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A Note on a Family of Generalized Pascal Matrices Defined by Riordan Arrays

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Abstract

We study the properties of a parameterized family of generalized Pascal matrices, defined by Riordan arrays. In particular, we characterize the central elements of these lower triangular matrices, which are analogues of the central binomial coefficients. We then specialize to the value 2 of the parameter, and study the inverse of the matrix in question, and in particular we study the sequences given by the first column and row sums of the inverse matrix. Links to moments and orthogonal polynomials are examined, and Hankel transforms are calculated. We study the effect of the powers of the binomial matrix on the family. Finally we posit a conjecture concerning determinants related to the Christoffel-Darboux bivariate quotients defined by the polynomials whose coefficient arrays are given by the generalized Pascal matrices.

1 Introduction

Pascal's triangle [19] holds a special place in mathematics. It has been generalized in many directions [7, 8, 11, 13, 36, 37]. One approach to generalizing Pascal's matrix is to use Riordan arrays [2]. This is very apt, as Pascal's triangle, as a mathematical object, is a Riordan array. In this note, we investigate some properties of the two-parameter generalized Pascal triangle $M^{(m)}(a,b)$ given by the Riordan array

$$M^{(m)}(a,b) = \left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^m}\right).$$

The binomial matrix $B = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ whose elements coincide with Pascal's triangle, is the case $a = 1$, $b = 0$ and $m = 1$ or $M^{(1)}(1,0)$. It is also given by $M^{(2)}(1,-1)$. The matrix $M^{(2)}(1,1)$ A114188 begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 9 & 7 & 1 & 0 & 0 & \cdots \\ 1 & 16 & 26 & 10 & 1 & 0 & \cdots \\ 1 & 25 & 70 & 52 & 13 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and factorizes as

$$B \cdot \left(\binom{k}{n-k} 2^{n-k}\right)_{0 \le n,k \le \infty}.$$

$M^{(2)}(-1,-2)$ is A110522. Our investigations lie in four principal directions. Firstly, we give characterizations of the central elements of the array $M^{(m)}(a,b)$. These elements generalize the central binomial coefficients $\binom{2n}{n}$ A000984 of Pascal's triangle. Secondly, we look at the specialized matrix with $m = 2$, and in particular we establish properties of the inverse matrix. These investigations cover such things as the Hankel transform of associated sequences, and links with moments and orthogonal polynomials. We then look at the images of $M^{(m)}(a,b)$ under the powers B^k of the binomial matrix. Finally we study matrices and determinants related to Christoffel-Darboux-type bivariate generating functions related to the matrices $M^{(m)}(a,b)$.

2 Preliminaries on integer sequences, Riordan arrays and Hankel transforms

In this section we define terms used later to discuss integer sequences, Riordan arrays, production matrices, orthogonal polynomials and Hankel transforms. Readers familiar with these areas and the links between them may skip this section.

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is called the *ordinary generating function* or g.f. of the sequence; a_n is thus the coefficient of x^n in this series. We denote this relationship by $a_n = [x^n]f(x)$. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $C_n = [x^n]\frac{1-\sqrt{1-4x}}{2x}$ is the n-th Catalan number A000108. We use the notation $0^n = [x^n]1$ for the sequence $1, 0, 0, 0, \ldots$, A000007. Thus $0^n = [n = 0] = \delta_{n,0} = \binom{0}{n}$. Here, we have used the Iverson bracket notation [21], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false.

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x)$ such that $f(\bar{f}(x)) = x$.

For a lower triangular matrix $(a_{n,k})_{n,k\geq 0}$ the row sums give the sequence with general term $\sum_{k=0}^{n} a_{n,k}$.

The *Riordan group* [28, 31], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1x + g_2x^2 + \cdots$ and $f(x) = f_1x + f_2x^2 + \cdots$ where $f_1 \neq 0$ [31]. We often require in addition that $f_1 = 1$. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) \cdot (h, l) = (g, f)(h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \ldots)^T$ is an integer sequence (expressed as an infinite column vector) with ordinary generating function $\mathcal{A}(x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbb{Z}^{\mathbb{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbb{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \mapsto (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

Example 1. The so-called *binomial matrix* B is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$, and hence as an array coincides with Pascal's triangle. More generally, B^m is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. It is easy to show that the inverse B^{-m} of B^m is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

The row sums of the matrix (g, f) have generating function

$$(g, f) \cdot \frac{1}{1-x} = \frac{g(x)}{1-f(x)}.$$

The production matrix of a matrix L is the matrix $P = L^{-1}\bar{L}$, where $\bar{L}$ is the matrix L with its top line removed. A Riordan array L whose inverse has a suitable tri-diagonal production matrix is the coefficient array of a family of orthogonal polynomials [4, 5]. Orthogonal polynomials [15, 20] are often associated with the three-term recurrences that can be used to define them.

