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Largest Values for the Stern Sequence

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Abstract

In 1858, Stern introduced an array, later called the diatomic array. The array is formed by taking two values a and b for the first row, and each succeeding row is formed from the previous by inserting $c+d$ between two consecutive terms with values c, d. This array has many interesting properties, such as the largest value in a row of the diatomic array is the $(r+2)$-th Fibonacci number, occurring roughly one-third and two-thirds of the way through the row. In this paper, we show each of the second and third largest values in a row of the diatomic array satisfy a Fibonacci recurrence and can be written as a linear combination of Fibonacci numbers. The array can be written in terms of a recursive sequence, denoted $s(n)$ and called the Stern sequence. The diatomic array also has the property that every third term is even. In function notation, we have $s(3n)$ is always even. We introduce and give some properties of the related sequence defined by $w(n) = s(3n)/2$.

1 Historical background

The Stern sequence, denoted by $s(n)$, satisfies the recurrences

$$s(2n) = s(n) \quad \text{and} \quad s(2n+1) = s(n+1) + s(n)$$

for $n \geq 1$, with $s(0) = 0$ and $s(1) = 1$. The first few values of this sequence are

$$0,\ 1,\ 1,\ 2,\ 1,\ 3,\ 2,\ 3,\ 1,\ 4,\ 3,\ 5, \ldots$$

This sequence has many properties, including $s(n)$ is even if and only if 3 divides n, and

$$\sum_{n=2^r}^{2^{r+1}-1} s(n) = 3^r.$$

In 1858, Stern [9] constructed an array by taking two values a and b, which give the first row, and then he formed the next row by rewriting the previous row and inserting the sum $a+b$ between its summands. Stern studied the properties of this general array, but he also examined the case where $a = b = 1$. The numerous properties of this $(1,1)$ array, later called the diatomic array by Lehmer, are summarized by Lehmer [4]. The diatomic array is given in Table 1.

															1		1															
															1	2	1															
														1	3	2	3	1														
												1	4	3	5	2	5	3	4	1												
								1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1								
1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	7	12	5	13	8	11	3	10	7	11	4	9	5	6	1

Table 1: Diatomic Array for $s(n)$

One of the most obvious properties of the diatomic array is the symmetry. In terms of the sequence notation, the symmetry is noted as

$$s(n) = s(n^*) \text{ where } n^* := 3 \cdot 2^r - n \text{ for } 2^r \leq n \leq 2^{r+1}. \tag{1}$$

Stern also considered the array starting with $(0,1)$, given in Table 2, and this array looks more like the sequence $s(n)$. In fact, the r-th row of the $(0,1)$ array is given by $s(n)$ for $0 \leq n \leq 2^r$, and the r-th row of the diatomic array is given by $s(n)$ for $2^r \leq n \leq 2^{r+1}$. For more information, Reznick [7] gives many recurrence and regularity properties of the Stern sequence, and Northshield [6] also gives a nice history of the sequence. See also sequence A002487 in Sloane's Online Encyclopedia of Integer Sequences [8].

															0		1															
															0	1	1															
														0	1	1	2	1														
												0	1	1	2	1	3	2	3	1												
								0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1								
0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1	5	4	7	3	8	5	7	2	7	5	3	3	7	4	5	1

Table 2: $(0,1)$ Array

2 Largest values

In 1878, Lucas [5] studied Stern's diatomic array, and noted that the largest value in the r-th row is a Fibonacci number, occurring roughly at one-third and two-thirds of the way in the row. We now ask, what are the second and third largest values in a row of the diatomic array?

Definition 1. Let $L_m(r)$ denote *the m-th largest distinct value in the r-th row of the Stern sequence*. The maximum of the r-th row is denoted by $L_1(r)$, the second largest value is denoted by $L_2(r)$, and so forth.

We restate Lucas's theorem, which was also later proved by Lehmer [4].

Theorem 2. *For all $r \geq 0$, we have $L_1(r) = F_{r+2}$. Moreover, this maximum occurs for the values $n_r = (4 \cdot 2^r - (-1)^r)/3$, as well as $n_r^* = (5 \cdot 2^r + (-1)^r)/3$ by symmetry.*

This theorem is proved by an easy induction, which we omit. The relationship between the position of the maximum of the r-th row and that of the $(r-1)$-th row is quite important. We have

$$n_r = 2n_{r-1} - (-1)^r, \tag{2}$$

which means the maximum for the r-th row will alternate between appearing on the left or right side of the previous maximum in the diatomic array. Similarly, $n_r^* = 2n_{r-1}^* + (-1)^r$, by the mirror symmetry of the second half of the row.

3 Second Largest Value for $s(n)$

Using Mathematica, we can easily compute the second largest value in a particular row of the diatomic array. The second largest values are given in Table 3, and they follow a Fibonacci recurrence relation, $L_2(r) = L_2(r-1) + L_2(r-2)$, for $r \geq 6$. However, starting in the 4th row, there are 4 occurrences of the second largest value. Two of them come from adding the second largest values from the two preceding rows, $L_2(r-1) + L_2(r-2)$, while the other two come from a linear combination of previous maximum values, $2L_1(r-2) + L_1(r-4)$. The second way of obtaining the second largest value, from $2L_1(r-2) + L_1(r-4)$, will occur either 2 to the left or right of where $L_1(r)$ occurs in the row. The Stern sequence achieves these second largest values for explicitly describable n.

