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On a Conjecture of Andrica and Tomescu

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Abstract

For positive integers n congruent to 0 or 3 (mod 4), let $S(n)$ be the coefficient of $x^{n(n+1)/4}$ in the expansion of $(1+x)(1+x^2)\cdots(1+x^n)$ We prove a conjecture of Andrica and Tomescu that $S(n)$ is asymptotically equal to $\sqrt{\frac{6}{\pi}} \cdot 2^n \cdot n^{-3/2}$.

1 Introduction

For positive integers n congruent to 0 or 3 (mod 4), let $S(n)$ denote the coefficient of the middle term of the expansion of the polynomial $(1+x)(1+x^2)\cdots(1+x^n)$. (In the case $n \equiv 1, 2 \pmod 4$, the quantity $n(n+1)/4$ is not an integer, and the expansion has no middle term.) This number also represents the number of partitions of $T_n/2 = n(n+1)/4$ into distinct parts less than or equal to n, where T_n is the nth triangular number. Andrica and Tomescu conjectured that as n approaches infinity, $S(n)$ behaves asymptotically like $\sqrt{\frac{6}{\pi}} \cdot 2^n \cdot n^{-3/2}$. More formally, writing $f(n) \sim g(n)$ as usual to denote

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1,$$

we have

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Conjecture 1. [Andrica, Tomescu [1]] $S(n) \sim \sqrt{\frac{6}{\pi}} \cdot 2^n \cdot n^{-3/2}$ for $n \equiv 0$ or 3 (mod 4).

From [1], one can write $S(n)$ in integral form via Cauchy's formula as

$$S(n) = \frac{2^{n-1}}{\pi} \int_0^{2\pi} \cos(t)\cos(2t)\cdots\cos(nt)\, dt.$$

We will use the Laplace method to estimate this integral [2]. Rewriting, we have $S(n) = \frac{2^{n-1}}{\pi} \int_0^{2\pi} f_n(t)\, dt$ where $f_n(t) = \prod_{k=1}^n \cos(kt)$. In Section 2, we analyze the behavior of $f_n(t)$ and note a technical lemma needed for the main proof of Conjecture 1, which is presented in Section 3. We note that a similar approach was suggested in a review of [1] by Hwang [3] published in *Mathematical Reviews*.

2 Behavior of $f_n(t)$

Lemma 2. *Let* $0 < \varepsilon < 1/4$, *and* $f_n(t) = \prod_{k=1}^n \cos(kt)$. *Then* $\int_{n^{-(3/2-\varepsilon)}<|t|<\pi/2} |f_n(t)|\, dt = o(n^{-3/2})$ *as* $n \to \infty$.

Proof. We break the integral into three pieces based on the value of $|t|$.

Case 1. $n^{-(\frac{3}{2}-\varepsilon)} \le |t| \le \frac{1}{n}$:

Since $\cos(x) = \cos(-x)$, and cos is a monotone decreasing function on $[0, \pi]$, $f_n(t) = f_n(-t)$ is also monotone decreasing for $t \in [0, 1/n]$, and it suffices to give an appropriate upper bound on $f_n(n^{-(3/2-\varepsilon)})$.

Since we need $\int_{n^{-(3/2-\varepsilon)}<|t|<\pi/2} f_n(t)\, dt = o(n^{-3/2})$, given that $0 < \varepsilon < 1/4$, it suffices to show that for a constant $c > 0$,

$$f_n(n^{-(3/2-\varepsilon)}) \le \exp(-cn^{2\varepsilon}(1 + o(1))).$$

Using the Taylor series expansion, we know $\cos(kt) \le 1 - \frac{(kt)^2}{2!} + \frac{(kt)^4}{4!}$. Substitution then yields

$$f_n(t) = \prod_{k=1}^n \cos(kt) \le \prod_{k=1}^n \left(1 - \frac{(kt)^2}{2!} + \frac{(kt)^4}{4!}\right),$$

since $k \le n$ and $|t| \le 1/n$ implies $kt \le 1$. When $t = n^{-(3/2-\varepsilon)}$, we have

$$f_n(t) \le \prod_{k=1}^n \left(1 - \frac{k^2 n^{-(3-2\varepsilon)}}{2} + \frac{k^4 n^{-(6-4\varepsilon)}}{24}\right).$$

