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Generalized Anti-Waring Numbers

Chris Fuller and Robert H. Nichols, Jr.
Labry School of Science, Technology, and Business
Cumberland University
1 Cumberland Square
Lebanon, TN 37087
USA
cfuller@cumberland.edu
rnichols@cumberland.edu

Abstract

The anti-Waring problem considers the smallest positive integer such that it and every subsequent integer can be expressed as the sum of the k^{th} powers of r or more distinct natural numbers. We give a generalization that allows elements from any nondecreasing sequence, rather than only the natural numbers. This generalization is an extension of the anti-Waring problem, as well as the idea of complete sequences. We present new anti-Waring and generalized anti-Waring numbers, as well as a result to verify computationally when a generalized anti-Waring number has been found.

1 Introduction

For positive integers k and r, the anti-Waring number $N(k, r)$ is defined to be the smallest positive integer such that $N(k, r)$ and every subsequent positive integer can be expressed as the sum of the k^{th} powers of r or more distinct positive integers. Several authors [3, 5, 7, 11] recently reported results on anti-Waring numbers.

Early results considered only $r = 1$. As early as 1948, Sprague found that $N(2, 1) = 129$ [15] and proved that $N(k, 1)$ exists for all $k \geq 2$ [16]. In 1964, Graham [6] reported that $N(3, 1) = 12759$ (Graham [6] references another Graham paper "On the Threshold of completeness for certain sequences of polynomial values" said to appear circa 1964). Dressler and Parker [4] also computed $N(3, 1)$ in 1974. Lin [10] used Graham's method to find that

$N(4,1) = 5134241$ with a computer in 1970. In 1992, Patterson [12, pp. 18–23] found that $N(5,1) = 67898772$. In this paper, we independently verify each of these numbers and show that $N(6,1) = 11146309948$.

More recently, Looper and Saritzky [11] proved that $N(k,r)$ exists for all positive integers k and r. Deering and Jamieson [3] found specific values of $N(2,r)$ for $1 \leq r \leq 10$ and $N(3,r)$ for $1 \leq r \leq 5$. Shortly afterwards, Fuller et al. [5] computed values of $N(2,r)$ for $1 \leq r \leq 50$ and $N(3,r)$ for $1 \leq r \leq 30$. We also verify these numbers and present $N(k,r)$ for more values of k and r. One can verify a suspected value of $N(k,r)$ using different sets of conditions [3, 5].

In an effort to generalize the anti-Waring results we consider a nondecreasing sequence of positive integers $A = (a_i)_{i \in \mathbb{N}}$. Here and throughout we use $\mathbb{N} = \{1, 2, 3, \dots\}$. For positive integers k, n, and r we define the *generalized anti-Waring number* $N(k,n,r,A)$ to be the smallest positive integer, if it exists, such that it and every subsequent positive integer can be expressed as the sum of the k^{th} powers of the a_i with $i \geq n$ ranging over r or more distinct values. If the sequence A has all distinct elements, we may use set notation for the last argument of the generalized anti-Waring number. The generalized anti-Waring number $N(k,n,r,A)$ does not exist for all sequences A (see Theorems 1 and 2 in Section 2). Looper and Saritzky [11] proved that both the anti-Waring number $N(k,r)$ and the generalized anti-Waring number $N(k,n,r,\mathbb{N})$ exist for all positive integers k, n, and r.

Early results of these generalized anti-Waring numbers when restricting r to 1 used different terminology. A nondecreasing sequence S of positive integers is *complete* if all sufficiently large positive integers can be written as a sum of distinct elements of S. If S is a complete sequence, the *threshold of completeness*, $\theta(S)$, is the largest positive integer that is not expressible as a sum of distinct elements of S. Therefore, the threshold of completeness, $\theta(S)$, is one less than the generalized anti-Waring number $N(1,1,1,S)$. Also, if $S = (s_i)_{i \in \mathbb{N}}$ is a nondecreasing sequence of positive integers such that the sequence $(s_i^k)_{i \geq n}$ is complete, then the generalized anti-Waring number $N(k,n,1,S)$ exists and $N(k,n,1,S) - 1 = \theta\left((s_i^k)_{i \geq n}\right)$. Brown [1] defined a sequence to be complete only when the threshold of completeness is zero; we use the more general definition.

