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On Reciprocity Formulas for Apostol's Dedekind Sums and Their Analogues

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Abstract

Using the Euler-MacLaurin summation formula, we give alternative proofs for the reciprocity formulas of Apostol's Dedekind sums and generalized Hardy-Berndt sums $s_{3,p}(b,c)$ and $s_{4,p}(b,c)$. We also obtain an integral representation for each sum.

1 Introduction

Let

$$((x)) = \begin{cases} x - [x] - 1/2, & \text{if } x \in \mathbb{R}\backslash\mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$$

with $[x]$ being the largest integer $\leq x$. For positive integers c and integers b the classical Dedekind sum $s(b,c)$, arising in the theory of Dedekind η-function, was introduced by R. Dedekind in 1892 as

$$s(b,c) = \sum_{m (\bmod c)} \left(\left(\frac{m}{c}\right)\right)\left(\left(\frac{bm}{c}\right)\right).$$

The most important property of Dedekind sums is the reciprocity theorem

$$s(b,c) + s(c,b) = -\frac{1}{4} + \frac{1}{12}\left(\frac{b}{c} + \frac{c}{b} + \frac{1}{bc}\right)$$

when $\gcd(b,c)=1$. The standard reference for Dedekind sums is Rademacher and Grosswald [7]. Several generalizations of Dedekind sums have been defined and the corresponding reciprocity formulas have been obtained. One of these generalizations, due to Apostol [1], is

$$s_p(b,c)=\sum_{m=1}^{c-1}\frac{m}{c}\overline{B}_p\left(\frac{bm}{c}\right),$$

where $\overline{B}_p(x)$ is the pth Bernoulli function defined by

$$\overline{B}_1(x)=((x)),\ \text{and } \overline{B}_p(x+m)=B_p(x) \text{ for } 0\leq x<1,\ m\in\mathbb{Z} \text{ and } p>1.$$

Here $B_p(x)$ is the pth Bernoulli polynomial. Apostol's reciprocity formula is as follows.

Theorem 1. *Let b and c be coprime positive integers. For odd $p\geq 1$, we have*

$$(p+1)\left(bc^p s_p(b,c)+cb^p s_p(c,b)\right)=\sum_{j=0}^{p+1}\binom{p+1}{j}(-1)^j b^{p+1-j}c^j B_{p+1-j}B_j+pB_{p+1}.$$

Here $B_p=B_p(0)$ is the pth Bernoulli number.

Similar arithmetic sums arise in the theory of logarithms of the classical theta functions. They were studied by Hardy and Berndt, and for this reason they are called Hardy or Hardy-Berndt sums. There are six such sums, two of which are [2, 6]

$$s_3(b,c)=\sum_{m=1}^{c-1}(-1)^m\,\overline{B}_1\left(\frac{bm}{c}\right),\ s_4(b,c)=-4\sum_{m=1}^{c-1}\overline{B}_1\left(\frac{bm}{2c}\right).$$

Goldberg [6] showed that these sums also arise in the theory of $r_m(n)$, the number of representations of n as a sum of m integral squares and in the study of the Fourier coefficients of the reciprocals of the classical theta functions. Like Dedekind sums, Hardy-Berndt sums also satisfy a reciprocity (or reciprocity-like) formula [2, 6]

$$2s_3(b,c)-s_4(c,b)=1-\frac{b}{c}$$

when c is odd and $\gcd(b,c)=1$. The generalizations of these sums in the sense of Apostol have been given in [3] by

$$s_{3,p}(b,c)=\sum_{m=1}^{c-1}(-1)^m\,\overline{B}_p\left(\frac{bm}{c}\right),\quad s_{4,p}(b,c)=-4\sum_{m=1}^{c-1}\overline{B}_p\left(\frac{bm}{2c}\right)$$

which satisfy the following reciprocity formula.

Theorem 2. *Let b and c be coprime positive integers with c odd. For odd $p \geq 1$, we have*

$$\begin{aligned}&(p+1)\left(2bc^{p}s_{3,p}\left(b,c\right)-2^{-1}c\left(2b\right)^{p}s_{4,p}\left(c,b\right)\right)\\&=4\sum_{j=1}^{p+1}\binom{p+1}{j}\left(-1\right)^{j}b^{j}c^{p+1-j}\left(1-2^{j}\right)B_{j}B_{p+1-j}.\end{aligned}$$

The reciprocity formulas in this concept are proved by employing various techniques and theories such as transformation formulas, residue theory, Franel integral and arithmetic methods.