An important feature of Riordan arrays is that they have a number of sequence characterizations [14, 22]. The simplest of these is as follows.

Proposition 2. *; [22, Theorem 2.1, Theorem 2.2]. Let $D = [d_{n,k}]$ be an infinite triangular matrix. Then D is a Riordan array if and only if there exist two sequences $A = [\alpha_0, \alpha_1, \alpha_2, \ldots]$ and $Z = [z_0, z_1, z_2, \ldots]$ with $\alpha_0 \neq 0$, $z_0 \neq 0$ such that*

- $d_{n+1,k+1} = \sum_{j=0}^{\infty} \alpha_j d_{n,k+j}, \quad (k, n = 0, 1, \ldots)$

- $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}, \quad (n = 0, 1, \ldots)$.

The coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots$ and $z_0, z_1, z_2, \ldots$ are called the A-sequence and the Z-sequence of the Riordan array $M = (g(x), f(x))$, respectively. Letting $A(x)$ be the generating function of the A-sequence and $Z(x)$ be the generating function of the Z-sequence, we have

$$A(x) = \frac{x}{\bar{f}(x)}, \quad Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{d_{0,0}}{g(\bar{f}(x))}\right). \tag{1}$$

Example 3. We can deduce from the last result that the A-sequence and the Z-sequence of the inverse Riordan array

$$(g(x), f(x))^{-1}$$

have generating functions $\frac{x}{f(x)}$ and $\frac{1}{f(x)}(1 - g(x))$, respectively.

Thus in the case of the inverse matrix

$$M^{(m)}(a,b)^{-1} = \left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^m}\right)^{-1},$$

we see that the corresponding generating functions are given by

$$A(x) = \frac{(1-ax)^m}{1+bx}, \qquad Z(x) = \frac{-a(1-ax)^{m-1}}{1+bx}.$$

Many interesting examples of sequences and Riordan arrays can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences (OEIS), [29, 30]. Sequences are frequently referred to by their OEIS number. For instance, the binomial matrix B ("Pascal's triangle") is A007318.

Given a sequence a_n, we denote by h_n the general term of the sequence with $h_n = |a_{i+j}|_{0 \le i,j \le n}$. The sequence h_n is called the Hankel transform of a_n [24, 25]. A well known example of Hankel transform is that of the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$, where we find that $h_n = 1$ for all n. Hankel determinants occur naturally in many branches of mathematics, from combinatorics [10] to number theory [26] and to mathematical physics [33].

There are a number of known ways of calculating Hankel transforms of sequences [12, 16, 23, 24, 27, 33]. One involves the theory of orthogonal polynomials, whereby we seek to represent the sequence under study as moments of a density function. Standard techniques of orthogonal polynomials then allow us to compute the desired Hankel transforms [6, 17]. These techniques are based on the following results (the first is the well-known "Favard's Theorem"), which we essentially reproduce from [24].

Theorem 4. *[24] (cf. [34, Théorème* 9*, p. I-4]; [35, Theorem 50.1]). Let $(p_n(x))_{n\ge 0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree $n = 0, 1, \ldots$. Then the sequence $(p_n(x))$ is (formally) orthogonal if and only if there exist sequences $(\alpha_n)_{n\ge 0}$ and $(\beta_n)_{n\ge 1}$ with $\beta_n \neq 0$ for all $n \ge 1$, such that the three-term recurrence*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad \textit{for} \quad n \ge 1,$$

holds, with initial conditions $p_0(x) = 1$ and $p_1(x) = x - \alpha_0$.

