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The Catalan Threshold Arrangement

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Abstract

The Catalan threshold arrangement is a hyperplane arrangement defined by $x_i + x_j = 0, 1, -1$. Using the finite field method, we obtain the number of regions and the characteristic polynomial of the Catalan threshold arrangement. We also give the exponential generating function for the characteristic polynomial of the Catalan threshold arrangement.

1 Introduction

Hyperplane arrangements are very interesting combinatorial objects and many results can be found in the literature. For instance, several papers [1, 2, 6, 7] are concerned with the characteristic polynomials and the number of regions of a hyperplane arrangement.

In his paper [9], Stanley reviewed various hyperplane arrangements raising interesting questions, one of which is related to the following hyperplane arrangement:

$$x_i + x_j = 0, 1, \quad 1 \le i < j \le n.$$

This is called the Shi threshold arrangement [9, p. 473]. Stanley asked to find the characteristic polynomial of the Shi threshold arrangement. In a recent paper [7], the author provided an answer to that question by applying the finite field method. The finite field method was first developed into a tool for computing characteristic polynomials by Athanasiadis [2].

In this paper, we study the following generalization of the Shi threshold arrangement:

$$x_i + x_j = 0, 1, -1, \quad 1 \le i < j \le n.$$

Throughout this paper, we will call this arrangement the Catalan threshold arrangement. The main result of this paper is the characteristic polynomial of the Catalan threshold arrangement. We also obtain the exponential generating function for the characteristic polynomial.

This paper is organized as follows. In Section 2, we recall some basic facts on hyperplane arrangements and related combinatorial objects that are used in the sequel. In Section 3, we prove our main result in Theorem 8 by the finite field method. The exponential generating function for the characteristic polynomial of the Catalan threshold arrangement is proven in Equation (17).

2 Preliminaries

We recall some notation and concepts of hyperplane arrangements and related combinatorial objects. For a more thorough introduction, see [4, 5, 9].

2.1 Basic concepts of hyperplane arrangements

Given a positive integer n and a field K, a finite set of affine hyperplanes in the vector space K^n is called a *hyperplane arrangement* (or *arrangement* for short). Now, let $K=\mathbb{R}$. A *region* of an arrangement $\mathcal{A}$ is a connected component of $\mathbb{R}^n - \bigcup_{H\in\mathcal{A}} H$. We denote the number of regions of $\mathcal{A}$ by $r(\mathcal{A})$.

Given an arrangement $\mathcal{A}$ in a vector space V, let $L(\mathcal{A})$ be the set of all nonempty intersections of hyperplanes in $\mathcal{A}$, including V. We define $x \le y$ in $L(\mathcal{A})$ if $x \supseteq y$. We call $L(\mathcal{A})$ the *intersection poset* of $\mathcal{A}$.

For a finite poset P with $\hat{0}$, the *Möbius function* $\mu : P \to \mathbb{Z}$ is defined by

$$\mu(\hat{0}) = 1 \quad \text{and} \quad \mu(x) = -\sum_{y<x} \mu(y).$$

Definition 1. The *characteristic polynomial* $\chi_{\mathcal{A}}(t)$ of the arrangement $\mathcal{A}$ is defined by

$$\chi_{\mathcal{A}}(t) := \sum_{x\in L(\mathcal{A})} \mu(x)\, t^{\dim(x)},$$

where $\dim(x)$ is the dimension of x as an affine subspace of V.

The characteristic polynomial plays an important role in analyzing arrangements. One of the fundamental result is the following theorem.