In the literature on complete sequences, some authors only report that a sequence is complete and hence the generalized anti-Waring number exists; some authors actually find the threshold of completeness. In 1952, Lekkerkerker [9] reported an account of the Zeckendorf representation (circa 1939 [17]), i.e., that every natural number is either a Fibonacci number or can be expressed as the sum of nonconsecutive Fibonacci numbers. Hence the generalized anti-Waring number for the Fibonacci sequence F is $N(1,1,1,F) = 1$. In 1975, Kløve [8] found thresholds of completeness for sequences of the form $\left(\lfloor i^\alpha \rfloor\right)_{i \in \mathbb{N}}$, where $\lfloor x \rfloor$ is the floor function, for $1 \leq \alpha \leq 4.18$ in increments of 0.02. In 1978, Porubský [13] proved that $N(k,1,1,\mathbb{P})$ exists for all positive integers k and the sequence of primes $\mathbb{P}$. Burr and Erdős [2] considered perturbations of complete sequences that resulted in noncomplete sequences and vice versa.

Generalized anti-Waring numbers extend the concept of anti-Waring numbers to sequences other than $\mathbb{N}$. The generalization also extends the concept of complete sequences to consider sums of r or more terms. We will present conditions needed to verify values of $N(k, n, r, A)$ computationally, sequences for which no $N(k, n, r, A)$ exists, and new values of $N(k, n, r, A)$ for various sequences.

2 Verifying $N(k, n, r, A)$, when it exists

For given positive integers k, n, r, and any nondecreasing sequence of positive integers $A = (a_i)_{i \in \mathbb{N}}$, we define a positive integer to be (k, n, r, A)-*good* if it can be written as a sum of the k^{th} powers of r or more distinct elements of the sequence $(a_i)_{i \geq n}$. We define a positive integer that is not (k, n, r, A)-good to be (k, n, r, A)-*bad*. Hence the generalized anti-Waring number $N(k, n, r, A)$ is the smallest positive integer such that it and every subsequent integer is (k, n, r, A)-good. Equivalently the threshold of completeness $N(k, n, r, A) - 1$ is the largest integer that is (k, n, r, A)-bad.

The generalized anti-Waring number $N(k, n, r, A)$ does not exist for all sequences A. For example, the sum of any elements of the sequence $(2, 4, 6, 8, \ldots)$ of positive even integers will never be odd. This is an instance of a more general phenomenon.

Theorem 1. *Let $A = (a_i)_{i \in \mathbb{N}}$ be a nondecreasing sequence of positive integers. If all a_i for $i \geq n$ have a common divisor $d > 1$, then for any positive integers k and r, the generalized anti-Waring number $N(k, n, r, A)$ does not exist.*

Proof. Every sum of positive powers of the a_i, $i \geq n$, is divisible by d. Since $d > 1$, arbitrarily large integers not divisible by d exist. Thus, arbitrarily large integers not representable in any way as a sum of powers of some of the $a_n, a_{n+1}, \ldots$ also exist. □

If instead the greatest common divisor is one, then the generalized anti-Waring number may or may not exist. We will consider examples of both cases.

As an additional example, the sequence of factorials has no generalized anti-Waring number.

Theorem 2. *Let $A = (i!)_{i \in \mathbb{N}}$, and let k, n, and r are any positive integers. Then the generalized anti-Waring number $N(k, n, r, A)$ does not exist.*

Proof. First notice that for each $a_i \in A$,

$$a_i^k \bmod 6 \equiv \begin{cases} 1, & \text{if } i = 1; \\ 2^k \bmod 6, & \text{if } i = 2; \\ 0, & \text{if } i > 2. \end{cases}$$

Consider any (k, n, r, A)-good number m. Distinct integers $i_1, i_2, \ldots, i_t$ exist such that

$$m = a_{i_1}^k + a_{i_2}^k + \cdots + a_{i_t}^k$$

where $t \geq r$ and $i_\alpha \geq n$ for each $\alpha \in \{1, 2, \ldots, t\}$. Thus the sum m must be 0, 1, 2^k, or $1 + 2^k$ modulo 6. Since we can have at most four consecutive (k, n, r, A)-good integers, no largest (k, n, r, A)-bad integer exists. □

On the other hand, in some cases the generalized anti-Waring number $N(k,n,r,A)$ is known to exist, but its value has not been found. As mentioned above, both the anti-Waring number $N(k,r)$ and the generalized anti-Waring number $N(k,n,r,\mathbb{N})$ exist for all k, n, and r [11]. A general formula for either of these is not known, but we present several values in the next section. We rewrite the following result related to complete sequences by Brown [1, Theorem 1] in terms of generalized anti-Waring numbers.