Theorem 5. *[24] (cf. [34, Proposition 1, (7), p. V-5]; [35, Theorem 51.1]). Let* $(p_n(x))_{n\geq 0}$ *be a sequence of monic polynomials, which is orthogonal with respect to some functional L. Let*

$$p_{n+1} = (x-\alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad \textit{for} \quad n \geq 1,$$

be the corresponding three-term recurrence which is guaranteed by Favard's theorem. Then the generating function

$$g(x) = \sum_{k=0}^{\infty} \mu_k x^k$$

for the moments $\mu_k = L(x^k)$ *satisfies*

$$g(x) = \cfrac{\mu_0}{1-\alpha_0 x - \cfrac{\beta_1 x^2}{1-\alpha_1 x - \cfrac{\beta_2 x^2}{1-\alpha_2 x - \cfrac{\beta_3 x^2}{1-\alpha_3 x - \cdots}}}}.$$

The Hankel transform of μ_n, which is the sequence with general term $h_n = |\mu_{i+j}|_{0\leq i,j\leq n}$, is then given by

$$h_n = \mu_0^{n+1}\beta_1^n \beta_2^{n-1}\cdots\beta_{n-1}^2\beta_n.$$

Other methods of proving Hankel transform evaluations include lattice path methods (the Lindström-Gessel-Viennot theorem) [1, 32] and Dodgson condensation (Desnanot-Jacobi adjoint matrix theorem) [1, 9, 18].

3 Central elements

We have studied the matrices $M^{(1)}(a,b)$ previously [2]. These are Pascal-like matrices (i.e., they are lower-triangular and have $T_{n,k} = T_{n,n-k}$ where $T_{n,k}$ is the n,k-th term, along with $T_{n,0} = T_{n,n} = 1$ for all n) and their central coefficients (the $(2n,n)$-elements) are of particular interest. In this section we propose to study the central elements of the general $M^{(m)}(a,b)$ matrices. A first task is to find an expression for the general (n,k)-th element of $M^{(m)}(a,b)$.

The general term $T_{n,k}^{(m)}$ of the Riordan array $M^{(m)}(a,b) = \left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^m}\right)$ can be calculated as follows.

$$\begin{aligned} T_{n,k}^{(m)}(a,b) &= [x^n](1+bx)^k x^k (1-ax)^{-(mk+1)} \\ &= [x^{n-k}] \sum_{j=0}^{k} \binom{k}{j} b^j x^j \sum_{i=0}^{\infty} \binom{-(mk+1)}{i} (-1)^i a^i x^i \\ &= [x^{n-k}] \sum_j \binom{k}{j} b^j x^j \sum_i \binom{mk+i}{i} a^i x^i \end{aligned}$$

$$= \sum_{j=0}^{k} \binom{k}{j} \binom{mk+n-k-j}{n-k-j} b^j a^{n-k-j}$$

$$= \sum_{j=0}^{k} \binom{k}{j} \binom{n+(m-1)k-j}{n-k-j} b^j a^{n-k-j}.$$

Note that by interchanging $k-j$ and j in the last summation we get

$$T_{n,k}^{(m)}(a,b) = \sum_{j=0}^{k} \binom{k}{j} \binom{n+(m-2)k+j}{n-2k+j} b^{k-j} a^{n-2k+j}.$$

Alternatively, we have

$$T_{n,k}^{(m)}(a,b) = \sum_{j=0}^{k} \binom{k}{j} \binom{n+(m-2)k+j}{mk} b^{k-j} a^{n-2k+j}.$$

We restate these results as a proposition.

Proposition 6. *The general term $T_{n,k}^{(m)}(a,b)$ of the generalized Pascal matrix*

$$M^{(m)}(a,b) = \left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^m} \right)$$

is given by

$$\begin{aligned} T_{n,k}^{(m)}(a,b) &= \sum_{j=0}^{k} \binom{k}{j} \binom{n+(m-1)k-j}{n-k-j} b^j a^{n-k-j} = \sum_{j=0}^{k} \binom{k}{j} \binom{n+(m-2)k+j}{n-2k+j} b^{k-j} a^{n-2k+j} \\ &= \sum_{j=0}^{k} \binom{k}{j} \binom{n+(m-2)k+j}{mk} b^{k-j} a^{n-2k+j}. \end{aligned}$$

Example 7. We have $M^{(2)}(1,-1) = B$. This implies that

$$\binom{n}{k} = \sum_{j=0}^{k} \binom{k}{j} \binom{n+j}{2k} (-1)^{k-j}$$

from which we can deduce that

$$\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j} \binom{2n+j}{j} (-1)^{n-j}.$$

