log-balanced. For $r > 2$, it follows from Theorem 2 that $\{\sqrt[r]{D_n}\}_{n\geq 0}$ is log-balanced. $\square$

3.11 Numbers counting a class of $n \times n$ symmetric matrices

Let τ_n denote the number of $n \times n$ symmetric $\mathbb{N}_0$-matrices with every row(and hence every column) sum equals to 2 with trace zero (i.e., all main diagonal entries are zero). The sequence $\{\tau_n\}_{n\geq 0}$ satisfies the recurrence

$$\tau_n = (n-1)\tau_{n-1} + (n-1)\tau_{n-2} - \binom{n-1}{2}\tau_{n-3}, \tag{29}$$

where $\tau_0 = 1, \tau_1 = 0, \tau_2 = \tau_3 = 1$; see Table 11 for some information about it. It is sequence A002137 in the OEIS. In particular, Došlić [7] showed that the sequence $\{\tau_n\}_{n\geq 6}$ is log-convex.

n	0	1	2	3	4	5	6	7	8
τ_n	1	0	1	1	6	22	130	822	6202

Table 11: Some initial values of $\{\tau_n\}_{n\geq 0}$

Theorem 14. *For $r \geq 2$, the sequence $\{\sqrt[r]{\tau_n}\}_{n\geq 6}$ is log-balanced.*

Proof. For $n \geq 2$, set $x_n = \frac{\tau_{n+1}}{\tau_n}$. We now prove by induction that

$$\lambda_n \leq x_n \leq \lambda_{n+1}, \quad n \geq 6, \tag{30}$$

where $\lambda_n = n$. It is clear that $\lambda_k \leq x_k \leq \lambda_{k+1}$ for $k = 6, 7$. Assume that $\lambda_k \leq x_k \leq \lambda_{k+1}$ for $k \geq 7$. It follows from (29) that

$$x_n = n + \frac{n}{x_{n-1}} - \frac{n(n-1)}{2x_{n-2}x_{n-1}}, \quad n \geq 4, \tag{31}$$

By applying (31), we get

$$x_{k+1} - \lambda_{k+1} = \frac{k+1}{x_k} - \frac{(k+1)k}{2x_{k-1}x_k} \quad \text{and} \quad x_{k+1} - \lambda_{k+2} = -1 + \frac{k+1}{x_k} - \frac{(k+1)k}{2x_{k-1}x_k}.$$

Due to $\frac{1}{\lambda_{k+1}} \leq \frac{1}{x_k} \leq \frac{1}{\lambda_k}$ ($k \geq 7$), we have

$$\begin{aligned} x_{k+1} - \lambda_{k+1} &\geq 1 - \frac{k+1}{2(k-1)} \\ &= \frac{k-3}{2(k-1)} \geq 0 \end{aligned}$$

and

$$\begin{aligned} x_{k+1} - \lambda_{k+2} &\leq \frac{1}{k} - \frac{1}{2} \\ &= -\frac{k-2}{2k} \leq 0. \end{aligned}$$

Then we derive $\lambda_n \leq x_n \leq \lambda_{n+1}$ for $n \geq 6$.

By using (30) and (31), we have

$$\begin{aligned}(n+2)^2 x_n - (n+1)^2 x_{n+1} &= (n+2)^2 x_n - (n+1)^3 - \frac{(n+1)^3}{x_n} + \frac{n(n+1)^3}{2x_n x_{n-1}} \\ &\geq (n+2)^2 \lambda_n - (n+1)^3 - \frac{(n+1)^3}{\lambda_n} + \frac{n(n+1)^3}{2\lambda_n \lambda_{n+1}} \\ &\geq \frac{3n^2 - 6n - 22}{6} > 0 \quad (n \geq 6).\end{aligned}$$

It follows from Theorem 1 that the sequence $\{\sqrt{\tau_n}\}_{n\geq 6}$ is log-balanced. For $r > 2$, it follows from Theorem 2 that $\{\sqrt[r]{\tau_n}\}_{n\geq 6}$ is log-balanced. $\square$

In the rest of this section, we discuss log-balancedness of some sequences by mens of Theorem 3.

3.12 The harmonic numbers

Let $\{H_n\}_{n\geq 1}$ be the sequence of harmonic numbers. It is well known that

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \geq 1.$$

Theorem 15. *The sequence $\{n!\sqrt{H_n}\}_{n\geq 1}$ is log-balanced.*

Proof. Using the definition of log-concavity, one can immediately prove that $\{H_n\}_{n\geq 1}$ is log-concave. Moreover, from Zhao [20], $\{n!H_n\}_{n\geq 1}$ is log-balanced. It follows from Theorem 3 that the sequence $\{n!\sqrt{H_n}\}_{n\geq 1}$ is log-balanced. $\square$

3.13 The Fibonacci and Lucas numbers

Let $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ denote the Fibonacci and Lucas sequence, respectively. The Binet's forms of F_n and L_n respectively are

$$F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}}, \quad L_n = \alpha^n + (-1)^n \alpha^{-n}, \quad n \geq 0,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$.

Theorem 16. *The sequences $\{n!\sqrt{F_{2n}}\}_{n\geq 1}$ and $\{n!\sqrt{L_{2n+1}}\}_{n\geq 1}$ are both log-balanced.*

Proof. Using the definition of log-concavity, we can prove that the sequences $\{F_{2n}\}_{n\geq 1}$ and $\{L_{2n-1}\}_{n\geq 1}$ are both log-concave. Zhao [20] showed that $\{n!F_{2n}\}_{n\geq 1}$ and $\{n!L_{2n+1}\}_{n\geq 1}$ are log-balanced. It follows from Theorem 3 that the sequences $\{n!\sqrt{F_{2n}}\}_{n\geq 1}$ and $\{n!\sqrt{L_{2n+1}}\}_{n\geq 1}$ are both log-balanced. $\square$

4 Conclusions

For a log-convex sequence $\{z_n\}_{n\geq 0}$, we have shown that the arithmetic root sequence $\{\sqrt[r]{z_n}\}_{n\geq 0}$ is log-balanced under suitable conditions. We have also derived the log-balancedness of a number of log-convex sequences related to many famous combinatorial numbers. However, we cannot give the minimum value of r such that $\{\sqrt[r]{z_n}\}_{n\geq 0}$ is log-balanced. We hope to solve this question in the future work. In addition, we also hope to find more functions f defined in $(-\infty, +\infty)$ such that $\{f(z_n)\}_{n\geq 0}$ is log-balanced.

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