To evaluate, we note the terms of this product are all in $[0, 1]$, and apply $\log(1 - x) \le -x$:

$$\log \prod_{k=1}^n \left(1 - \left(\frac{k^2 n^{-(3-2\varepsilon)}}{2} - \frac{k^4 n^{-(6-4\varepsilon)}}{24}\right)\right) \le \sum_{k=1}^n \left(-\frac{k^2 n^{-(3-2\varepsilon)}}{2} + \frac{k^4 n^{-(6-4\varepsilon)}}{24}\right).$$

Writing $\sum_{k=1}^{n} k^2 = (1/3 + o(1))n^3$ and $\sum_{k=1}^{n} k^4 = (1/5 + o(1))n^5$, we have

$$\log f_n = -\left(\frac{1}{6} + o(1)\right) n^3 n^{-(3-2\varepsilon)} + \left(\frac{1}{120} + o(1)\right) n^5 n^{-(6-4\varepsilon)}.$$

Letting $c = 1/6$, $f_n \leq \exp(-c(1+o(1))n^{2\varepsilon} + c(1+o(1))n^{-1+4\varepsilon})$, and recalling $\varepsilon < 1/4$, $f_n \leq \exp(-c(1+o(1))n^{2\varepsilon})$ as desired.

Case 2. $\frac{1}{n} \leq |t| \leq \frac{\pi}{n}$:

Here we use will the monotonicity of $f_n(t)$ in n. It follows directly from $f_n(t) = \prod_{k=1}^{n} \cos(kt)$ and $0 \leq \cos(x) \leq 1$ that $|f_n(t)| \leq |f_m(t)|$ for $n \geq m$. Let $h_n = \lfloor n/4 \rfloor$ be the greatest integer in $n/4$. Then $|f_n(t)| \leq |f_{h_n}(t)|$. From Case 1, $f_{h_n}(t) \leq \exp(-ch_n^{2\varepsilon}(1+o(1)))$ for $h_n^{-(\frac{3}{2}-\varepsilon)} \leq |t| \leq 1/h_n$. Since $1/n > h_n^{-5/4} \geq h_n^{-(\frac{3}{2}-\varepsilon)}$ for $n > 1050$ and $h_n \leq n/4 \leq n/\pi$ implies $\pi/n \leq 1/h_n$, we get $|f_n(t)| \leq \exp(-ch_n^{2\varepsilon}(1+o(1)))$ for $t \leq \pi/n$ as $n \to \infty$.

Case 3. $\frac{\pi}{n} \leq |t| \leq \frac{\pi}{2}$:

Note that it suffices to show that $|f_n(t)| \leq c^n$ for a constant $c < 1$, since then

$$\int_{\pi/n \leq |t| < \pi/2} |f_n(t)|\, dt \leq \pi \cdot c^n = o(n^{-3/2}).$$

To accomplish this, we first transform $f_n(t)$ from a product to a sum using the arithmetic-geometric mean inequality:

$$(f_n^2(t))^{1/n} = \left(\prod_{k=1}^{n} \cos^2(kt)\right)^{1/n} \leq \frac{1}{n}\sum_{k=1}^{n} \cos^2(kt). \tag{1}$$

The sum on the right-hand side can be simplified as

$$\sum_{k=1}^{n} \cos^2(kt) = \frac{n}{2} + \frac{1}{2}\sum_{k=1}^{n} \cos(2kt) = \frac{n}{2} + \frac{\cos((n+1)t)}{2}\frac{\sin(nt)}{\sin(t)}. \tag{2}$$

Combining equations 1 and 2, we can write

$$|f_n(t)| \leq \left(\frac{1}{2} + \frac{1}{2n}\frac{1}{\sin(t)}\right)^{n/2}. \tag{3}$$

We will now apply the Jordan-style concavity inequality $|\sin(t)| \geq \frac{2|t|}{\pi}$ for $0 \leq |t| \leq \pi/2$. For $\pi/n \leq |t| \leq \pi/2$, substitution in equation 3 gives

$$|f_n(t)| \leq \left(\frac{1}{2} + \frac{1}{2n}\frac{\pi}{2|t|}\right)^{n/2} = \left(\frac{1}{2} + \frac{\pi}{4n|t|}\right)^{n/2}.$$

Observing that the right-hand side is monotonically decreasing in $|t|$, we have $|f_n(t)| \leq f_n(\pi/n)$. Evaluating, we see

$$|f_n(t)| \leq \left(\frac{1}{2} - \frac{1}{2n}\right)^{n/2}$$

proving $|f_n(t)| \leq (\sqrt{7/16})^n$ (since we may assume $2n \geq 16$ as $n \to \infty$).