Definition 3. For $r \geq 4$, let

$$n_{2,1}(r) := \frac{17 \cdot 2^{r-2} - (-1)^{r-1}}{3} \quad \text{and} \quad n_{2,2}(r) := \frac{16 \cdot 2^{r-2} - 7(-1)^r}{3}.$$

By the symmetry defined earlier in (1), we have

$$n_{2,1}^*(r) = \frac{19 \cdot 2^{r-2} + (-1)^{r-1}}{3} \quad \text{and} \quad n_{2,2}^*(r) = \frac{20 \cdot 2^{r-2} + 7(-1)^r}{3}.$$

Table 3: Second Largest Values of $s(n)$ in rows

row r	n	$L_2(r)$
1	2, 4	1
2	6	2
3	9, 15	4
4	19, 23, 25, 29	7
5	45, 51	12
6	83, 91, 101, 109	19
7	173, 181, 203, 211	31
8	339, 363, 405, 429	50
9	685, 725, 811, 851	81
10	1363, 1451, 1621, 1709	131
11	2733, 2901, 3243, 3411	212
12	5459, 5803, 6485, 6829	343

Note that $n_{2,1}(r)$ and $n_{2,2}(r)$ coalesce at $r = 5$ so that there are only two occurrences of $L_2(5)$.

We observe $s(n_{2,1}(r))$ and $s(n^*_{2,1}(r))$ give the second largest value in the r-th row as a sum of preceding second largest values, and $s(n_{2,2}(r))$ and $s(n^*_{2,2}(r))$ give the second largest value of the r-th row as a linear combination of previous maximum values. Also note $n_{2,1}(r)$ has a similar recurrence relation as n_r in (2):

$$n_{2,1}(r) = \frac{17 \cdot 2^{r-2} + 2(-1)^{r-1} - 3(-1)^{r-1}}{3} = 2n_{2,1}(r-1) + (-1)^r. \tag{3}$$

We will also need

$$n_{2,2}(r) = 4n_{r-2} - (-1)^r = n_r - 2(-1)^r, \tag{4}$$

and the last equality makes it explicit that the second way of obtaining the second largest value, from
$2L_1(r-2) + L_1(r-4)$, will occur either two to the left or right of where $L_1(r)$ occurs in the row.

Theorem 4. (i) *$L_2(r) = F_{r+2} - F_{r-3} = L_1(r) - F_{r-3}$, for $r \geq 4$.*

(ii) *$s(n_{2,1}(r)) = s(n^*_{2,1}(r)) = s(n_{2,2}(r)) = s(n^*_{2,2}(r)) = L_2(r)$, for $r \geq 4$.*

(iii) *For $n_{2,1}(r)$ and $n^*_{2,1}(r)$, $L_2(r)$ arises from the sum $L_2(r-1) + L_2(r-2)$, for $r \geq 6$.*

(iv) *For $n_{2,2}(r)$ and $n^*_{2,2}(r)$, $L_2(r)$ arises from the sum $2L_1(r-2) + L_1(r-4)$, for $r \geq 8$.*

Proof. The induction base is clear and easily verified. Assume (i)-(iv) hold for all rows before and including the r-th row, with $r \geq 8$.

We first show that besides the maximum value, there is no element larger than $L_2(r) + L_2(r-1)$ in the $(r+1)$-th row. If $2k \in [2^{r+1}, 2^{r+2}]$, then using the induction hypothesis from part (i), we find that for all the even values in the $(r+1)$-th row

$$s(2k) = s(k) \leq L_1(r) = F_{r+2} < F_{r+2} + 2F_{r-1} = F_{r+3} - F_{r-2} = L_2(r) + L_2(r-1).$$

We now want to obtain a bound on the odd values in the $(r+1)$-th row, and more specifically $s(4k \pm 1)$. Now let $k \in [2^{r-1}, 2^r]$ such that $4k \pm 1 \in (2^{r+1}, 2^{r+2})$. We examine some special cases first. If $k = n_{r-1}$, or $k = n^*_{r-1}$, we have

$$s(2n_{r-1} - (-1)^r) = s(n_r) = L_1(r) = s(n^*_r) = s(2n^*_{r-1} + (-1)^r),$$

which implies $s(2n_{r-1} + (-1)^r) < L_1(r)$ and $s(2n^*_{r-1} - (-1)^r) < L_1(r)$. Similarly, if $k = 2n_{r-2}$ or $k = 2n^*_{r-2}$, we have

$$s(2 \cdot 2n_{r-2} + (-1)^r) = s(n_r) = L_1(r) = s(n^*_r) = s(2 \cdot 2n^*_{r-2} - (-1)^r),$$

which means $s(2 \cdot 2n_{r-2} - (-1)^r) < L_1(r)$ and $s(2 \cdot 2n^*_{r-2} + (-1)^r) < L_1(r)$. Except for these special cases where $k = n_{r-1}$, n^*_{r-1}, $2n_{r-2}$, or $2n^*_{r-2}$, we have $s(2k \pm 1) < L_1(r)$, and therefore $s(2k \pm 1) \leq L_2(r)$. Then we have

$$s(4k \pm 1) = s(2k) + s(2k \pm 1) = s(2k \pm 1) + s(k) \leq L_2(r) + L_2(r-1).$$

We now examine the special cases more closely. If $k = n_{r-1}$, we have

$$s(4n_{r-1} - (-1)^r) = s(2n_{r-1}) + s(2n_{r-1} - (-1)^r) = s(n_{r-1}) + s(n_r) = L_1(r+1),$$

but we can disregard this (largest) value, since we are looking for the second largest value. Now, we also have

$$\begin{aligned} s(4n_{r-1} + (-1)^r) &= s(2n_{r-1}) + s(2n_{r-1} + (-1)^r) \\ &= s(n_{r-1}) + s(n_{r-1}) + s(n_{r-1} + (-1)^r) \\ &= 2s(n_{r-1}) + s(2n_{r-2} + 2(-1)^r) \\ &= 2s(n_{r-1}) + s(n_{r-2} + (-1)^r) \\ &= 2s(n_{r-1}) + s(2n_{r-3} - (-1)^r + (-1)^r) \\ &= 2s(n_{r-1}) + s(n_{r-3}) \\ &= 2L_1(r-1) + L_1(r-3). \end{aligned} \tag{5}$$