In this study we give rather elementary but new proofs for Theorems 1 and 2 when $p>1$ by applying the Euler-MacLaurin summation formula to Bernoulli function.

The method presented in the sequel is motivated by [4].

2 Proofs of the reciprocity theorems

Let us state the Euler-MacLaurin summation formula, which can be found in various books, for example, [5, p. 22].

Theorem 3. ***(Euler-MacLaurin)*** *Let α and β be real numbers such that $\alpha \leq \beta$ and assume that $f \in C^{(l)}\left[\alpha,\beta\right]$ for some $l \geq 1$. Then*

$$\begin{aligned}\sum_{\alpha<m\leq\beta}f(m)=&\int_{\alpha}^{\beta}f(u)du+\sum_{j=1}^{l}\frac{(-1)^{j}}{j!}\left(\overline{B}_{j}\left(\beta\right)f^{(j-1)}(\beta)-\overline{B}_{j}\left(\alpha\right)f^{(j-1)}(\alpha)\right)\\&+\frac{(-1)^{l-1}}{l!}\int_{\alpha}^{\beta}f^{(l)}(u)\overline{B}_{l}\left(u\right)du.\end{aligned}$$

We will also need the facts $\overline{B}_{2r+1}(0)=\overline{B}_{2r+1}(1/2)=0$ for all $r \geq 0$ and Raabe's formula

$$\sum_{m=0}^{n-1}\overline{B}_{r}\left(x+\frac{m}{n}\right)=n^{1-r}\overline{B}_{r}\left(nx\right). \tag{1}$$

Firstly, we consider the function $f\left(x\right)=\overline{B}_{p}\left(xy\right)$, $y \in \mathbb{R}$. The property

$$\frac{d}{dx}\overline{B}_{p}\left(x\right)=p\overline{B}_{p-1}\left(x\right),\ p>2$$

entails that

$$\frac{d^{j}}{dx^{j}}f\left(x\right)=\frac{d^{j}}{dx^{j}}\overline{B}_{p}\left(xy\right)=y^{j}\frac{p!}{(p-j)!}\overline{B}_{p-j}\left(xy\right) \tag{2}$$

for $1 \leq j \leq p-2$ and $f \in C^{(p-2)}[\alpha, \beta]$. For $\alpha = 0$ and $1 \leq l \leq p-2$, Theorem 3 can be written as

$$\begin{aligned}
&\sum_{1 \leq m \leq \beta} \overline{B}_p(my) \\
&\quad = \frac{1}{p+1} \sum_{j=1}^{l} (-1)^j \binom{p+1}{j} y^{j-1} \left(\overline{B}_j(\beta) \overline{B}_{p-j+1}(\beta y) - \overline{B}_{p-j+1}(0) \overline{B}_j(0) \right) \\
&\qquad + \frac{1}{y} \frac{\overline{B}_{p+1}(\beta y) - \overline{B}_{p+1}(0)}{p+1} - (-y)^l \binom{p}{l} \int_0^\beta \overline{B}_l(u) \overline{B}_{p-l}(yu)\, du. \qquad (3)
\end{aligned}$$

Let $y = b/c$ and $\beta = c \in \mathbb{N}$. Then (3) becomes

$$\sum_{m=1}^{c} \overline{B}_p\left(\frac{bm}{c}\right) = -\binom{p}{l}\left(-\frac{b}{c}\right)^l \int_0^c \overline{B}_l(u) \overline{B}_{p-l}\left(\frac{b}{c}u\right) du. \qquad (4)$$

- For $b = c$ in (4) we have the well-known relation [5, p. 120]

$$\int_0^1 \overline{B}_l(u) \overline{B}_r(u)\, du = (-1)^{l-1} \frac{l!r!}{(r+l)!} B_{l+r}, \text{ for } l \geq 1 \text{ and } p - l = r \geq 2.$$

- Let $\gcd(b, c) = 1$. Then, it follows from (1) and (4) that

$$\int_0^1 \overline{B}_l(cu) \overline{B}_{p-l}(bu)\, du = \frac{(-1)^{l-1}}{\binom{p}{l}} \frac{c^{l-p}}{b^l} B_p. \qquad (5)$$

- Now assume that $\gcd(b, c) = q$ and put $c = qc_1$, $b = qb_1$. From (5), we have

$$\int_0^1 \overline{B}_l(cu) \overline{B}_{p-l}(bu)\, du = c \int_0^1 \overline{B}_l(c_1 u) \overline{B}_{p-l}(b_1 u)\, du = cq^p \frac{(-1)^{l-1}}{\binom{p}{l}} \frac{c^{l-p}}{b^l} B_p.$$