Theorem 3. *Let k and n be positive integers, and let $A=(a_i)_{i\in\mathbb{N}}$ be a nondecreasing sequence of positive integers. The generalized anti-Waring number $N(k,n,1,A)$ both exists and equals one if and only if (i) $a_n=1$ and (ii) for all integers $p\geq n$, $a_{p+1}^k\leq 1+\sum_{i=n}^{p} a_i^k$.*

This result only considers $r=1$. Also since Brown [1] defined complete sequences requiring the threshold of completeness to be zero, he requires $a_n=1$. Theorem 3 proves that all positive integers are representable as a sum of different elements of sequences such as the natural numbers, the Fibonacci numbers, and the powers of two (including 2^0). We must consider different conditions for the more general definition of complete sequences with any threshold of completeness.

The next result from Graham [6, Theorem 4] establishes completeness conditions for sequences generated by polynomials.

Theorem 4. *Let $f(x)$ be a polynomial with real coefficients expressed in the form*

$$f(x)=\alpha_0+\alpha_1\binom{x}{1}+\cdots+\alpha_n\binom{x}{n},\quad \alpha_n\neq 0.$$

The sequence $S(f)=(f(1),f(2),\cdots)$ is complete if and only if

1. *$\alpha_k=p_k/q_k$ for some integers p_k and q_k with $\gcd(p_k,q_k)=1$ and $q_k\neq 0$ for $0\leq k\leq n$,*

2. *$\alpha_n>0$, and*

3. *$\gcd(p_0,p_1,\ldots,p_n)=1$.*

Again, in terms of generalized anti-Waring numbers Theorem 4 only considers the case of $r=1$ and can only be used to establish that a given generalized anti-Waring number exists. As a remark to this theorem, Graham notes that a sequence $(f(1),f(2),f(3),\ldots)$ is complete if and only if $(f(n),f(n+1),f(n+2),\ldots)$ is complete for any n. The next theorem shows that nothing like this can be expected in general.

Theorem 5. *Let k, n, and r be positive integers, and let A be a sequence of nondecreasing positive integers. If the generalized anti-Waring number $N(k,n,r,A)$ exists, then so does $N(k,j,r,A)$ for $j\in\{1,2,\ldots,n-1\}$ and $N(k,j,r,A)\leq N(k,n,r,A)$. Furthermore, the converse is false.*

Proof. The implication is clear. If all positive integers greater than or equal to $N(k,n,r,A)$ can be written as a sum k^{th} powers of r or more distinct elements of $(a_i)_{i\geq n}$, then, with the same elements, each positive integer can be written as a sum k^{th} powers of r or more distinct elements of $(a_i)_{i\geq j}$ for $j \in \{1,2,\ldots,n-1\}$. Therefore, we have $N(k,j,r,A) \leq N(k,n,r,A)$ for $j \in \{1,2,\ldots,n-1\}$.

To see that the converse is false, consider the sequence $A = (2^{i-1})_{i\in\mathbb{N}}$. From the binary representation of the positive integers, the generalized anti-Waring number $N(1,1,1,A)$ clearly exists and equals one. However, the generalized anti-Waring number $N(1,2,1,A)$ does not exist because no odd integer can be expressed as a sum of elements from $(2^{i-1})_{i\geq 2}$. $\square$

In general, whether $N(k,n,r,A)$ exists or not cannot easily be determined. However, we can validate a suspect value of $N(k,n,r,A)$ if enough consecutive integers are (k,n,r,A)-good and certain other conditions are met. Theorem 6 is a generalization of a recent result for anti-Waring numbers [5, Theorem 2.2].

Theorem 6. *Let k, n, r, b, and $\hat{N}$ be positive integers, and let $A = (a_i)_{i\in\mathbb{N}}$ with $0 < a_i \leq a_{i+1}$ and $a_i \in \mathbb{N}$ for all i. If the consecutive integers $\{\hat{N},\ldots,b^k\}$ are all (k,n,r,A)-good, the number $\hat{N}-1$ is (k,n,r,A)-bad, and there exists a positive integer x such that the conditions*

1. $\hat{N} \leq b^k + 1 - (b-x)^k$,
2. $a_n \leq b - x$,
3. $0 < \left(\sum_{i=n}^{n+r-2} a_i^k\right) + 2(m-x)^k - (m+1)^k$ *for all* $m \geq b$, *and*
4. $(m+1)^k - (m-x)^k \leq m^k$ *for all* $m \geq b$

hold, then the generalized anti-Waring number $N(k,n,r,A)$ exists and equals $\hat{N}$. Note: The sum in condition 3 is zero if $r = 1$.

Proof. We want to prove that if $\ell \leq m^k$ and ℓ is (k,n,r,A)-bad, then $\ell \leq \hat{N}-1$ by induction on m with $m \geq b$.

This is clearly true for $m = b$ as we know the consecutive integers $\{\hat{N},\ldots,b^k\}$ are all (k,n,r,A)-good.