Note that by symmetry we have $s(4n^*_{r-1} - (-1)^r) = 2L_1(r-1) + L_1(r-3)$ and $s(4n^*_{r-1} +$

$(-1)^r) = L_1(r)$. Lastly, if $k = 2n_{r-2}$ we have

$$\begin{aligned}
s(4 \cdot 2n_{r-2} \pm 1) &= 2s(n_{r-2}) + s(2n_{r-2} \pm 1)\\
&= \begin{cases} 2s(n_{r-2}) + s(2n_{r-2} + (-1)^r) \\ 2s(n_{r-2}) + s(2n_{r-2} - (-1)^r) \end{cases}\\
&= \begin{cases} 2s(n_{r-2} + s(n_{r-1}) = 2L_1(r-2) + L_1(r-1) \\ 3s(n_{r-2}) + s(n_{r-4}) = 3L_1(r-2) + L_1(r-4) \end{cases}\\
&< 2L_1(r-1) + L_1(r-3).
\end{aligned}$$

So now we only need compare $2L_1(r-1) + L_1(r-3)$ to $L_2(r) + L_2(r-1)$. However, using part (i) from the induction hypothesis, we have

$$\begin{aligned}
L_2(r) + L_2(r-1) &= F_{r+2} - F_{r-3} + F_{r+1} - F_{r-4}\\
&= F_{r+3} - F_{r-2} = L_1(r+1) - L_1(r-4)\\
&= 2L_1(r-1) + L_1(r-3).
\end{aligned}$$

Thus, all elements in the $(r+1)$-th row, besides the maximum, are less than or equal to $L_2(r) + L_2(r-1)$, and so we have

$$L_2(r+1) = F_{r+3} - F_{r-2} = L_2(r) + L_2(r-1) = 2L_1(r-1) + L_1(r-3).$$

All that is left is to verify the second largest values occur for the n given earlier. Evaluating $s(n_{2,1}(r+1))$ and using (3), we have

$$\begin{aligned}
s(n_{2,1}(r+1)) &= s(2n_{2,1}(r) + (-1)^{r+1}) = s(n_{2,1}(r)) + s(n_{2,1}(r) - (-1)^r)\\
&= s(n_{2,1}(r)) + s(2n_{2,1}(r-1))\\
&= s(n_{2,1}(r)) + s(n_{2,1}(r-1))\\
&= L_2(r) + L_2(r-1).
\end{aligned}$$

We also note that by (4) we have $n_{2,2}(r+1) = 4n_{r-1} + (-1)^r$, and in (5) we see that $s(4n_{r-1} + (-1)^r)$ also gives $L_2(r)$. Therefore, the second largest values occur where we expect. Finally, by the symmetry of the Stern sequence in rows, we have

$$s(n^*_{2,1}(r+1)) = s(n_{2,1}(r+1)) = L_2(r+1) = s(n_{2,2}(r+1)) = s(n^*_{2,2}(r+1)). \qquad \square$$

4 Third Largest Value for $s(n)$

The third largest values in a row for $s(n)$, given in Table 4, also eventually satisfy a Fibonacci recurrence. This recurrence starts in the 10th row, and rows 8 and 9 give the two initial values. Similar to the second largest value in a row, there are 4 occurrences of the third

Table 4: Third Largest Values of $s(n)$ in rows

row r	n	$L_3(r)$
1	N/A	N/A
2	4	1
3	10, 14	3
4	17, 22, 26, 31	5
5	37, 41, 55, 59	11
6	75, 87, 105, 117	18
7	165, 219	30
8	331, 347, 421, 437	49
9	693, 843	80
10	1355, 1387, 1685, 1717	129
11	2741, 2773, 3371, 3403	209
12	5451, 5547, 6741, 6837	338

largest value. By symmetry, two of them come from $L_3(r-1) + L_3(r-2)$, and the other two come from the sum of $(2L_1(r-4) + L_1(r-6)) + (3L_1(r-4) + 2L_1(r-6))$, which adds to $5L_1(r-4) + 3L_1(r-6)$.

The values of n where the third largest values occur also follow a pattern, starting in the eighth row. For the outside left and right values, it appears that the next value is two times the previous value plus or minus 31. The inside right and left values follow the pattern, two times the previous value plus or minus one. Taking this recurrences and iterating them, we obtain the following apparent closed form for these values.