2.1 Proof of Theorem 1

Let $f(x) = x\overline{B}_p(xy)$, $y \in \mathbb{R}$. Then $f \in C^{(p-2)}[\alpha, \beta]$. From (2) and the Leibniz rule for the derivative we have, for $1 \leq j \leq p-2$,

$$\frac{d^j}{dx^j}\left[x\overline{B}_p(xy)\right] = y^j \frac{p!}{(p-j)!} x\overline{B}_{p-j}(xy) + y^{j-1} \frac{p!}{(p+1-j)!} j\overline{B}_{p+1-j}(xy).$$

For $f(x) = x\overline{B}_p(xy)$, $\alpha = 0$, $\beta = c$ and $y = b/c$ with $\gcd(b,c) = 1$ in Theorem 3, we have

$$\begin{aligned}&\sum_{m=1}^{c} m\overline{B}_p\left(m\frac{b}{c}\right)\\&= c^2\int_0^1 x\overline{B}_p(bx)\,dx + \frac{c}{p+1}\sum_{j=1}^{l}(-1)^j\binom{p+1}{j}\left(\frac{b}{c}\right)^{j-1}\overline{B}_j(0)\,\overline{B}_{p+1-j}(0)\\&+\binom{p}{l}\left(-\frac{b}{c}\right)^{l-1}c\int_0^1\overline{B}_l(cx)\left(bx\overline{B}_{p-l}(bx)+\frac{l}{p+1-l}\overline{B}_{p+1-l}(bx)\right)dx,\end{aligned}\tag{6}$$

where $1 \leq l \leq p-2$. It follows from integration by parts that

$$\int_0^1 x\overline{B}_p(bx)\,dx = \frac{1}{b(p+1)}\overline{B}_{p+1}(0)\tag{7}$$

and from (5)

$$(-1)^{l-1}\binom{p+1}{l}\left(\frac{b}{c}\right)^{l}c\int_0^1\overline{B}_l(cx)\,\overline{B}_{p+1-l}(bx)\,dx = c^{-p}\overline{B}_{p+1}(0).\tag{8}$$

Combining (6), (7) and (8), we have

$$\begin{aligned}&\sum_{m=1}^{c} m\overline{B}_p\left(\frac{bm}{c}\right)\\&= \frac{c}{p+1}\sum_{j=0}^{l}(-1)^j\binom{p+1}{j}\left(\frac{b}{c}\right)^{j-1}\overline{B}_{p+1-j}(0)\,\overline{B}_j(0) + \frac{l}{p+1}\frac{c^{1-p}}{b}B_{p+1}\\&+(-1)^{l-1}\binom{p}{l}\left(\frac{b}{c}\right)^{l-1}bc\int_0^1 x\overline{B}_l(cx)\,\overline{B}_{p-l}(bx)\,dx.\end{aligned}\tag{9}$$

It can be easily seen from (1) and the fact $\overline{B}_p(-x) = (-1)^p\,\overline{B}_p(x)$ that $2s_p(b,c) = (c^{1-p}-1)\,B_p$ when p is even. We then have

$$\sum_{m=1}^{c} m\overline{B}_p\left(\frac{bm}{c}\right) = \sum_{m=1}^{c-1} m\overline{B}_p\left(\frac{bm}{c}\right) + c\overline{B}_p(0) = \begin{cases} cs_p(b,c), & \text{if } p \text{ is odd;}\\ (c^{1-p}+1)\,\dfrac{c}{2}B_p, & \text{if } p \text{ is even.}\end{cases}\tag{10}$$

- Let $p>1$ be odd. Putting $l=2$ in (9) and using (10), we get

$$\begin{aligned} cs_p(b,c) =& \frac{c}{p+1}\left(\frac{c}{b}\overline{B}_{p+1}(0)+\binom{p+1}{2}\frac{b}{c}\overline{B}_{p-1}(0)\overline{B}_2(0)\right) \\ &+\frac{2}{p+1}\frac{c^{1-p}}{b}B_{p+1}-\binom{p}{2}b^2\int_0^1 x\overline{B}_{p-2}(bx)\overline{B}_2(cx)\,dx \end{aligned} \tag{11}$$