We are now in a position to calculate an expression for the central terms $T^{(m)}_{2n,n}(a,b)$. We have

$$\begin{aligned} T^{(m)}_{2n,n}(a,b) &= \sum_{j=0}^{n} \binom{n}{j}\binom{2n+(m-1)n-j}{2n-n-j} b^j a^{2n-n-j} \\ &= \sum_{j=0}^{n} \binom{n}{j}\binom{(m+1)n-j}{n-j} b^j a^{n-j}. \end{aligned}$$

Exchanging $n-j$ for j in the above summation we also get

$$T^{(m)}_{2n,n}(a,b) = \sum_{j=0}^{n} \binom{n}{j}\binom{nm+j}{j} a^j b^{n-j}.$$

Note that when $b=0$, only the term with $n=j$ survives in the above sum. Thus in the case $b=0$, we find that the central terms of $\left(\frac{1}{1-ax}, \frac{x}{1-ax}\right)$ are given by

$$T^{(m)}_{2n,n}(a,0) = \binom{(m+1)n}{n} a^n.$$

We gather these results in the following proposition.

Proposition 8. *The central terms of the Riordan array* $\left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^m}\right)$ *are given by*

$$T^{(m)}_{2n,n}(a,b) = \sum_{j=0}^{n} \binom{n}{j}\binom{(m+1)n-j}{n-j} b^j a^{n-j} = \sum_{j=0}^{n} \binom{n}{j}\binom{nm+j}{j} a^j b^{n-j}.$$

For the case $b=0$, *the central terms of* $\left(\frac{1}{1-ax}, \frac{x}{1-ax}\right)$ *are given by*

$$T^{(m)}_{2n,n}(a,0) = \binom{(m+1)n}{n} a^n.$$

Example 9. We take the case of $b=-a$. Then we have

$$M^{(m)} = \left(\frac{1}{1-ax}, \frac{x(1-ax)}{(1-ax)^m}\right) = \left(\frac{1}{1-ax}, \frac{x}{(1-ax)^{m-1}}\right).$$

The proposition then implies that

$$\binom{mn}{n} a^n = \sum_{j=0}^{n} \binom{n}{j}\binom{mn+j}{j} a^j (-a)^{n-j}.$$

We conclude that

$$\binom{mn}{n} = \sum_{j=0}^{n} \binom{n}{j}\binom{mn+j}{j} (-1)^{n-j}. \tag{2}$$

We have the following alternative characterization of $T^{(m)}_{2n,n}(a,b)$.

Proposition 10.

$$T^{(m)}_{2n,n}(a,b) = [x^n]((1+ax)^{m+1}+bx(1+ax)^m)^n = [x^n](1+ax)^{mn}(1+(a+b)x)^n.$$

Proof. Let

$$S_n = [x^n]((1+ax)^{m+1}+bx(1+ax)^m)^n.$$

Then

$$\begin{aligned} S_n &= [x^n]\sum_{j=0}^n \binom{n}{j}(1+ax)^{(m+1)j}(bx)^{n-j}(1+ax)^{m(n-j)} \\ &= [x^n]\sum_{j=0}^n \binom{n}{j}(1+ax)^{mn+j}(bx)^{n-j} \\ &= [x^n]\sum_{j=0}^n \binom{n}{j}\sum_i \binom{mn+j}{i} a^i x^i b^{n-j} x^j \\ &= \sum_{j=0}^n \binom{n}{j}\binom{mn+j}{j} a^j b^{n-j} \\ &= T^{(m)}_{2n,n}(a,b). \end{aligned}$$

□

We next define a polynomial $P_n(x;a,b)$ of degree n as follows.