Definition 5. For $r \geq 8$, let

$$n_{3,1}(r) := \frac{64 \cdot 2^{r-4} - 31(-1)^r}{3} \quad \text{and} \quad n_{3,2}(r) := \frac{65 \cdot 2^{r-4} + (-1)^r}{3}.$$

By symmetry, we have

$$n^*_{3,1}(r) = \frac{80 \cdot 2^{r-4} + 31(-1)^r}{3} \quad \text{and} \quad n^*_{3,2}(r) = \frac{79 \cdot 2^{r-4} - (-1)^r}{3}.$$

We will show that $n_{3,1}(r)$, $n^*_{3,1}(r)$, $n_{3,2}(r)$, and $n^*_{3,2}(r)$ give the third largest values for $s(n)$ in the r-th row. It is useful to note

$$n_{3,1}(r) = \frac{4 \cdot 2^r - (-1)^r - 30(-1)^r}{3} = n_r - 10(-1)^r. \tag{6}$$

This tells us that the first (and by symmetry, the last) occurrence of the third largest value in the row will be either 10 to the left or 10 to the right of the largest values, depending on the parity of the row. We also remark that $n_{3,2}(r)$ satisfies the recurrence relations

$$n_{3,2}(r) = 2n_{3,2}(r-1) + (-1)^r \quad \text{and} \quad n_{3,2}(r) - (-1)^r = 2n_{3,2}(r-1). \tag{7}$$

Theorem 6. (i) $L_3(r) = L_2(r) - F_{r-7} = F_{r+1} + 5F_{r-4} = L_1(r) - 3F_{r-5}$, *for* $r \geq 8$.

(ii) $s(n_{3,1}(r)) = s(n^*_{3,1}(r)) = s(n_{3,2}(r)) = s(n^*_{3,2}(r)) = L_3(r)$, *for* $r \geq 8$.

(iii) *For* $n_{3,1}(r)$ *and* $n^*_{3,1}(r)$, $L_3(r)$ *arises from the sum* $5L_1(r-4) + 3L_1(r-6)$, *for* $r \geq 8$.

(iv) *For* $n_{3,2}(r)$ *and* $n^*_{3,2}(r)$, $L_3(r)$ *arises from the sum* $L_3(r-1) + L_3(r-2)$, *for* $r \geq 10$.

Proof. We proceed by induction. Again, the base case is easily verified. Assume the induction hypotheses hold for all values below $r > 10$. We will first verify that $L_3(r) + L_3(r-1)$ and $5L_1(r-3) + 3L_1(r-5)$ are achieved at the expected values. We will then show that all values in the $(r+1)$-th row, except for $L_1(r+1)$ and $L_2(r+1)$, are less than or equal to $L_3(r) + L_3(r-1) = F_{r+2} + 5F_{r-3}$.

We first verify that $s(n_{3,2}(r+1)) = L_3(r) + L_3(r-1)$. Using (7), we have

$$\begin{aligned} s(n_{3,2}(r+1)) &= s(2n_{3,2}(r) - (-1)^r) = s(n_{3,2}(r)) + s(n_{3,2}(r) - (-1)^r) \\ &= L_3(r) + s(2n_{3,2}(r-1)) \\ &= L_3(r) + L_3(r-1). \end{aligned}$$

Now considering $s(n_{3,1}(r+1))$, and using (6) and (2) we have

$$\begin{aligned} s(n_{3,1}(r+1)) &= s(n_{r+1} + 10(-1)^r) = s(2n_r + (-1)^r + 10(-1)^r) \\ &= s(n_r + 5(-1)^r) + s(n_r + 5(-1)^r + (-1)^r) \\ &= s(2n_{r-1} - (-1)^r + 5(-1)^r) + s(2n_{r-1} + 6(-1)^r - (-1)^r) \\ &= s(n_{r-1} + 2(-1)^r) + s(n_{r-1} + 3(-1)^r) + s(n_{r-1} + 3(-1)^r - (-1)^r) \\ &= 2s(n_{r-1} + 2(-1)^r) + s(n_{r-1} + 3(-1)^r) \\ &= 2s(2n_{r-2} + (-1)^r + 2(-1)^r) + s(2n_{r-2} + (-1)^r + 3(-1)^r) \\ &= 2s(n_{r-2} + (-1)^r) + 2s(n_{r-2} + 2(-1)^r) + s(n_{r-2} + 2(-1)^r) \\ &= 2s(n_{r-3}) + 3s(2n_{r-3} + (-1)^r) \\ &= 2s(n_{r-3}) + 3s(n_{r-3}) + 3s(n_{r-3} + (-1)^r) \\ &= 5s(n_{r-3}) + 3s(2n_{r-4} + 2(-1)^r) \\ &= 5s(n_{r-3}) + 3s(n_{r-4} + (-1)^r) \\ &= 5s(n_{r-3}) + 3s(2n_{r-5}) \\ &= 5L_1(r-3) + 3L_1(r-5). \end{aligned}$$

We also note by hypothesis (i), we have

$$L_3(r) + L_3(r-1) = F_{r+3} - 3F_{r-4} = 5F_{r-1} + 3F_{r-3} = 5L_1(r-3) + 3L_1(r-5). \tag{8}$$

We now show that all values in the $(r+1)$-th row, except for $L_1(r+1)$ and $L_2(r+1)$, are less than or equal to $L_3(r) + L_3(r-1) = F_{r+2} + 5F_{r-3}$. First note that for $2k \in [2^{r+1}, 2^{r+2}]$, we have

$$s(2k) = s(k) \leq L_1(r) = F_{r+2} < F_{r+2} + 5F_{r-3} = L_3(r) + L_3(r-1).$$

For the odd values in the $(r+1)$-th row, we eliminate the cases that anything larger than $L_3(r) + L_3(r-1)$ can come from the first, second or third largest values and a neighbor. Now consider $s(2k+1) = s(k) + s(k+1)$, and let b represent the larger of $s(k)$ and $s(k+1)$, which comes from the r-th row, and c denote the smaller value coming from the $(r-1)$-th row.