Putting $l=p-2$ and interchanging b and c in (9)

$$\begin{aligned} bs_p(c,b) =& \frac{b}{p+1}\sum_{j=0}^{p-2}(-1)^j\binom{p+1}{j}\left(\frac{c}{b}\right)^{j-1}\overline{B}_{p+1-j}(0)\overline{B}_j(0)+\frac{p-2}{p+1}\frac{b^{1-p}}{c}B_{p+1} \\ &+(-1)^{p-1}\binom{p}{2}\left(\frac{c}{b}\right)^{p-3}bc\int_0^1 x\overline{B}_{p-2}(bx)\overline{B}_2(cx)\,dx. \end{aligned} \tag{12}$$

Then, from (11) and (12), we arrive at the reciprocity formula

$$(p+1)\left(bc^p s_p(b,c)+cb^p s_p(c,b)\right)=\sum_{j=0}^{p+1}\binom{p+1}{j}(-1)^j b^{p+1-j}c^j B_{p+1-j}B_j+pB_{p+1}.$$

Note that for $l=1$ in (9) we have the following integral representation

$$s_p(b,c)=\left(c+c^{-p}\right)\frac{1}{b}\frac{B_{p+1}}{p+1}+pb\int_0^1 x\overline{B}_1(cx)\overline{B}_{p-1}(bx)\,dx, \text{ for odd } p>1.$$

- Let $p>2$ be even. From (9) and (10) we have

$$b\int_0^1 x\overline{B}_l(cx)\overline{B}_{p-l}(bx)\,dx=\frac{1}{2}\left(1+c^{1-p}\right)\left(-\frac{c}{b}\right)^{l-1}\binom{p}{l}^{-1}B_p, \text{ for } 1\le l\le p-2.$$

2.2 Proof of Theorem 2

We will use (3) for the following cases;

I) $y=b/(2c)$ and $\beta=c$,

II) $y=2b/c$ and $\beta=c/2$.

I) Let b be odd and consider $y=b/2c$, $\beta=c$ with $\gcd(b,c)=1$ in (3). From (1), we have

$$\overline{B}_{p-j+1}\left(\frac{b}{2}\right)=\overline{B}_{p-j+1}\left(\frac{1}{2}\right)=\left(2^{j-p}-1\right)\overline{B}_{p-j+1}(0). \tag{13}$$

Therefore, (3) becomes

$$
\begin{aligned}
\sum_{n=1}^{c}\overline{B}_p\left(\frac{b}{2c}n\right) &= -\frac{1}{4}s_{4,p}\left(b,c\right)+\overline{B}_p\left(\frac{1}{2}\right)\\
&= \frac{2^{1-p}}{p+1}\sum_{m=p-l+1}^{p+1}(-1)^{p+1-m}\binom{p+1}{m}\left(\frac{b}{c}\right)^{p-m}(1-2^m)\overline{B}_m(0)\overline{B}_{p+1-m}(0)\\
&\quad - c\binom{p}{l}\left(-\frac{b}{2c}\right)^{l}\int_0^1\overline{B}_l\left(cu\right)\overline{B}_{p-l}\left(\frac{b}{2}u\right)du \qquad (14)
\end{aligned}
$$

by setting $j=p+1-m$.

- Let $p>1$ be odd and put $l=2$ in (14). Then,

$$
\begin{aligned}
-\frac{2^p}{4}s_{4,p}\left(b,c\right) =&\frac{2}{p+1}\sum_{m=p-1}^{p+1}\binom{p+1}{m}(-1)^m\left(\frac{b}{c}\right)^{p-m}\left(1-2^m\right)\overline{B}_m(0)\overline{B}_{p+1-m}(0)\\
&-2^{p-3}p(p-1)\frac{b^2}{c}\int_0^1\overline{B}_2\left(cu\right)\overline{B}_{p-2}\left(\frac{b}{2}u\right)du. \qquad (15)
\end{aligned}
$$

For $l=1$ in (14) we have the following integral representation

$$
\begin{aligned}
-\frac{1}{4}s_{4,p}\left(b,c\right) =&\frac{2^{1-p}}{p+1}\frac{c}{b}\left(1-2^{p+1}\right)B_{p+1}\\
&+p\frac{b}{2}\int_0^1\overline{B}_1\left(cu\right)\overline{B}_{p-1}\left(\frac{b}{2}u\right)du,\ \text{for odd } p>1.
\end{aligned}
$$

- If $p>2$ is even, it is seen from (14), (13) and

$$s_{4,p}(b,c)=2^{2-p}\left(1-c^{1-p}\right)B_p,\ \text{for even } p \text{ and odd } b$$