$$P_n(x;a,b) = (1+ax)^n + bx(1+ax)^{n-1}.$$

Proposition 11. *Let*

$$P_n(x;a,b) = \sum_{k=0} p_{n,k}x^{n-k}.$$

Then the coefficient array $(p_{n,k})_{0\le n,k\le\infty}$ *coincides with the Riordan array*

$$\left(\frac{1+bx}{1-ax}, \frac{x}{1-ax}\right).$$

Proof. We let $\tilde{p}_{n,k}$ denote the (n,k)-th element of $\left(\frac{1+bx}{1-ax}, \frac{x}{1-ax}\right)$. Then

$$\begin{aligned}
\tilde{p}_{n,k} &= [x^n](1+bx)x^k(1-ax)^{-(k+1)} \\
&= [x^{n-k}](1+bx)\sum_{j=0}^{\infty}\binom{-(k+1)}{j}(-1)^j a^j x^j \\
&= [x^{n-k}](1+bx)\sum_{j=0}\binom{k+j}{j}a^j x^j \\
&= [x^{n-k}]\sum_j \binom{k+j}{j}a^j x^j + b[x^{n-k-1}]\sum_j \binom{k+j}{j}a^j x^j \\
&= \binom{k+n-k}{n-k}a^{n-k} + b\binom{k+n-k-1}{n-k-1}a^{n-k-1} \\
&= \binom{n}{k}a^{n-k} + b\binom{n-1}{n-k-1}a^{n-k-1}.
\end{aligned}$$

We thus have

$$\sum_{k=0}^{n}\tilde{p}_{n,k}x^{n-k} = (1+ax)^n + bx(1+ax)^{n-1}.$$

□

We note that the decomposition

$$\left(\frac{1+bx}{1-ax}, \frac{x}{1-ax}\right) = \left(\frac{1}{1-ax}, \frac{x}{1-ax}\right) + b\left(\frac{x}{1-ax}, \frac{x}{1-ax}\right)$$

shows again that

$$\tilde{p}_{n,k} = \binom{n}{k}a^{n-k} + b\binom{n-1}{n-k-1}a^{n-k-1}.$$

Corollary 12. *Let $T^{(m)}_{2n,n}(a,b)$ be the central elements of the Riordan array $\left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^m}\right)$ and let $P_n(x;a,b)$ be the family of polynomials with coefficient array $\left(\frac{1+bx}{1-ax}, \frac{x}{1-ax}\right)$. Then*

$$T^{(m)}_{2n,n}(a,b) = [x^n](P_{m+1}(x;a,b))^n.$$

Thus the central terms $T^{(m)}_{2n,n}(a,b)$ are defined by the elements of the $(m+1)$-th row of the Riordan array $\left(\frac{1+bx}{1-ax}, \frac{x}{1-ax}\right)$.

Example 13. The $(2,1)$-Pascal triangle $\left(\frac{1+x}{1-x}, \frac{x}{1-x}\right)$ A029653 begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 5 & 4 & 1 & 0 & 0 & \cdots \\ 2 & 7 & 9 & 5 & 1 & 0 & \cdots \\ 2 & 9 & 16 & 14 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We then have, since $a = b = 1$ in this case,

$$\begin{aligned} T^{(2)}_{2n,n}(1,1) &= \sum_{k=0}^{n} \binom{n}{k}\binom{2n+k}{k} = [x^n](1+4x+5x^2+2x^3)^n, \\ T^{(3)}_{2n,n}(1,1) &= \sum_{k=0}^{n} \binom{n}{k}\binom{3n+k}{k} = [x^n](1+5x+9x^2+7x^3+2x^4)^n, \\ T^{(4)}_{2n,n}(1,1) &= \sum_{k=0}^{n} \binom{n}{k}\binom{4n+k}{k} = [x^n](1+6x+14x^2+16x^3+9x^4+2x^5)^n, \end{aligned}$$

for instance (A114496, A156886, and A156887, respectively).

Example 14. We have

$$T^{(2)}_{2n,n}(-1,1) = [x^n](1-x)^{2n} = (-1)^n\binom{2n}{n}.$$

In general, we have

$$T^{(m)}_{2n,n}(a,-a) = [x^n](1+ax)^{mn} = \binom{mn}{n}a^n.$$

4 The matrices $M^{(2)}(a,b)$, moments and orthogonal polynomials

In this section, we look at some properties of $M^{(2)}(a,b)$ and of the inverse Riordan array

$$M^{(2)}(a,b)^{-1} = \left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^2}\right)^{-1}.$$

The presence of the term $(1-ax)^2$ in the denominator is an indicator that links to orthogonal polynomials may be possible [4], and indeed, we show that this is the case. We will in particular study the first column elements and the row sum elements of the inverse matrix, and show that they constitute moments for appropriate families of orthogonal polynomials. We indicate the form of the defining densities on the real number line that define these families. We take advantage of these links to orthogonal polynomials (and the corresponding continued faction expressions for the generating functions) to calculate the Hankel transforms of these moment sequences.