Now suppose $\ell \leq (m+1)^k$ and ℓ is (k,n,r,A)-bad. If $\ell \leq m^k$, then by induction $\ell \leq \hat{N}-1$. Next, consider ℓ such that

$$m^k + 1 \leq \ell \leq (m+1)^k. \tag{1}$$

Notice $b^k - (b-x)^k \leq m^k - (m-x)^k$ for $m \geq b$. Using this along with (1) and condition 1, we have

$$\hat{N} \leq \ell - (m-x)^k. \tag{2}$$

To see that $\ell - (m-x)^k$ is (k,n,r,A)-bad, suppose it is (k,n,r,A)-good. Then

$$\ell - (m-x)^k = a_{i_1}^k + a_{i_2}^k + a_{i_3}^k + \cdots + a_{i_t}^k$$

where $t \geq r$, $i_\alpha \neq i_\beta$ for all $\alpha \neq \beta$, and $i_\alpha \geq n$ for all $\alpha \in \{1, 2, \ldots, t\}$. Since ℓ is (k, n, r, A)-bad and

$$\ell = a_{i_1}^k + a_{i_2}^k + a_{i_3}^k + \cdots + a_{i_t}^k + (m - x)^k,$$

either $m - x < a_n$, which contradicts condition 2, or $a_{i_\alpha} = m - x$ for some $\alpha \in \{1, 2, \ldots, t\}$. Therefore,

$$\ell \geq a_n^k + a_{n+1}^k + a_{n+2}^k + \cdots + a_{n+r-2}^k + 2(m - x)^k.$$

If $r = 1$, this is just $\ell \geq 2(m - x)^k$. Combining with (1), we get

$$\left(\sum_{i=n}^{n+r-2} a_i^k \right) + 2(m - x)^k - (m + 1)^k \leq 0.$$

This contradicts condition 3 and means that $\ell - (m - x)^k$ must be (k, n, r, A)-bad.

Now from (1) and condition 4,

$$\ell - (m - x)^k \leq (m + 1)^k - (m - x)^k \leq m^k.$$

By induction we then have $\ell - (m - x)^k \leq \hat{N} - 1$. This contradicts (2). Hence there are no ℓ that are (k, n, r, A)-bad and satisfy (1).

□

Most of the threshold of completeness results in the literature of complete sequences rely on work by Richert [14], where different sufficient conditions imply that a sequence is complete when restricting $r = 1$. Our algorithms for computing generalized anti-Waring numbers were designed to stop when x and b are found satisfying Theorem 6.

3 Values of $N(k, n, r, A)$

As a result of Theorems 1 and 2, we know that $N(k, n, r, A)$ does not exist for all values of k, n, and r and all sequences A. Ideally, if the generalized anti-Waring number $N(k, n, r, A)$ exists, a formula for it can be derived. We have found such a formula for some cases. For other cases, we have computationally found and verified $N(k, n, r, A)$ with Theorem 6.

Johnson and Laughlin [7, Theorem 1] proved a first result

$$N(1, 1, r, \mathbb{N}) = \sum_{i=1}^{r} i = \frac{r}{2}(r + 1) \tag{3}$$

for the case of $k = n = 1$. A similar argument is valid for general values of n.

Theorem 7. *For positive integers n and r, the generalized anti-Waring number is given by*

$$N(1, n, r, \mathbb{N}) = \sum_{i=n}^{n+r-1} i = \frac{r}{2}(r + 1) + r(n - 1).$$

Proof. Clearly, the sum $\sum_{i=n}^{n+r-1} i$ is the smallest integer expressible as the sum of r or more distinct integers greater than or equal to n. For any positive integer x greater than the sum $\sum_{i=n}^{n+r-1} i$, we have

$$x - \sum_{i=n}^{n+r-2} i > n + r - 1.$$

Finally, we have that

$$x = \sum_{i=n}^{n+r-2} i + \left(x - \sum_{i=n}^{n+r-2} i \right)$$

so the integer x is the sum of r distinct integers greater than or equal to n. □

Theorem 8. *For positive integers n, r, and s and integers t such that $|t| < s$ and $\gcd(s,t) = 1$, the generalized anti-Waring number is given by*

$$N(1, n, r, (si + t)_{i\in\mathbb{N}}) = 1 - s + \sum_{i=n}^{n+r+s-2} (si + t). \tag{4}$$

Note: For the case of $s = 1$ and $t = 0$, this reduces to $N(1, n, r, \mathbb{N})$ and agrees with Theorem 7.