If b is $L_1(r)$, then c is either $L_1(r-1)$ or $L_1(r-2)$. However, $L_1(r) + L_1(r-1)$ gives the largest value in the $(r+1)$-th row, so we can ignore this value. Observe that

$$L_1(r) + L_1(r-2) = F_{r+2} + F_r < F_{r+2} + 5F_{r-3}$$

is too small. So then it must be the case that $b \leq L_2(r)$. Now, there are 2 distinct ways of obtaining $L_2(r)$. If $b = L_2(r)$, then c could be $L_2(r-1)$ or $L_2(r-2)$. But $L_2(r) + L_2(r-1)$ gives $L_2(r+1)$ and we can ignore this value. We also have

$$\begin{aligned} L_2(r) + L_2(r-2) &= F_{r+2} - F_{r-3} + F_r - F_{r-4} = F_{r+2} + F_{r-2} + 2F_{r-4} \\ &< F_{r+2} + 5F_{r-3} = L_3(r) + L_3(r-1), \end{aligned}$$

so this value is too small. But if $b = 2L_1(r-2) + L_1(r-4)$, then c is $L_1(r-2)$ or $L_1(r-2) + L_1(r-4)$, and we only need to consider the latter. We have

$$\begin{aligned} 2L_1(r-2) + L_1(r-4) + L_1(r-2) + L_1(r-4) &= 3F_r + 2F_{r-2} \\ &< F_{r+2} + 5F_{r-3} = L_3(r) + L_3(r-1), \end{aligned}$$

which implies $b \leq L_3(r)$. If $b = L_3(r)$, there are 2 distinct ways of obtaining $L_3(r)$. If $b = L_3(r)$, then c could be $L_3(r-1)$ or $L_3(r-2)$, but we can disregard these because we want to find something larger. If $b = 5L_1(r-4) + 3L_1(r-6)$, then c could be $2L_1(r-4) + L_1(r-6)$ or $3L_1(r-4) + 2L_1(r-6)$. We need only consider the latter, and we have

$$\begin{aligned} 5L_1(r-4) + 3L_1(r-6) + 3L_1(r-4) + 2L_1(r-6) &= 8F_{r-2} + 5F_{r-4} \\ &< F_{r+2} + 5F_{r-3} = L_3(r) + L_3(r-1), \end{aligned}$$

which implies $b < L_3(r)$. However c might be a large value in the $(r-1)$-the row to make up for b being so small. Now, we know $c < b < L_3(r)$, that c comes from the $(r-1)$-th row, and that $c > L_3(r-1)$. If $c = L_1(r-1)$, then b could only be $L_1(r-2) + L_1(r-3)$ (since we already eliminated $b = L_1(r)$). But then $b + c = 2L_1(r-1) + L_1(r-3) = L_2(r+1)$, and we can also ignore this value. Then c must be $L_2(r-1)$, and the only possibility for b is $L_2(r-1) + L_2(r-3)$ (as we already eliminated $b = L_2(r)$). By Theorem 4 (i) and the induction hypothesis (i), we have

$$\begin{aligned} L_2(r+1) - F_{r-6} &= L_2(r) + L_2(r-1) - F_{r-6} \\ &= L_3(r) + F_{r-7} + L_3(r-1) + F_{r-8} - F_{r-6} \\ &= L_3(r) + L_3(r-1). \end{aligned} \tag{9}$$

Using (9), we have

$$\begin{aligned}
b+c &= 2L_2(r-1)+L_2(r-3)\\
&= 2L_1(r-1)-2F_{r-4}+L_1(r-3)-F_{r-6}\\
&= L_2(r+1)-2F_{r-4}-F_{r-6}\\
&< L_2(r+2)-F_{r-6} = L_3(r)+L_3(r-1).
\end{aligned}$$

Finally, we have that c could be $L_2(r-1)=2L_1(r-3)+L_1(r-5)$ and b could be $3L_1(r-3)+2L_1(r-5)$ or $3L_1(r-3)+L_1(r-5)$. Then by (8) we see

$$L_3(r)+L_3(r-1)=5L_1(r-3)+3L_1(r-5)>5L_1(r-3)+2L_1(r-5),$$

so that $5L_1(r-3)+2L_1(r-5)$ is too small, while $5L_1(r-3)+3L_1(r-5)$ gives $L_3(r)+L_3(r-1)$. This eliminates all possibilities and shows nothing can be between $L_3(r)+L_3(r-1)$ and $L_2(r+1)$, so that $s(2k+1)\le L_3(r)+L_3(r-1)$. Thus, we have $L_3(r+1)=L_3(r)+L_3(r-1)$. $\square$

Since the next row in the diatomic array is formed by inserting the sum of two consecutive terms in between them, it makes sense that the k-th largest value in the r-th row will satisfy a Fibonacci recurrence. This appears to hold true for the 4th, 5th, and 6th (distinct) largest value in a row, for a sufficiently large row value. We see in Table 5 a Fibonacci recurrence starts at the 14th row for L_4, the 18th for L_5, and the 22nd row for L_6. This leads us to the following conjecture.

Conjecture 7. For all $r\ge 4m-2$, the m-th largest distinct value satisfies the recurrence $L_m(r)=L_m(r-1)+L_m(r-2)$.

Remark 8. By Theorems 4 (i) and 6 (i), we have $L_2(r)=F_{r+2}-F_{r-3}$ and $L_3(r)=L_2(r)-F_{r-7}$. Note that $L_3(r)=F_{r+2}-F_{r-3}-F_{r-7}$. Inspecting Table 5, we notice the following:

$$\begin{aligned}
L_4(r) =& L_3(r)-F_{r-11}=F_{r+2}-F_{r-3}-F_{r-7}-F_{r-11} \quad \text{for } r\ge 12,\\
L_5(r) =& L_4(r)-F_{r-15}=F_{r+2}-F_{r-3}-F_{r-7}-F_{r-11}-F_{r-15} \quad \text{for } r\ge 16, \text{ and}\\
L_6(r) =& L_5(r)-F_{r-19}=F_{r+2}-F_{r-3}-F_{r-7}-F_{r-11}-F_{r-15}-F_{r-19} \quad \text{for } r\ge 20.
\end{aligned}$$

This leads us to another conjecture.