Theorem 2 (Zaslavsky [12]). *For any arrangement $\mathcal{A}$ in $\mathbb{R}^n$, we have*

$$r(\mathcal{A}) = (-1)^n\, \chi_{\mathcal{A}}(-1).$$

In general, it is hard to compute the characteristic polynomial of an arrangement $\mathcal{A}$. However, if $\mathcal{A}$ is in $\mathbb{Q}^n$ (i.e., all coefficients of hyperplanes in $\mathcal{A}$ are rational), then there is a powerful method in computing its characteristic polynomial. For a prime number p, let $\mathbb{F}_p$ be the finite field of order p. If H is a hyperplane of $\mathcal{A}$ in $\mathbb{Q}^n$, by multiplying a proper integer to the equation of H, we can make all the coefficients of the equation of H integers. We then reduce the coefficients modulo p to get an arrangement $\mathcal{A}_p$ in $\mathbb{F}_p^n$. It is well known that there are all but finitely many primes p such that $L(\mathcal{A})$ is isomorphic to $L(\mathcal{A}_p)$.

Theorem 3 (Athanasiadis [2]). *Let $\mathcal{A}$ be an arrangement in $\mathbb{Q}^n$. If $L(\mathcal{A}) \cong L(\mathcal{A}_q)$ for some prime q, then*

$$\chi_{\mathcal{A}}(q) = \left| \mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}_q} H \right| = q^n - \left| \bigcup_{H \in \mathcal{A}_q} H \right|,$$

which is called the finite field method.

We now consider two special hyperplane arrangements: the *Catalan arrangement* and the *threshold arrangement*. The *Catalan arrangement* $\mathcal{C}_n$ is given by

$$x_i - x_j = 0, 1, -1 \quad 1 \le i < j \le n.$$

The characteristic polynomial of $\mathcal{C}_n$ is

$$\chi_{\mathcal{C}_n}(t) = t(t-n-1)(t-n-2)\cdots(t-2n+1).$$

Applying Zaslavsky's theorem, we have

$$r(\mathcal{C}_n) = (2n)(2n-1)\cdots(n+2) = n!C_n,$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number A000108.

The *threshold arrangement* $\mathcal{T}_n$ is given by

$$x_i + x_j = 0, \quad 1 \le i < j \le n.$$

The "threshold" comes from threshold graphs introduced by Chvátal and Hammer [3]. There is a canonical bijection between the set of regions of $\mathcal{T}_n$ and the set of threshold graphs on the vertex set $[n] := \{1, 2, \ldots, n\}$. The number of threshold graphs appears in A005840. Stanley [9, p. 473] showed that the exponential generating function for the characteristic polynomial of $\mathcal{T}_n$ is given by

$$\sum_{n \ge 0} \chi_{\mathcal{T}_n}(t) \frac{x^n}{n!} = (1+x)(2e^x - 1)^{(t-1)/2}.$$

2.2 Delannoy numbers and Schröder numbers

Given nonnegative integers p and q, consider a lattice path P in the plane from $(0,0)$ to (p,q) using steps $(1,0)$, $(0,1)$, or $(1,1)$. The total number $D(p,q)$ of such paths is called the *Delannoy number* A008288. For an nonnegative integer n, $D(n,n)$ is called the *central Delannoy number* A001850. The following properties of Delannoy numbers are well known [4, p. 81].

Proposition 4. *The Delannoy number $D(p,q)$ satisfies the following:*

1. *The Delannoy number $D(p,q)$ has two expressions*

$$D(p,q) = \sum_{d\geq 0} \binom{q}{d}\binom{p+q-d}{q}, \tag{1}$$

$$D(p,q) = \sum_{d\geq 0} 2^d \binom{p}{d}\binom{q}{d}. \tag{2}$$

2. *The Delannoy number $D(p,q)$ satisfies the recurrence relation*

$$D(p,q) = D(p-1,q) + D(p,q-1) + D(p-1,q-1). \tag{3}$$

3. *The generating function $D(x)$ for $D(n,n)$ is*

$$D(x) := \sum_{n\geq 0} D(n,n)x^n = \frac{1}{\sqrt{1-6x+x^2}}. \tag{4}$$

For a nonnegative integer n, consider a lattice path P in the plane from $(0,0)$ to (n,n) using steps $(1,0)$, $(0,1)$, or $(1,1)$, such that P never passes above the line $y=x$. The total number r_n of such paths is called the *Schröder number* A006318. The following properties of Schröder numbers are well known [10].