Proof. The sequence $B = (si + t)_{i\geq n}$ consists of all positive integers equivalent to $t \bmod s$ that are greater than or equal to $sn+t$. For any positive integer p, the sum of any p elements of B is equivalent to $pt \bmod s$. In order to express all sufficiently large integers as the sum of r or more distinct elements of B, we need sums with the number of summands covering all equivalence classes of $\mathbb{Z}_s$. The list $r, r+1, r+2, \ldots, r+s-1$ contains representatives of each equivalence class in $\mathbb{Z}_s$. Since the integers s and t are relatively prime, the same is true for the list $rt, (r+1)t, (r+2)t, \ldots, (r+s-1)t$. Hence, all sums containing between r and $r+s-1$ distinct elements of B will account for all sufficiently large positive integers, as we shall see. We must determine the smallest integer not expressible by one of these sums.

For $p \in \{r, r+1, r+2, \ldots, r+s-1\}$, let m_p be the sum of the first p elements of B, i.e.,

$$m_p = \sum_{i=n}^{n+p-1} (si + t) = s \left(\sum_{i=n}^{n+p-1} i \right) + pt.$$

As noted before, we have $m_p \equiv pt \pmod{s}$. We also know that m_p is the smallest integer equivalent to $pt \bmod s$ expressible as the sum of r or more distinct elements of B. Hence the integer $m_p - s$ is $(1, n, r, (si+t)_{i\in\mathbb{N}})$-bad. If a positive integer $x \geq m_p$ is also equivalent to $pt \bmod s$, then we have $x = m_p + \ell s$ for some positive $\ell \in \mathbb{Z}$ or, equivalently,

$$x = \ell s + \sum_{i=n}^{n+p-1} (si + t) = (s(\ell + n + p - 1) + t) + \sum_{i=n}^{n+p-2} (si + t).$$

Thus, all integers equivalent to pt mod s greater than m_p are expressible as the sum of r or more distinct elements of B. Since we have $m_p < m_{p+1}$ for all p, the last $(1, n, r, (si+t)_{i\in\mathbb{N}})$-bad integer is $m_{r+s-1} - s$. Therefore, the generalized anti-Waring number is $N(1, n, r, (si+t)_{i\in\mathbb{N}}) = m_{r+s-1} - s + 1$ which is (4). $\square$

k	$N(k,1)$	x	b	bad count
1	1	1	4	0
2	129	4	18	31
3	12759	5	32	2788
4	5134241	8	59	889576
5	67898772	4	45	13912682
6	11146309948	5	55	2037573096

Table 1: Values of $N(k, 1, 1, \mathbb{N})$

k	$N(k,1,1,\mathbb{P})$	x	b	bad count
1	7	6	14	3
2	17164	54	187	2438
3	1866001	31	157	483370

Table 2: Values of $N(k, 1, 1, \mathbb{P})$

For most cases, a formula for $N(k, n, r, A)$ is not known, but we can compute particular values. In the Tables 1 to 6 we list values of $N(k, n, r, A)$ along with the corresponding x and b that satisfy the conditions for Theorem 6 hence confirming the given generalized anti-Waring number. Tables 1, 3, and 4 use $A = \mathbb{N}$. In Table 1 we consider $n = r = 1$, i.e., the first positive integer such that it and every subsequent integer can be written as the sum k^{th} powers of distinct integers. For each k we also include a *bad count*, i.e., the number of positive integers that cannot be written as a sum of k^{th} powers. Table 2 lists the corresponding values over the sequence of primes $\mathbb{P}$. Table 3 lists generalized anti-Waring numbers for fixed $n = 1$ and varying k and r. We stopped the table at $r = 36$ but were able to compute some $N(k, 1, r, \mathbb{N})$ for much larger r. For example, we found that $N(2, 1, 1000, \mathbb{N}) = 333951595$ with $x = 12898$ and $b = 19395$. Table 4 lists generalized anti-Waring numbers for varying k, n, and r. Tables 3 and 4 omit generalized anti-Waring numbers when $k = 1$ because a formula for $N(1, n, r, \mathbb{N})$ for all n and r in $\mathbb{N}$ exists by Theorem 7. Tables 5 and 6 list generalized anti-Waring numbers for fixed $n = 1$ and $r = 1$ over various sequences of the form $(si + t)_{i\in\mathbb{N}}$.