Conjecture 9. For $r\ge 4(m-1)$, we have

$$L_m(r)=L_{m-1}(r)-F_{r-(4m-5)}=F_{r+2}-\sum_{j=2}^{m}F_{r-(4j-5)}.$$

Table 5: L_m of $s(n)$ in rows

row r	$L_1(r)$	$L_2(r)$	$L_3(r)$	$L_4(r)$	$L_5(r)$	$L_6(r)$
3	5	4	3	2	1	N/A
4	8	7	5	4	3	2
5	13	12	11	10	9	8
6	21	19	18	17	16	15
7	34	31	30	29	27	26
8	55	50	49	47	46	45
9	89	81	80	79	76	75
10	144	131	129	128	123	121
11	233	212	209	208	207	199
12	377	343	338	337	335	322
13	610	555	547	546	545	542
14	987	898	885	883	882	877
15	1597	1453	1432	1429	1428	1427
16	2584	2351	2317	2312	2311	2309
17	4181	3804	3749	3741	3740	3739
18	6765	6155	6066	6053	6051	6050
19	10946	9959	9815	9794	9791	9790
20	17711	16114	15881	15847	15842	15841
21	28657	26073	25696	25641	25633	25632
22	46368	42187	41577	41488	41475	41473
23	75025	68260	67273	67129	67108	67105

5 The Distribution of Gaps

In a previous paper, the author [2] considered the classical question of the distribution of values for the Stern sequence. A more modern question to consider is, what is the distribution of gaps? There are numerous ways we could consider the gaps of the Stern sequence. First, we can simply compute $s(n+1)-s(n)$. Figure 2 plots the normalized difference for the 14th row. Figure 2 looks like the plot of $s(n)$, given in Figure 1, with also a reflection over the x-axis. If $n=2\ell$, we have

$$s(n+1)-s(n)=s(2\ell+1)-s(2\ell)=s(\ell+1).$$

If $n=2\ell+1$, then we have

$$s(n+1)-s(n)=s(2\ell+2)-s(2\ell+1)=-s(\ell).$$

From this perspective, the distribution of gaps is the same problem as the distribution of values.

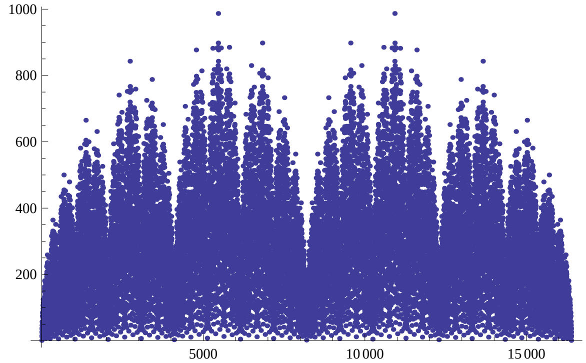

Figure 1: Values of $s(n)$ in the 14th row

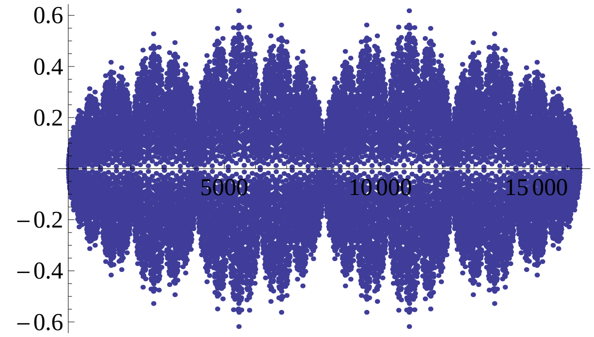

Figure 2: $s(n+1) - s(n)$ for the 14th row

Another way to approach this problem is to order a row from largest to smallest and then take the difference of consecutive terms. Since values occur at least twice in a row, we would have a lot of zeros. If we eliminate repeated values when taking the difference, then we have a type of step function. Figures 3 and 4 give the differences of terms for the 10th

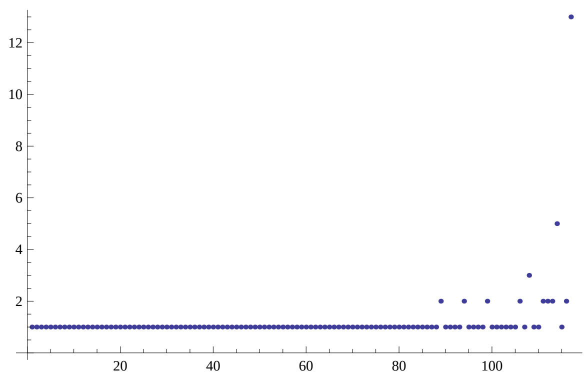

Figure 3: Length of gaps for the 10th row

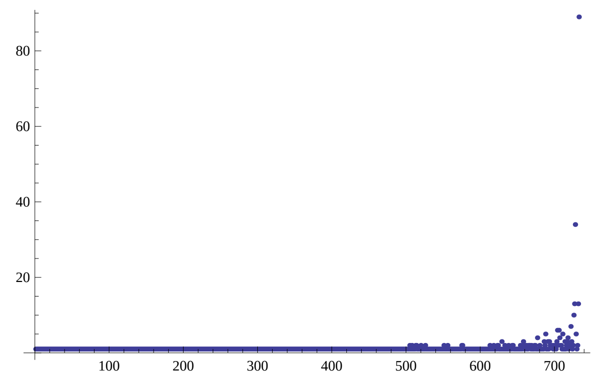

Figure 4: Length of gaps for the 14th row

row and the 14th row, after they have been sorted and repeated elements deleted.