$\square$

We will also need the following straightforward lemma from analysis.

Lemma 3. *Let $c \in \mathbb{R}$ and $a(c), b(c)$ be real-valued functions such that*

$$\lim_{c\to\infty} -a(c)\sqrt{c} = \lim_{c\to\infty} b(c)\sqrt{c} = \infty.$$

Then

$$\int_{a(c)}^{b(c)} e^{-ct^2}\,dt \sim \int_{-\infty}^{\infty} e^{-ct^2}\,dt$$

as $c \to \infty$.

3 Main Result

We now prove Conjecture 1.

Theorem 4. *For $n \equiv 0$ or $3 \pmod 4$, we have $S(n) \sim \sqrt{\frac{6}{\pi}} \cdot 2^n \cdot n^{-3/2}$.*

Proof. When $n \equiv 0$ or $3 \pmod 4$, $f_n(t + m\pi) = f_n(t)$ for any integer m, so

$$S(n) = \frac{2 \cdot 2^{n-1}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_n(t)\,dt, \tag{4}$$

and we may assume $|t| \leq \pi/2$ when evaluating $f_n(t)$.

By Lemma 2, $\int_{n^{-(3/2-\varepsilon)}<|t|<\pi/2} |f_n(t)|\,dt = o(n^{-3/2})$, so it suffices to consider $|t| < n^{-(3/2-\varepsilon)}$ when estimating $f_n(t)$ around $t = 0$. Recalling

$$f_n(t) = \prod_{k=1}^{n} e^{\ln(\cos(kt))},$$

we first use Taylor series to approximate $g_k(t) = \ln(\cos(kt))$ at $t = 0$. We have $g_k(t) = -k^2t^2/2 + R_2$, where R_2 is the Lagrange remainder. Then R_2 is bounded by a constant times $t^3 g_k^{(3)}(t_0)$ for some t_0 near 0. Since $g_k^{(3)}(t) = -2k^3\sin(kt)/\cos^3(kt)$, and t_0 is small (since $|t| < n^{-(3/2-\varepsilon)}$), we have that $R_2 \leq ak^3t^3$ where a is constant. The absolute error for $g_k(t)$ is thus bounded by $ak^3n^{-(9/2-3\varepsilon)}$.

Around $t = 0$, $f_n(t)$ can be approximated as $\delta \prod_{k=1}^{n} e^{-\frac{k^2t^2}{2}}$ with error $\delta \leq \prod_{k=1}^{n} e^{ak^3n^{-(9/2-3\varepsilon)}}$. This simplifies to

$$f_n(t) \approx e^{-t^2/2\sum_{k=1}^{n} k^2} = e^{-t^2 n(n+1)(2n+1)/12}. \tag{5}$$

Our error bound simultaneously simplifies to

$$\delta \leq e^{an^{-(9/2-3\varepsilon)}\sum_{k=1}^{n} k^3} = e^{an^{-(9/2-3\varepsilon)}n^2(n+1)^2/4}.$$

This proves that the error goes to one as n approaches infinity whenever $\varepsilon < \frac{1}{6}$.

Substituting (5) for $f_n(t)$ in equation 4, and applying Lemma 2, we find that

$$\frac{\pi S(n)}{2^n} = (1+o(1)) \int_{-n^{-(3/2-\varepsilon)}}^{n^{-(3/2-\varepsilon)}} e^{-n(n+1)(2n+1)t^2/12}\, dt + o(n^{-3/2}).$$

By Lemma 3, this implies

$$\frac{\pi S(n)}{2^n} = (1+o(1)) \int_{-\infty}^{\infty} e^{-n(n+1)(2n+1)t^2/12}\, dt + o(n^{-3/2}).$$

Using

$$\int_{-\infty}^{\infty} e^{-Ct^2}\, dt = \sqrt{\frac{\pi}{C}}$$

for any constant $C > 0$ and $n(n+1)(2n+1) \sim 2n^3$, we have

$$S(n) \sim \frac{2^n}{\pi}\sqrt{\frac{12\pi}{2n^3}} = \sqrt{\frac{6}{\pi}} \cdot 2^n \cdot n^{-3/2},$$

as desired.

$\square$

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