r	$N(2,r)$	x	b	$N(3,r)$	x	b	$N(4,r)$	x	b	$N(5,r)$	x	b
1	129	4	18	12759	5	32	5134241	8	59	67898772	4	45
2	129	4	18	12759	5	32	5134241	8	59	67898772	4	45
3	129	4	18	12759	5	32	5134241	8	59	67898772	4	45
4	129	4	18	12759	5	32	5134241	8	59	67898772	4	45
5	198	6	22	12759	5	32	5134241	8	59	67898772	4	45
6	238	6	23	15279	6	33	5134241	8	59	67898772	4	45
7	331	8	26	15279	6	33	5134241	8	59	67898772	4	45
8	383	9	27	15279	6	33	5134241	8	59	67898772	4	45
9	528	10	32	16224	6	33	5134241	8	59	67898772	4	45
10	648	12	33	18149	6	35	5134241	8	59	67898772	4	45
11	889	14	39	22398	7	37	5191473	8	59	67898772	4	45
12	989	15	41	24855	7	38	5626194	8	60	67898772	4	45
13	1178	17	44	28887	8	39	6018930	8	62	71780055	4	46
14	1398	19	47	36951	9	42	6408466	9	62	74729904	4	46
15	1723	21	52	39660	9	43	6664722	9	62	81846431	5	45
16	1991	24	54	49083	10	46	6938867	9	63	92894512	5	47
17	2312	26	58	56076	11	47	8077523	9	66	95723448	5	47
18	2673	28	62	66534	12	50	8592323	9	67	112031630	5	49
19	3048	31	65	75912	12	52	9269124	10	67	124811198	5	50
20	3493	34	69	87567	13	54	10418260	10	69	142118181	5	52
21	4094	36	75	101093	14	56	10589380	10	70	163637305	6	52
22	4614	39	79	122064	15	60	12852837	11	72	189572962	6	54
23	5139	42	83	138696	16	62	13199973	11	73	210715205	6	55
24	5719	44	87	156498	17	64	15148358	11	76	247073537	6	57
25	6380	48	91	179520	18	67	16526214	12	76	285744830	7	57
26	7124	51	96	201921	19	69	17803895	12	78	319712379	7	59
27	7953	54	101	227400	20	72	20499591	13	81	374237223	7	61
28	8677	57	105	256254	22	73	21202776	13	81	430026890	7	63
29	9538	61	109	289869	23	76	24306872	13	84	491665093	8	64
30	10394	63	114	325590	24	79	25670088	14	84	558015873	8	65
31	11559	67	120	359358	25	82	29819129	14	88	640101337	8	68
32	12603	71	125	401496	26	85	31126025	15	88	737104155	9	68
33	13744	74	130	448503	27	88	35677050	15	92	839165455	9	71
34	14864	78	135	496257	29	90	38187306	16	92	950792455	9	73
35	16253	81	141	554217	30	93	43256507	16	96	1070200765	10	73
36	17529	85	146	611736	30	97	46180043	17	97	1215652918	10	76

Table 3: Values of $N(k,1,r,\mathbb{N})$ and the corresponding x and b that satisfy Theorem 6. Values of $N(1,n,r,\mathbb{N})$ are given by Theorem 7.

r	$N(2,2,r)$	x	b	$N(3,2,r)$	x	b	$N(4,2,r)$	x	b	$N(5,2,r)$	x	b
1	193	5	22	19310	6	36	6659841	9	62	84038312	5	46
2	193	5	22	19310	6	36	6659841	9	62	84038312	5	46
3	193	5	22	19310	6	36	6659841	9	62	84038312	5	46
4	213	6	22	19310	6	36	6659841	9	62	84038312	5	46
5	318	7	27	19310	6	36	6659841	9	62	84038312	5	46
6	334	8	26	19310	6	36	6692881	9	62	84038312	5	46
7	398	9	27	19310	6	36	6692881	9	62	84038312	5	46
8	527	10	32	19310	6	36	6692881	9	62	84038312	5	46
9	647	12	33	20885	7	36	6778897	9	62	84038312	5	46
10	888	14	39	24098	7	38	6778897	9	62	84038312	5	46

r	$N(2,3,r)$	x	b	$N(3,3,r)$	x	b	$N(4,3,r)$	x	b	$N(5,3,r)$	x	b
1	224	6	23	23775	7	38	7076321	9	63	110100822	5	49
2	224	6	23	23775	7	38	7076321	9	63	110100822	5	49
3	233	6	23	23775	7	38	7076321	9	63	110100822	5	49
4	314	7	26	23775	7	38	7076321	9	63	110100822	5	49
5	330	8	26	23775	7	38	7076321	9	63	110100822	5	49
6	418	9	28	23775	7	38	7076321	9	63	110100822	5	49
7	523	10	32	23775	7	38	7103505	9	63	110100822	5	49
8	643	12	33	24756	7	38	7103505	9	63	110100822	5	49
9	884	14	39	28221	7	41	7103505	9	63	110100822	5	49
10	984	15	41	28950	8	40	7103505	9	63	110100822	5	49