In the previous three sections, we discussed the largest three values in a row of the diatomic array, and we can use this information here. First, we normalize the data by dividing by the largest possible value F_{r+2}. By Theorems 4 and 6, we have the biggest gap is F_{r-3}/F_{r+2}, and the next gap is F_{r-7}/F_{r+2}. If the conjecture $L_k(r) - L_{k-1}(r) = F_{r-(4k-5)}$ is true, then the distribution of gaps is $F_{r-(4k-5)}/F_{r+2}$ for $2 \le k \le (r+5)/4$, with the smallest value being $1/F_{r+2}$. The largest gaps are fixed, regardless of the row. The smallest gaps then approach 0, for the later rows.

This would mean the distribution of the gaps is not uniformly distributed. If Conjecture 9 is true, then we have the following conjecture.

Conjecture 10. The normalized distribution of gaps for the Stern sequence is

$$\phi^{-(4k-3)} \text{ for } 2 \leq k \leq (r+5)/4.$$

6 A related sequence

We introduce a new and related sequence. As noted earlier, $s(3n)$ is always even, so we study the related sequence defined by

$$w(n) := \frac{1}{2}s(3n).$$

How does this sequence behave? It has some similarities to the Stern sequence, such as symmetry of terms in a row in the diatomic array, the same average order of magnitude, and that the sum over powers of 2 is a power of 3. However, $w(n)$ has a much more complicated structure; it has no simple generating function and the recursive definition is not as short.

Table 6 gives a comparison of the first 64 values of $s(n)$ and $w(n)$. Analogous to the

Table 6: Values for $s(n)$ and $w(n)$

n	$s(n)$	$w(n)$	n	$s(n)$	$w(n)$	n	$s(n)$	$w(n)$	n	$s(n)$	$w(n)$
1	1	1	17	5	6	33	6	8	49	9	13
2	1	1	18	4	4	34	5	6	50	7	9
3	2	2	19	7	5	35	9	9	51	12	12
4	1	1	20	3	2	36	4	4	52	5	5
5	3	2	21	8	3	37	11	7	53	13	8
6	2	2	22	5	3	38	7	5	54	8	7
7	3	4	23	7	7	39	10	9	55	11	15
8	1	1	24	2	2	40	3	2	56	3	4
9	4	4	25	7	9	41	11	7	57	10	17
10	3	2	26	5	5	42	8	3	58	7	9
11	5	3	27	8	7	43	13	4	59	11	11
12	2	2	28	3	4	44	5	3	60	4	6
13	5	5	29	7	9	45	12	8	61	9	13
14	3	4	30	4	6	46	7	7	62	5	8
15	4	6	31	5	8	47	9	11	63	6	10
16	1	1	32	1	1	48	2	2	64	1	1

Stern sequence, we can arrange $w(n)$ into a triangular array, where the k-th row is given by $w(n)$ with $2^k/3 < n < 2^{k+1}/3$, with the convention of when $k = 0$, $n = 0$. The entries in this triangular array are one half the even elements from the diatomic array of the Stern

										0										
										1										
										1										
									2	1	2									
								2	4	1	4	2								
					3	2	5	4	6	1	6	4	5	2	3					
3	7	2	9	5	7	4	9	6	8	1	8	6	9	4	7	5	9	2	7	3

Table 7: Triangular Array for $w(n)$

sequence. What properties does this sequence have? How often is $w(n)$ even? What is the largest value in a row? What type of recurrence relations does this sequence have? Does it have a nice expression for its generating function?

Looking at Table 6, we see many similarities to the Stern sequence. We notice the first five values are the same, and that at powers of 2, the sequence is 1. The Stern sequence has lots of beautiful recurrence relations, and while $w(n)$ is a little more complicated, it has some similar structure.

We now show the sequence $w(n)$ can be defined independently of $s(n)$.

Theorem 11. *Let $w(0) = 0$, $w(1) = 1$ and $w(3) = 2$. For $n \geq 1$, we have*

$$\begin{aligned} w(2n) &= w(n), \\ w(8n \pm 1) &= w(4n \pm 1) + 2w(n), && (10) \\ w(8n \pm 3) &= w(4n \pm 1) + w(2n \pm 1) - w(n). && (11) \end{aligned}$$

Proof. Using the recurrence properties for $s(n)$ and the definition of $w(n)$, we immediately see

$$w(2^k n) = \frac{1}{2} s(3 \cdot 2^k n) = \frac{1}{2} s(3n) = w(n).$$

Looking at $w(2n+1)$ we see

$$w(2n+1) = \frac{1}{2} s(3n+1) + \frac{1}{2} s(3n+2),$$

which is not very enlightening. However, if we consider arithmetic progressions modulo 4, we see

$$w(4n+1) = \frac{1}{2} s(3n) + s(3n+1) = w(n) + s(3n+1), \tag{12}$$

$$w(4n+3) = \frac{1}{2} s(3n+3) + s(3n+2) = w(n+1) + s(3n+2), \tag{13}$$

which is better, but still not completely in terms of $w(n)$. We can use a more compact notation and write (12) and (13) as

$$w(4n \pm 1) = w(n) + s(3n \pm 1).$$

Since we still do not have any recurrence formulas strictly in terms of $w(n)$, we consider arithmetic progressions modulo 8. Replace n with $2n$ and then $2n+1$ in (12) and (13). We then use the definition and recurrence relations for $s(n)$, and rearranging the relations in (12) and (13) to substitute in for $s(3n+1)$ and $s(3n+2)$, we obtain the recurrence relations in (10) and (11), which completes the proof. □

Remark 12. We remark that the Stern sequence is a 2-regular sequence. The referee observes that, from the point of view of 2-regular sequences, the existence of recurrence formulas such as in Theorem 11 are not surprising. For more on 2-regular sequences, see Allouche and Shallit [1, Ch. 16].

