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Fermat Numbers in Multinomial Coefficients

Shane Chern
Department of Mathematics
Zhejiang University
Hangzhou, 310027
China
chenxiaohang92@gmail.com

Abstract

In 2001 Luca proved that no Fermat number can be a nontrivial binomial coefficient. We extend this result to multinomial coefficients.

1 Introduction

Let $F_m = 2^{2^m} + 1$ be the m^{th} Fermat number for any nonnegative integer m. Several authors studied the Diophantine equation

$$\binom{n}{k} = 2^{2^m} + 1 = F_m, \tag{1}$$

where $n \geq 2k \geq 2$, and $m \geq 0$. We refer to the articles [2, 3, 5, 6, 8] for further details. In 2001, Luca [6] completely solved Eq. (1) and proved that it has only the trivial solutions $k = 1, n-1$ and $n = F_m$. The proof is mainly based on a congruence given by Lucas [7]. For more about Fermat numbers, see [4].

For a positive integer t, let $n, k_1, \ldots, k_t$ be nonnegative integers, and define the t-order multinomial coefficient as follows:

$$\binom{n}{k_1, \ldots, k_t} = \frac{n(n-1)\cdots(n-k_1-\cdots-k_t+1)}{k_1!\cdots k_t!},$$

with $\sum_{i=1}^{t} k_i < n+1$. In particular, $\binom{n}{0,\ldots,0} = 1$. Note that for $t \geq 2$, if $\sum_{i=1}^{t} k_i = n$, then the t-order multinomial coefficient equals a $(t-1)$-order multinomial coefficient

$$\binom{n}{k_1, \ldots, k_t} = \binom{n}{k_1, \ldots, k_{t-1}}.$$

There are many papers concerning the Diophantine equations related to multinomial coefficients. For example, Yang and Cai [9] proved that the Diophantine equation

$$\binom{n}{k_1,\ldots,k_t} = x^l$$

has no positive integer solutions for $n, t \geq 3$, $l \geq 2$, and $\sum_{i=1}^{t} k_i = n$.

In this paper, we consider the Diophantine equation

$$\binom{n}{k_1,\ldots,k_t} = 2^{2^m} + 1 = F_m, \quad \text{for } t \geq 2, \text{ and } \sum_{i=1}^{t} k_i < n, \tag{2}$$

and prove the following theorem.

Theorem 1. *The Diophantine equation (2) has no integer solutions $(m, n, k_1, \ldots, k_t)$ for nonnegative m and positive $n, k_1, \ldots, k_t$.*

2 Two Lemmas

To prove Theorem 1, we need the following two lemmas.

Lemma 2 (Euler [1])**.** *Any prime factor p of the Fermat number F_m satisfies*

$$p \equiv 1 \pmod{2^{m+1}}.$$

Lemma 3 (Luca [6])**.** *If $F_m = s\binom{n}{k}$, with $m \geq 5$, $s \geq 1$, and $1 \leq k \leq \frac{n}{2}$, we have the following two properties.*

(i) Let $n = n'd$, where

$$A = \left\{p : \text{prime } p \mid n, \text{ and } p \equiv 1 \pmod{2^{m+1}}\right\},$$

and

$$n' = \prod_{p \in A} p^{\alpha_p}.$$

Then $k = d < 2^m$.

(ii) $k - i \mid n - i$ for any $i = 0, \ldots, k-1$.

Remark 4. Lemma 3 is summarized from Luca's proof [6] of Diophantine equation (1). Although Luca only proved the case $s = 1$, he indicated that the result also holds for all positive integers s.

3 Proof of Theorem 1

The first five Fermat numbers are primes, which cannot be a multinomial coefficient in Eq. (2). Therefore, we only need to consider $m \geq 5$.

Moreover, for any multinomial coefficient $\binom{n}{k_1,\ldots,k_t}$ with $t > 0$, $k_1, \ldots, k_t \geq 1$, and $\sum_{i=1}^{t} k_i < n$, there exists a multinomial coefficient

$$\binom{n}{k'_1, \ldots, k'_t} = \binom{n}{k_1, \ldots, k_t},$$

such that $1 \leq k'_1, \ldots, k'_t \leq \frac{n}{2}$.

Hence, Eq. (2) becomes

$$F_m = \binom{n}{k_1, \ldots, k_t}, \quad \text{for } m \geq 5,\ 1 \leq k_i \leq \frac{n}{2}, \text{ and } \sum_{i=1}^{t} k_i < n. \tag{3}$$

Let $n = n'd$, where

$$A = \left\{p : \text{prime } p \mid n, \text{ and } p \equiv 1 \pmod{2^{m+1}}\right\},$$

and

$$n' = \prod_{p \in A} p^{\alpha_p}.$$

For any $i = 1, \ldots, t$, we have

$$F_m = \binom{n}{k_1, \ldots, k_t} = \binom{n}{k_i}\binom{n-k_i}{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_t}, \tag{4}$$

where $\binom{n-k_i}{k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_t}$ is a positive integer. By Lemma 3 (i), we have $k_i = d < 2^m$ for $i = 1, \ldots, t$. Then Eq. (3) becomes

$$F_m = \binom{n}{\underbrace{d, \ldots, d}_{t}}, \quad n > td, \text{ and } t \geq 2 \tag{5}$$

$$= \binom{n}{d}\binom{n-d}{d}\binom{n-2d}{\underbrace{d, \ldots, d}_{t-2}}. \tag{6}$$

Note that $d \geq 1$. We study Eq. (6) in the following three cases.

Case 1: $d > 2$. Since $n > 2d$ and $d \mid n$, we have $n \geq 3d$. Then,

$$d \leq \frac{n-d}{2} < \frac{n}{2}.$$

In Eq. (6), applying Lemma 3 (ii) to $\binom{n}{d}$ and $\binom{n-d}{d}$ respectively, and setting $i = 1$, we have

$$d - 1 \mid n - 1$$

and

$$d - 1 \mid n - d - 1.$$

Thus, $d - 1 \mid d$, which is impossible.

Case 2: $d = 2$. Let $n = 2n'$. Then Eq. (5) becomes

$$F_m = \binom{2n'}{\underbrace{2, \ldots, 2}_{t}} = n'(2n' - 1)(n' - 1)(2n' - 3)\binom{2n' - 4}{\underbrace{2, \ldots, 2}_{t-2}}.$$

Then n' and $n' - 1$ are both F_m's factors. According to Lemma 2, we obtain $n' \equiv n' - 1 \equiv 1 \pmod{2^{m+1}}$, which is impossible.

Case 3: $d = 1$. Eq. (5) becomes

$$F_m = \binom{n}{\underbrace{1, \ldots, 1}_{t}} = n(n - 1)\binom{n - 2}{\underbrace{1, \ldots, 1}_{t-2}}.$$

Then n and $n - 1$ are both F_m's factors. According to Lemma 2, we obtain $n \equiv n - 1 \equiv 1 \pmod{2^{m+1}}$, which is also impossible.

This completes the proof of Theorem 1.

Remark 5. One can even find that the multinomial coefficient in Eq. (2) could not divide a Fermat number. Otherwise, assume that there exists a positive integer s such that

$$F_m = s\binom{n}{k_1, \ldots, k_t}.$$

Note that in Eq. (4) we still have

$$\binom{n}{k_i} \mid F_m,$$

and in Eqs. (5) and (6) similar results hold. Hence, we can get the proof in the same way.

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