r	$N(2,4,r)$	x	b	$N(3,4,r)$	x	b	$N(4,4,r)$	x	b	$N(5,4,r)$	x	b
1	385	8	30	26862	7	40	8912545	9	68	129436797	5	51
2	385	8	30	26862	7	40	8912545	9	68	129436797	5	51
3	385	8	29	26862	7	40	8912545	9	68	129436797	5	51
4	385	8	28	26862	7	40	8912545	9	68	129436797	5	51
5	453	9	30	26862	7	40	8912545	9	68	129436797	5	51
6	558	10	33	26862	7	40	8912545	9	68	129436797	5	51
7	634	12	34	27528	7	40	8912545	9	68	129436797	5	51
8	875	14	39	28194	7	41	8912545	9	68	129436797	5	51
9	999	15	41	30111	8	40	8912545	9	68	129436797	5	51
10	1164	17	43	33234	8	42	8912545	9	68	130964972	5	51

r	$N(2,5,r)$	x	b	$N(3,5,r)$	x	b	$N(4,5,r)$	x	b	$N(5,5,r)$	x	b
1	493	9	34	34844	8	43	9292705	9	69	167956256	5	54
2	493	9	33	34844	8	43	9292705	9	69	167956256	5	54
3	493	9	32	34844	8	43	9292705	9	69	167956256	5	54
4	494	9	32	34844	8	43	9292705	9	69	167956256	5	54
5	542	10	33	34844	8	43	9292705	9	69	167956256	5	54
6	670	12	35	34844	8	43	9377041	9	69	167956256	5	54
7	883	14	39	35060	8	43	9377041	9	69	167956256	5	54
8	983	15	41	35060	8	43	9377041	10	68	167956256	5	54
9	1188	17	44	38048	8	44	9377041	10	68	167956256	6	53
10	1412	19	47	43880	9	45	9377041	10	68	167956256	5	54

r	$N(2,6,r)$	x	b	$N(3,6,r)$	x	b	$N(4,6,r)$	x	b	$N(5,6,r)$	x	b
1	637	10	37	40416	8	45	11728881	10	72	191116579	6	54
2	637	10	37	40416	8	45	11728881	10	72	191116579	6	54
3	637	10	37	40416	8	45	11728881	10	72	191116579	6	54
4	637	11	35	40416	8	45	11728881	10	72	191116579	6	54
5	834	13	40	40416	8	45	11728881	10	72	191116579	6	54
6	870	13	40	40416	8	45	11728881	10	72	191116579	6	54
7	958	15	40	40416	8	45	11728881	10	72	191116579	6	54
8	1163	17	43	41450	9	44	11728881	10	72	191116579	6	54
9	1387	19	46	48066	9	47	11728881	10	72	191116579	6	54
10	1668	21	51	49893	10	46	11728881	10	72	191116579	6	54

Table 4: Values of $N(k,n,r,\mathbb{N})$ for $n>1$ and the corresponding x and b that satisfy Theorem 6. Values of $N(1,n,r,\mathbb{N})$ are given by Theorem 7.

(s,t)	$k=2$	x	b	$k=3$	x	b
$(2,-1)$	1923	18	64	212595	15	77
$(2,+1)$	2355	20	71	266459	16	83
$(3,-2)$	3250	23	83	942316	25	126
$(3,-1)$	3014	22	80	957226	25	126
$(3,+1)$	4093	26	92	1103569	26	132
$(3,+2)$	4414	27	96	1181758	27	135
$(4,-3)$	10588	42	148	2576040	35	174
$(4,-1)$	11268	43	153	3026615	37	184
$(4,+1)$	13708	48	167	3152462	37	187
$(4,+3)$	14948	50	175	3534459	39	193
$(5,-4)$	14900	50	174	6146241	47	232
$(5,-3)$	14121	49	170	6373428	47	236
$(5,-2)$	16810	53	186	6672804	48	239
$(5,-1)$	16379	52	184	7077048	49	244
$(5,+1)$	17242	54	187	7165274	49	245
$(5,+2)$	19090	57	198	7526193	50	249
$(5,+3)$	19690	58	201	7821959	51	252
$(5,+4)$	19799	58	201	8326652	52	257
$(6,-5)$	255964	209	717	32025571	82	402
$(6,-1)$	261868	211	727	35431051	85	416
$(6,+1)$	270796	215	738	38008681	87	426
$(6,+5)$	282028	219	754	40622251	88	436
$(7,-6)$	44329	87	300	24233667	74	367
$(7,-5)$	45769	88	305	23668124	74	363
$(7,-4)$	49737	92	317	25473560	76	373
$(7,-3)$	49009	91	315	26139255	76	376
$(7,-2)$	48989	91	315	27035708	77	380
$(7,-1)$	49537	92	317	27348027	77	382
$(7,+1)$	51889	94	324	28963994	79	389
$(7,+2)$	55884	97	337	28297320	78	387
$(7,+3)$	54217	96	331	30183369	80	394
$(7,+4)$	60377	101	350	28992218	79	389
$(7,+5)$	58292	99	344	31374203	81	400
$(7,+6)$	63453	104	358	31015095	81	397
$(8,-7)$	183828	177	608	43603746	91	445
$(8,-5)$	186684	178	614	44323025	91	448
$(8,-3)$	192748	181	623	44594177	91	449
$(8,-1)$	199124	184	634	49916598	95	466
$(8,+1)$	208164	188	648	51794250	96	472
$(8,+3)$	216940	192	661	53940372	97	479
$(8,+5)$	223884	195	672	53774817	97	478
$(8,+7)$	227204	197	676	55157135	98	482
$(9,-8)$	104873	134	460	316621582	176	861
$(9,-7)$	114857	140	481	317215246	176	862
$(9,-5)$	114653	140	481	327375655	178	871
$(9,-4)$	118829	142	490	329700964	179	872
$(9,-2)$	120113	143	492	338139583	180	880
$(9,-1)$	130217	149	512	339498184	180	882
$(9,+1)$	134681	151	522	352115215	183	891
$(9,+2)$	129149	148	511	358747834	184	897
$(9,+4)$	137873	153	528	371854375	186	908
$(9,+5)$	141329	155	534	365220868	185	902
$(9,+7)$	142825	156	536	383482411	188	917
$(9,+8)$	149990	160	549	376489804	187	911