Proposition 5. *The Schröder number r_n satisfies the following:*

1. *The Schröder number r_n satisfies the recurrence relation*

$$r_n = r_{n-1} + \sum_{k=0}^{n-1} r_k\, r_{n-1-k}\,, \quad r_0 = 1.$$

2. *The generating function $R(x)$ for r_n is*

$$R(x) := \sum_{n\geq 0} r_n x^n = \frac{1-x-\sqrt{1-6x+x^2}}{2x}. \tag{5}$$

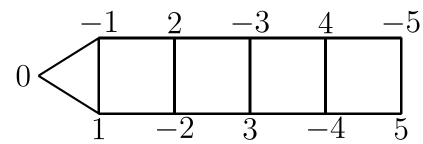

Figure 1: Graph G_{11}

3 Main result

Recall that the Catalan threshold arrangement is defined by

$$x_i + x_j = 0, 1, -1, \quad 1 \le i < j \le n. \tag{6}$$

We denote by $\mathcal{CT}_n$ the Catalan threshold arrangement. In this section, we will find the characteristic polynomial of Catalan threshold arrangement and its exponential generating function.

3.1 The characteristic polynomial of $\mathcal{CT}_n$

Let X be the set defined by

$$X = \{(a_1, \dots, a_n) \in \mathbb{F}_q^n \mid a_i + a_j \ne 0, 1, -1, \text{ for } i < j\}. \tag{7}$$

By the finite field method, there exist infinitely many odd primes $q = 2r + 1$ such that

$$\chi_{\mathcal{CT}_n}(q) = |X|.$$

Let G_q be a simple graph such that

$$V(G_q) = \mathbb{F}_q \quad \text{and} \quad E(G_q) = \{\, \{u, v\} \mid u, v \in \mathbb{F}_q \text{ with } u + v = 0, 1, -1 \,\}.$$

See Figure 1 for G_{11}. We can regard an element $(a_1, \dots, a_n)$ in $\mathbb{F}_q^n$ as a function $f : [n] \to \mathbb{F}_q$ satisfying $f(i) = a_i$ for all $i = 1, 2, \dots, n$. It is obvious that $f \in \mathbb{F}_q^n$ belongs to X if and only if f satisfies the following two conditions.

(C1) The image of f, i.e., $f([n])$ is an independent set of G_q, and

(C2) $|f^{-1}(0)| \le 1$, $|f^{-1}(r)| \le 1$, and $|f^{-1}(-r)| \le 1$ hold.

Here, an independent set of G_q is a subset I of $V(G_q)$ such that no two vertices of I represent an edge of G_q. Note that $I = \emptyset$ is always an independent set. Now, we will count the number of functions $f : [n] \to \mathbb{F}_q$ satisfying two conditions (C1) and (C2).

To consider the condition (C1), we need to find the number of independent sets of the following graph. Given a nonnegative integer m, let A_m be a graph on the vertex set $V(A_m) = \{1, 2, \dots, m\} \cup \{1', 2', \dots, m'\}$ with the edge set

$$E(A_m) = \{\{i, i+1\} \mid 1 \le i \le m-1\} \cup \{\{i', (i+1)'\} \mid 1 \le i \le m-1\} \cup \{\{i, i'\} \mid 1 \le i \le m\},$$

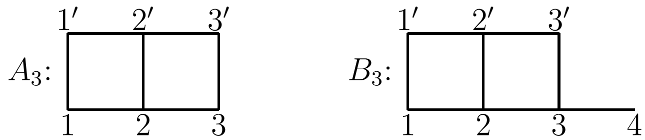

Figure 2: Graphs A_3 and B_3

for $m \geq 1$ and A_0 be the empty graph. We also define the graph B_m by $A_{m+1} - (m+1)'$. See Figure 2 for A_3 and B_3.