To begin this study, we have

$$\left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^2}\right)^{-1} = \left(\frac{a\sqrt{1+4x(a+b)}-a-2b}{2(a^2x-b)}, \frac{1+2ax-\sqrt{1+4x(a+b)}}{2(a^2x-b)}\right).$$

If we write this inverse matrix as $M^{(2)}(a,b)^{-1} = (g(x), f(x))$, then

$$g(x) = \sum_{k=0}^{\infty} g_n x^n = \frac{a\sqrt{1+4x(a+b)}-a-2b}{2(a^2x-b)}$$

is the g.f. of the first column sequence g_n.

Proposition 15. *The Hankel transform of the first column elements of the inverse array is given by*

$$h_n = a^n (a+b)^{n^2}.$$

Proof. We consider the continued fraction [35]

$$\tilde{g}(x) = \cfrac{1}{1+ax-\cfrac{a(a+b)x^2}{1+2(a+b)x-\cfrac{(a+b)^2x^2}{1+2(a+b)x-\cfrac{(a+b)^2x^2}{1-\cdots}}}}. \tag{3}$$

We then have

$$\tilde{g}(x) = \frac{1}{1+ax-a(a+b)x^2u}$$

where u satisfies the equation

$$u = \frac{1}{1+2(a+b)x-(a+b)^2x^2u}.$$

Solving for u and substituting into $\tilde{g}(x)$, we find that

$$\tilde{g}(x) = \frac{a\sqrt{1+4x(a+b)}-a-2b}{2(a^2x-b)} = g(x),$$

the g.f. of the first column elements. It follows that

$$h_n = \frac{a^n}{(a+b)^n}(a+b)^{2\binom{n+1}{2}} = a^n(a+b)^{n^2}.$$

$\square$

Proposition 16. *We have the following expression for $g(x)$, the g.f. of the first column elements of $M^{(2)}(a,b)^{-1}$.*

$$g(x) = \frac{1}{1+axc(-(a+b)x)} = \cfrac{1}{1+\cfrac{ax}{1+\cfrac{(a+b)x}{1+\cfrac{(a+b)x}{1+\cdots}}}},$$

where

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}$$

is the g.f for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof. The continued fraction is an equivalent form of (3). Substituting the expression for $c(x)$ into $\frac{1}{1+axc(-(a+b)x)}$ and simplifying shows that this is indeed $g(x)$. We can recover the continued fraction expression again by noting that

$$c(x) = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cdots}}}.$$

□

Corollary 17. *We have*

$$M^{(2)}(a,b)^{-1} = \left(\frac{1}{1+axc(-(a+b)x)}, \frac{xc(-(a+b)x)}{1+axc(-(a+b)x)}\right).$$

We can express the first column elements g_n as moments in the following way:

$$g_n = \frac{1}{2\pi}\int_{-4(a+b)}^{0} x^n \frac{a\sqrt{-x(x+4(a+b))}}{x(bx-a^2)}\,dx,$$

where we have taken the case $a+b>0$.

Example 18. We let $a=2$, $b=3$. Then the sequence g_n has g.f. $\frac{\sqrt{1+20x}-4}{4x-3}$ and begins

$$1, -2, 14, -148, 1886, -26652, \ldots.$$

In this case we have

$$g_n = \frac{1}{\pi}\int_{-20}^{0} x^n \frac{\sqrt{-x(x+20)}}{x(3x-4)}\,dx.$$

The first column elements g_n of the inverse array can be expressed as follows.

$$g_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{n-k}\frac{2k+1}{n+k+1}(a+b)^{n-k}b^k.$$

An alternative expression is

$$g_n=\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{2n}{j}\binom{j}{k}(-1)^j\frac{n-2j+1}{2n-j+1}a^kb^{n-k}.$$