Table 5: Values of $N(k,1,1,(si+t)_{i\in\mathbb{N}})$ and the corresponding x and b that satisfy Theorem 6. The generalized anti-Waring number $N(k,n,r,(si+t)_{i\in\mathbb{N}})$ does not exist if $\gcd(s,t)>1$ by Theorem 1, and values of $N(1,n,r,(si+t)_{i\in\mathbb{N}})$ are given by Theorem 8.

(s,t)	$k=2$	x	b	$k=3$	x	b
$(10,-1)$	2866844	701	2396	167900541	142	698
$(10,+1)$	2770803	689	2356	164930981	142	693
$(11,-1)$	251377	207	711	188148921	148	724
$(11,+1)$	260001	211	723	200560127	151	740
$(12,-1)$	1186948	451	1543	1871937463	320	1555
$(12,+1)$	1207948	455	1556	1897625923	321	1562
$(13,-1)$	484333	288	986	427144568	195	951
$(13,+1)$	498269	292	1000	434996727	196	957
$(14,-1)$	14209388	1561	5333	718660158	232	1130
$(14,+1)$	14254244	1563	5342	750996509	235	1148
$(15,-1)$	878885	388	1328	7192487965	501	2434
$(15,+1)$	890945	390	1338	7247153841	502	2440
$(16,-1)$	4345668	863	2950	1162662009	272	1328
$(16,+1)$	4411364	869	2973	1188105593	274	1337
$(17,-1)$	1468737	501	1717	1528625985	298	1454
$(17,+1)$	1487777	505	1727	1574453445	302	1468
$(18,-1)$	47752420	2862	9774	23390399911	742	3606
$(18,+1)$	47891524	2866	9789	23431535880	743	3607
$(19,-1)$	2296953	627	2146	2670453204	360	1750
$(19,+1)$	2330393	632	2161	2654207231	359	1746
$(20,-1)$	12065164	1438	4915	3392160594	390	1895
$(20,+1)$	12241324	1449	4950	3426870488	391	1901

Table 6: Additional values of $N(k,1,1,(si+t)_{i\in\mathbb{N}})$ and the corresponding x and b that satisfy Theorem 6. The generalized anti-Waring number $N(k,n,r,(si+t)_{i\in\mathbb{N}})$ does not exist if $\gcd(s,t)>1$ by Theorem 1, and values of $N(1,n,r,(si+t)_{i\in\mathbb{N}})$ are given by Theorem 8.

4 Future work

With enough time and computing power, we can compute any values of $N(k,n,r,A)$ that exist. However, we have only found a formula for cases with $k=1$.

Some simple inequalities involving $N(k,n,r,A)$ are clear. For example, for $i\le j$ we have the inequalities $N(k,i,r,A)\le N(k,j,r,A)$ and $N(k,n,i,A)\le N(k,n,j,A)$ when each exists. We are unable to prove the inequality $N(k,n,r,A)\le N(k+1,n,r,A)$ even though all data seem to emphatically support it.

We have found and considered several algorithms for generating good numbers. However, none reveal a formula for the largest bad number, i.e., threshold of completeness for $k>1$.

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