For nonnegative integers k and m, let $a(m,k)$ be the number of k-independent sets of A_m, and $b(m,k)$ be the number of k-independent sets of B_m. By convention, we set $a(0,k) = \delta_{0,k}$, for the empty graph A_0 which has the unique independent set $\emptyset$.

Lemma 6. *For nonnegative integers k and m, we have*

$$b(m,k) = D(m-k+1,k), \tag{8}$$

$$a(m,k+1) = D(m-k,k+1) - D(m-k,k). \tag{9}$$

Here $D(p,q)$ is the Delannoy number.

Proof. To select a k-independent set I of B_m, we can consider three cases: (i) $m \notin I$ and $m+1 \notin I$, (ii) $m \notin I$ and $m+1 \in I$, (iii) $m \in I$. From this, we see that

$$b(m,k) = b(m-1,k) + b(m-1,k-1) + b(m-2,k-1),$$

which shows that $b(m,k)$ satisfies the same recursion as $D(m-k+1,k)$ in (3).

We now check the initial conditions. Note that

$$b(m,k) = \begin{cases} 0, & k-m \geq 2; \\ 1, & k-m = 1 \text{ or } k = 0; \\ 3, & k = m = 1, \end{cases} \quad \text{and} \quad D(p,q) = \begin{cases} 0, & p \leq -1; \\ 1, & p = 0 \text{ or } q = 0; \\ 3, & p = q = 1. \end{cases}$$

Thus, $b(m,k) = D(m-k+1,k)$ for all nonnegative integer m and k.

Next, we prove (9). To select a $(k+1)$-independent set J of B_m, we can consider two cases: (i) $m+1 \notin J$ and (ii) $m+1 \in J$. Thus we have

$$b(m,k+1) = a(m,k+1) + b(m-1,k),$$

which yields $a(m,k+1) = b(m,k+1) - b(m-1,k)$. □

From Lemma 6 we can easily get a number of functions f satisfying the condition (C1).

Lemma 7. *For positive integers m and n, the number of $f : [n] \to V(A_m)$ satisfying that $f([n])$ is an independent set of A_m is*

$$\sum_{1\le j\le \min(m,n)} a(m,j)\, S(n,j)\, j!,$$

and the number of $f : [n] \to V(B_m)$ satisfying that $f([n])$ is an independent set of B_m is

$$\sum_{1\le j\le \min(m,n)} b(m,j)\, S(n,j)\, j!,$$

where $S(n,j)$ is the Stirling number of the second kind.

For a nonnegative integers m and n, let $a_m(n)$ and $b_m(n)$ be

$$a_m(n) := \sum_{0\le j\le \min(m,n)} a(m,j)\, S(n,j)\, j!\,, \tag{10}$$

$$b_m(n) := \sum_{0\le j\le \min(m,n)} b(m,j)\, S(n,j)\, j!\,. \tag{11}$$

Now we go back to Catalan threshold arrangements. To compute the size of X in (7), we should also regard the second condition (C2). Among $f = (f(1), \dots, f(n)) \in X$, each 0, r and $-r$ can be chosen at most once. So we have four cases to consider.

(i) None of them are selected.

(ii) Only 0 is selected.

(iii) 0 is not selected and either r or $-r$ is selected.

(iv) 0 is selected and either r or $-r$ is selected.

By Lemma 7, the number of elements f of the case (i) is

$$\sum_{0\le j\le \min(r-1,n)} a(r-1,j)\, S(n,j)\, j!\,,$$

which reduces to $a_{r-1}(n)$. Similarly, the case (ii) is $n\, a_{r-2}(n-1)$. The case (iii) is

$$2n \sum_{0\le j\le \min(r-2,n-1)} b(r-2,j)\, S(n-1,j)\, j!,$$

which reduces to $2n\, b_{r-2}(n-1)$. Similarly, the case (iv) is $2n(n-1)\, b_{r-3}(n-2)$. Thus we have the following result.