A natural question to ask is, what is the largest value for $w(n)$ in a row of its triangular array? Not surprisingly, the largest value of $w(n)$ depends on the three largest values for $s(n)$. Since $L_1(r)$, $L_2(r)$, and $L_3(r)$ each satisfy a Fibonacci recurrence which is not all even, every third term in the sequence is even. This simply comes from the fact $odd + even = odd$, $even + odd = odd$, and $odd + odd = even$. So the maximum of $w(n)$ in a row occurs precisely when one of $L_1(r)$, $L_2(r)$,or $L_3(r)$ is even and then it is divided by two. Please note Table 8 gives a comparison of the three largest values for $s(n)$ for easy reference. The boxed even values in Table 8 are divided by 2 to yield the M_k's in Table 9.

Let M_k be the maximum of $w(n)$ in the k-th row. The maximum values for $w(n)$ and where they occur are given in Table 9. There is a pattern for M_k and where they occur, based on every third row.

The Fibonacci numbers F_r are even when r is a multiple of 3, which implies $L_1(r) = F_{r+2}$ will be even when $r \equiv 1 \pmod 3$. Now, $L_2(r) = F_{r+2} - F_{r-3}$, and this will be even only when $r \equiv 2 \pmod 3$. If $r \equiv 1 \pmod 3$, then F_{r+2} is even and F_{r-3} is odd, so that their sum is odd. If $r \equiv 0 \pmod 3$, then F_{r+2} is odd, and F_{r-3} would be even. If $r \equiv 2 \pmod 3$, then both are odd, so that $F_{r+2} - F_{r-3}$ is even. We have $L_3(r) = F_{r+2} - 3F_{r-5}$ is even when $r \equiv 0 \pmod 3$.

However, it is important to note the row numbers for $w(n)$ are shifted up one relative to the row numbers for $s(n)$. To avoid confusion, we denote the row numbers for $s(n)$ with r and the row numbers for $w(n)$ with k (with $k = r+1$). We then have the following theorem.

Theorem 13. *For $k \geq 5$, the maximum value of $w(n)$ in the k-th row is*

$$\begin{aligned} \frac{1}{2}F_{k+1} - \frac{1}{2}F_{k-4} \quad &\textit{when } k \equiv 0 \pmod 3, \\ \frac{1}{2}F_{k+1} - \frac{3}{2}F_{k-6} \quad &\textit{when } k \equiv 1 \pmod 3, \\ \frac{1}{2}F_{k+1} \quad &\textit{when } k \equiv 2 \pmod 3. \end{aligned}$$

Table 8: Comparison of the Three Largest Values of $s(n)$ in rows

row r	$L_1(r)$	$L_2(r)$	$L_3(r)$
1	$\boxed{2}$	1	N/A
2	3	$\boxed{2}$	1
3	5	$\boxed{4}$	3
4	$\boxed{8}$	7	5
5	13	$\boxed{12}$	11
6	21	19	$\boxed{18}$
7	$\boxed{34}$	31	30
8	55	$\boxed{50}$	49
9	89	81	$\boxed{80}$
10	$\boxed{144}$	131	129
11	233	$\boxed{212}$	209
12	377	343	$\boxed{338}$

Table 9: Maxima of $w(n)$ in rows

row	n	M_k	row	n	M_k
2	1	1	13	1817, 1849, 2247, 2279	169
3	2	1	14	3641, 4551	305
4	3, 5	2	15	7281, 7737, 8647, 9103	449
5	7,9	4	16	14567, 14791, 17977, 18201	716
6	15, 17	6	17	29127, 36409	1292
7	25, 29, 35, 39	9	18	58255, 61895, 69177, 72817	1902
8	57, 71	17	19	116505, 118329, 143815, 145639	3033
9	113, 121, 135, 143	25	20	233017, 291271	5473
10	231, 281	40	21	466033, 495161, 533415, 582543	8057
11	455, 569	72	22	932071, 946631, 1150521, 1165081	12848
12	911, 967, 1081, 1137	106	23	1864135, 2330169	23184

Proof. This is a consequence of Theorems 2, 4, and 6. Note when the three largest values for $s(n)$ begin their Fibonacci recurrence (starting at their initial values): $L_1(r)$ for all r,

$L_2(r)$ for $r \geq 4$ ($k \geq 5$), and $L_3(r)$ for $r \geq 8$ ($k \geq 9$). However, in the $r = 6$-th row ($k = 7$-th), L_3 is 18 and the largest even value among L_1, L_2, and L_3. We also see that $\frac{1}{2}(F_8 - 3F_1) = \frac{1}{2}(21 - 3) = \frac{1}{2} \cdot 18 = 9$, so the last condition is also true for $k = 7$. This completes the proof. $\square$

Using Theorem 13, the generating function for M_k is

$$\sum_{n=0}^{\infty} M_k x^k = \frac{x^2 + x^3 + 2x^4 + 2x^6 + x^7 + 2x^{10}}{1 - 4x^3 - x^6}.$$

The results in this paper are contained in the author's University of Illinois Ph.D. dissertation [3], which has much more information on $w(n)$.

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