[3, Proposition 2.5] that

$$c\int_0^1\overline{B}_l\left(cu\right)\overline{B}_{p-l}\left(\frac{b}{2}u\right)du=2^{l-p}(2^p-c^{1-p}-1)\left(-\frac{c}{b}\right)^l\binom{p}{l}^{-1}B_p.$$

II) Let c be odd and consider $y=2b/c$, $\beta=c/2$ with $\gcd\left(b,c\right)=1$. Then, from (3) and

(13) we have

$$\sum_{0<n\leq c/2}\overline{B}_p\left(\frac{2b}{c}n\right)=\sum_{n=1}^{(c-1)/2}\overline{B}_p\left(\frac{2b}{c}n\right)$$
$$=\frac{1}{p+1}\sum_{j=1}^{l}(-1)^j\binom{p+1}{j}\left(\frac{2b}{c}\right)^{j-1}\left(2^{1-j}-2\right)\overline{B}_j(0)\overline{B}_{p+1-j}(0)$$
$$-\left(-\frac{2b}{c}\right)^l\binom{p}{l}\frac{c}{2}\int_0^1\overline{B}_l\left(\frac{c}{2}u\right)\overline{B}_{p-l}(bu)\,du. \tag{16}$$

By the definition of the sum $s_{3,p}(b,c)$ we have

$$s_{3,p}(b,c)=\sum_{n=1}^{c-1}(-1)^n\overline{B}_p\left(\frac{bn}{c}\right)=2\sum_{n=1}^{(c-1)/2}\overline{B}_p\left(\frac{2bn}{c}\right)-\sum_{n=1}^{c-1}\overline{B}_p\left(\frac{bn}{c}\right). \tag{17}$$

- Let $p>1$ be odd. Put $l=p-2$ in (16). Then, (1), (16) and (17) yield

$$(p+1)\,bc^p s_{3,p}(b,c)=2\sum_{j=1}^{p-2}\binom{p+1}{j}(-1)^j b^j c^{p+1-j}\left(1-2^j\right)\overline{B}_{p+1-j}(0)\overline{B}_j(0)$$
$$+2^{p-3}b^{p-1}c^3p(p-1)(p+1)\int_0^1\overline{B}_{p-2}\left(\frac{c}{2}u\right)\overline{B}_2(bu)\,du. \tag{18}$$

Combining (15) and (18) we obtain the reciprocity formula

$$(p+1)\left(bc^p s_{3,p}(b,c)-2^{-2}c\,(2b)^p\,s_{4,p}(c,b)\right)$$
$$=2\sum_{j=1}^{p+1}\binom{p+1}{j}(-1)^j b^j c^{p+1-j}\left(1-2^j\right)B_jB_{p+1-j},$$

when c and $p>1$ are odd.

- If $p>2$ is even, then from (1), (16) and $s_{3,p}(b,c)=0$ for odd $(p+c)$ we get

$$c\binom{p}{l}\int_0^1\overline{B}_l\left(\frac{c}{2}u\right)\overline{B}_{p-l}(bu)\,du=\left(-\frac{c}{2b}\right)^l\left(1-c^{1-p}\right)B_p.$$

Putting $l=1$ in (16) gives an integral representation for $s_{3,p}(b,c)$ as

$$s_{3,p}(b,c)=2bp\int_0^1\overline{B}_1\left(\frac{c}{2}u\right)\overline{B}_{p-1}(bu)\,du,\ \text{for odd } p>1.$$

References

[1] T. M. Apostol, Generalized Dedekind sums and transformation formulae of certain Lambert series. *Duke Math. J.* **17** (1950) 147–157.

[2] B. C. Berndt, Analytic Eisenstein series, theta functions and series relations in the spirit of Ramanujan, *J. Reine Angew. Math.* **303/304** (1978) 332–365.

[3] M. Can, M. Cenkci, and V. Kurt, Generalized Hardy-Berndt sums, *Proc. Jangjeon Math. Soc.* **9** (2006) 19–38.

[4] M. Can and V. Kurt, Character analogues of certain Hardy-Berndt sums, *Int. J. Number Theory*, **10** (2014), 737–762.

[5] H. Cohen, *Number Theory Volume II: Analytic and Modern Tools*, Springer, 2007.

[6] L. A. Goldberg, Transformations of theta-functions and analogues of Dedekind sums, thesis, University of Illinois, 1981.

[7] H. Rademacher and E. Grosswald, *Dedekind Sums*, Carus Math. Monographs, Vol. 16, Math. Assoc. Amer., 1972.

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