Theorem 8 (Main result)**.** *The characteristic polynomial of the Catalan threshold arrangement $\mathcal{CT}_n$ is given by*

$$\chi_{\mathcal{CT}_n}(t) = n! \sum_{k=0}^{n} \sum_{\ell=0}^{k} \alpha_{n,k,\ell} \, \frac{(\!(t-2k-1)\!)_\ell}{\ell!} \,, \tag{12}$$

where $(\!(x)\!)_k$ is defined by $(\!(x)\!)_0 = 1$ and $(\!(x)\!)_k = x(x-2)(x-4)\cdots(x-2k+2)$ for $k \geq 1$. Here $\alpha_{n,k,\ell}$ is given by $\alpha_{0,0,0} = \alpha_{1,0,0} = 1$, and for $n \geq 2$ or $k^2 + \ell^2 > 0$,

$$\alpha_{n,k,\ell} = \binom{k-1}{\ell-1} s_{n,k} + \binom{k-2}{\ell-1} s_{n-1,k-1} + 2\binom{k-1}{\ell} s_{n-1,k-1} + 2\binom{k-2}{\ell} s_{n-2,k-2} \,,$$

where $s_{n,k}$ is defined by $s_{n,k} = \frac{k!}{n!} S(n,k)$.

Proof. Since we can easily check that (12) holds for $n \leq 2$, we may assume $n \geq 3$. By considering all the cases (i)–(iv), we have

$$\begin{aligned}
\chi_{\mathcal{CT}_n}(q) =& a_{r-1}(n) + n\, a_{r-2}(n-1) + 2n\, b_{r-2}(n-1) + 2n(n-1)\, b_{r-2}(n-2) \\
=& \sum_{j \geq 0} a\left(\frac{q-3}{2}, j\right) j!\, S(n,j) + n \sum_{j \geq 0} a\left(\frac{q-5}{2}, j\right) j!\, S(n-1,j) \\
& + 2n \sum_{j \geq 0} b\left(\frac{q-5}{2}, j\right) j!\, S(n-1,j) + 2n(n-1) \sum_{j \geq 0} b\left(\frac{q-7}{2}, j\right) j!\, S(n-2,j) \,,
\end{aligned} \tag{13}$$

for infinitely many primes $q = 2r+1$. Since $n \geq 3$, we can ignore the case $j = 0$. By applying (8), (9), and (2), we have

$$\begin{aligned}
\chi_{\mathcal{CT}_n}(t) =& \sum_{j=1}^{n} \sum_{d=1}^{j} S(n,j) \frac{j!}{d!} \binom{j-1}{d-1} (\!(t-2j-1)\!)_d \\
& + \sum_{j=1}^{n} \sum_{d=1}^{j} n\, S(n-1,j) \frac{j!}{d!} \binom{j-1}{d-1} (\!(t-2j-3)\!)_d \\
& + \sum_{j=1}^{n} \sum_{d=0}^{j} 2n\, S(n-1,j) \frac{j!}{d!} \binom{j}{d} (\!(t-2j-3)\!)_d \\
& + \sum_{j=1}^{n} \sum_{d=0}^{j} 2n(n-1)\, S(n-2,j) \frac{j!}{d!} \binom{j}{d} (\!(t-2j-5)\!)_d \,.
\end{aligned}$$

Let $[(\!(t-2k-1)\!)_\ell]\, F(x)$ be the "coefficient" of $(\!(t-2k-1)\!)_\ell$ in $F(x)$. Then

$$\begin{aligned}
[(\!(t-2k-1)\!)_\ell] \chi_{\mathcal{CT}_n}(t) \;=\;& S(n,k) \frac{k!}{\ell!} \binom{k-1}{\ell-1} + nS(n-1,k-1) \frac{(k-1)!}{\ell!} \binom{k-2}{\ell-1} \\
& + 2nS(n-1,k-1) \frac{k!}{\ell!} \binom{k-1}{\ell} + 2n(n-1) S(n-2,k-2) \frac{(k-2)!}{\ell!} \binom{k-2}{\ell}.
\end{aligned}$$

Since $\alpha_{n,k,\ell} = \frac{\ell!}{n!}\left[((t-2k-1))_\ell\right]\chi_{\mathcal{CT}_n}(t)$, we have

$$\begin{aligned}\alpha_{n,k,\ell} =& \frac{k!}{n!}S(n,k)\binom{k-1}{\ell-1} + \frac{(k-1)!}{(n-1)!}S(n-1,k-1)\binom{k-2}{\ell-1} \\ &+ 2\frac{(k-1)!}{(n-1)!}S(n-1,k-1)\binom{k-1}{\ell} + 2\frac{(k-2)!}{(n-2)!}S(n-2,k-2)\binom{k-2}{\ell}.\end{aligned}$$

□

For instance $\chi_{\mathcal{CT}_0}(t) = 1$, $\chi_{\mathcal{CT}_1}(t) = t$, $\chi_{\mathcal{CT}_2}(t) = t^2 - 3t$, and

$$\begin{aligned}\chi_{\mathcal{CT}_3}(t) &= t^3 - 9t^2 + 27t - 27 \\ \chi_{\mathcal{CT}_4}(t) &= t^4 - 18t^3 + 135t^2 - 483t + 675 \\ \chi_{\mathcal{CT}_5}(t) &= t^5 - 30t^4 + 405t^3 - 2955t^2 + 11340t - 17993.\end{aligned}$$

Applying Zaslavsky's theorem, we have the following result.

Corollary 9. *The number of regions of the Catalan threshold arrangement $\mathcal{CT}_n$ is given by*

$$r(\mathcal{CT}_n) = (-1)^n n! \sum_{k=0}^{n}\sum_{\ell=0}^{k} \alpha_{n,k,\ell}\,(-2)^\ell \binom{k+1}{\ell},$$

where $\alpha_{n,k,\ell}$ is defined in Theorem 8.

The sequence $(r(\mathcal{CT}_n))_{n\geq 0}$ starts with

$$1,\ 1,\ 4,\ 64,\ 1312,\ 32724,\ 979316,\ \ldots$$

which is not listed in the On-Line Encyclopedia of Integer Sequences [8].

3.2 The exponential generating function for $\chi_{\mathcal{CT}_n}(t)$

Recall the definition of $b_m(n)$ in (11). By the exponential formula we have

$$b_m(n) = \sum_{j\geq 0} b(m,j)\left[\frac{x^n}{n!}\right](e^x-1)^j,$$

where $\left[\frac{x^n}{n!}\right]F(x)$ is the coefficient of $\frac{x^n}{n!}$ in the power series $F(x)$. Thus

$$\sum_{n\geq 0} b_m(n)\frac{x^n}{n!} = \sum_{j\geq 0} b(m,j)(e^x-1)^j. \tag{14}$$

It is easy to show that

$$\sum_{j\geq 0} D(j+\alpha,j)z^j = \frac{R(z)^\alpha}{D(z)}, \tag{15}$$

where $D(z)$ and $R(z)$ are given in (4) and (5). Note that this is an extended version of the following well known (for example [11, p. 54]) identity

$$\sum_{j\geq 0}\binom{2j+\alpha}{j}z^j = \frac{C(z)^\alpha}{\sqrt{1-4z}}\,,$$

where $C(z)$ is the ordinary generating function for the Catalan number.

Recall $b(m,j) = D(m-j+1,j)$. By (1), we see that $b(m,j)$ can be considered as a monic polynomial in m with degree j. By simple calculation, we have

$$\begin{aligned}
D(m-j+1,j) &= \sum_{0\leq d\leq j}\binom{j}{d}\binom{m+1-d}{j} \\
&= (-1)^j\sum_{0\leq d\leq j}\binom{j}{d}\binom{j-m-2+d}{j} \\
&= (-1)^j\sum_{0\leq c\leq j}\binom{j}{c}\binom{2j-m-2-c}{j} && (c=j-d) \\
&= (-1)^j D(j-m-2,j).
\end{aligned}$$

By replacing $\alpha = -m-2$ and $z = -(e^x-1)$, we obtain

$$\sum_{j\geq 0} D(j+\alpha,j)z^j = \sum_{j\geq 0} b(m,j)(e^x-1)^j\,. \tag{16}$$

Combining (14), (15), and (16) yields that

$$\sum_{n\geq 0} b_m(n)\frac{x^n}{n!} = \frac{R(1-e^x)^{-m-2}}{D(1-e^x)}\,.$$

Meanwhile, from (10), we deduce that

$$\begin{aligned}
\sum_{n\geq 0} a_m(n)\frac{x^n}{n!} &= \sum_{n\geq 0}\sum_{j\geq 0} a(m,j)S(n,j)j!\frac{x^n}{n!} \\
&= \sum_{n\geq 0}\sum_{j\geq 0} b(m-1,j)S(n,j)j!\frac{x^n}{n!} + \sum_{n\geq 0}\sum_{j\geq 1} b(m-2,j-1)S(n,j)j!\frac{x^n}{n!} \\
&= \sum_{n\geq 0} b_{m-1}(n)\frac{x^n}{n!} + \sum_{j\geq 1} b(m-2,j-1)(e^x-1)^j \\
&= \sum_{n\geq 0} b_{m-1}(n)\frac{x^n}{n!} + (e^x-1)\sum_{n\geq 0} b_{m-2}(n)\frac{x^n}{n!}.
\end{aligned}$$

Thus we have

$$\sum_{n\geq 0} a_m(n)\frac{x^n}{n!} = \frac{R(1-e^x)^{-m-1} - R(1-e^x)^{-m}}{D(1-e^x)}\,.$$

As seen in (13), the characteristic polynomial $\chi_{\mathcal{CT}_n}(t)$ can be expressed as a linear combination of $a_m(n)$ and $b_m(n)$. So, with simple calculations, we can deduce the exponential generating function for $\chi_{\mathcal{CT}_n}(t)$:

$$\sum_{n\geq 0} \chi_{\mathcal{CT}_n}(t)\frac{x^n}{n!} \;=\; \frac{R(1-e^x)^{\frac{1-t}{2}}}{D(1-e^x)}\Big(1+xR(1-e^x)\Big)\Big(1+2x-(1-e^x)R(1-e^x)\Big). \tag{17}$$

Put $t=-1$ and $x=-x$ in (17) to get the exponential generating function for $r(\mathcal{CT}_n)$:

$$\sum_{n\geq 0} r(\mathcal{CT}_n)\frac{x^n}{n!} \;=\; \frac{R(1-e^{-x})}{D(1-e^{-x})}\Big(1-xR(1-e^{-x})\Big)\Big(1-2x-(1-e^{-x})R(1-e^{-x})\Big). \tag{18}$$

3.3 Remark

Consider a generalized threshold arrangement such as

$$x_i + x_j = -\ell, -\ell+1, \ldots, m-1, m \quad 1 \leq i < j \leq n,$$

for nonnegative integers ℓ and m. We denote the arrangement by $\mathcal{T}_n^{[-\ell,m]}$. By parallel translation, $\mathcal{T}_n^{[-\ell,m]}$ is transformed to either $\mathcal{T}_n^{[-k+1,k]}$ or $\mathcal{T}_n^{[-k,k]}$, which can be called an *extended Shi threshold arrangement* or an *extended Catalan threshold arrangement*. In particular, $\mathcal{T}_n^{[0,1]} = \mathcal{ST}_n$ and $\mathcal{T}_n^{[-1,1]} = \mathcal{CT}_n$. It would be interesting to find the characteristic polynomials of these two arrangements for $k \geq 2$.

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