The first column elements begin

$$1,-a,a(2a+b),-a(5a^2+6ab+2b^2),a(14a^3+28a^2b+20ab^2+5b^3),\ldots.$$

We can represent this as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & b & 2 & 0 & 0 & 0 & \cdots \\ 0 & -2b^2 & -6b & -5 & 0 & 0 & \cdots \\ 0 & 5b^3 & 20b^2 & 28b & 14 & 0 & \cdots \\ 0 & -14b^4 & -70b^3 & -135b^2 & -120b & -42 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} 1 \\ a \\ a^2 \\ a^3 \\ a^4 \\ a^5 \\ \vdots \end{pmatrix}.$$

Here, the infinite matrix has bivariate generating function

$$\cfrac{1}{1+\cfrac{xy}{1+\cfrac{bx+xy}{1+\cfrac{bx+xy}{1+\cdots}}}},$$

or in other words, the matrix is given by the Deleham construct [3]

$$[0,-b,-b,-b,\ldots]\,\Delta\,[-1,-1,-1,\ldots].$$

See A094385 and A157491.

Proposition 19. *The Hankel transform h_n^* of the shifted sequence g_{n+1} is given by*

$$h_n^*=(-a)^{n+1}(a+b)^{n(n+1)}.$$

Proof. The generating function of g_{n+1} is given by

$$\frac{g(x)-1}{x} = \frac{-ac(-(a+b)x)}{1+axc(-(a+b)x)}.$$

By the theory of Hankel transforms, g_{n+1} then has the same Hankel transform as

$$[x^n](-ac(-(a+b)x))$$

(this is an example of invariance under the INVERT transform [25]). The result follows from this. □

We now turn our attention to the row sum elements s_n of the inverse array.

Proposition 20. *The Hankel transform of the row sums sequence of the inverse Riordan array* $\left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^2}\right)^{-1}$ *is given by*

$$h_n = (a-1)^n (a+b)^{n^2}.$$

Proof. The generating function of the row sum sequence of the inverse array is given by $\frac{g(x)}{1-f(x)}$, where the inverse array is expressed as $(g(x), f(x))$. We find that this g.f. is given by

$$s(x) = \frac{1+a+2b-(a-1)\sqrt{1+4x(a+b)}}{2(1+b-(a-1)^2 x)}.$$

But this can be expressed as the following continued fraction.

$$s(x) = \cfrac{1}{1+(a-1)x - \cfrac{(a-1)(a+b)x^2}{1+2(a+b)x - \cfrac{(a+b)^2x^2}{1+2(a+b)x - \cfrac{(a+b)^2x^2}{1-\cdots}}}}. \tag{4}$$

Thus we obtain

$$h_n = (a-1)^n (a+b)^{n^2}.$$

□

We note that the generating function $s(x)$ of the row sum sequence can be written as

$$s(x) = \frac{1}{1+(a-1)xc(-(a+b)x)} = (1, xc(-(a+b)x)) \cdot \frac{1}{1-(1-a)x}.$$

We now recall that the general element of the Riordan array $(1, xc(\lambda x))$ is given by

$$\binom{2n-k-1}{n-k} \frac{k+0^{n+k}}{n+0^{nk}} \lambda^{n-k}.$$

Thus we obtain the following expression for the row sum elements.

$$s_n = \sum_{k=0}^{n} \binom{2n-k-1}{n-k} \frac{k+0^{n+k}}{n+0^{nk}} (-a-b)^{n-k}(1-a)^k.$$

We can express the row sum elements s_n as moments as follows

$$s_n = \frac{1}{2\pi} \int_{-4(a+b)}^{0} x^n \frac{(a-1)\sqrt{-x(x+4(a+b))}}{x(x(1+b)-(a-1)^2)}\, dx,$$

where we take the case $a+b>0$.

Example 21. We let $a=b=-1$. Then $s_n = 2^n C_n$ and we obtain

$$s_n = \frac{1}{\pi} \int_0^8 x^n \frac{\sqrt{x(8-x)}}{4x}\, dx.$$

As with the first column elements, we have the following alternative continued fraction form of the generating function $s(x)$.

Proposition 22. *We have the following expression for $s(x)$, the g.f. of the row sum elements of $M^{(2)}(a,b)^{-1}$.*

$$s(x) = \frac{1}{1+(a-1)xc(-(a+b)x)} = \cfrac{1}{1+\cfrac{(a-1)x}{1+\cfrac{(a+b)x}{1+\cfrac{(a+b)x}{1+\cdots}}}},$$

where

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}$$

is the g.f for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proposition 23. *The Hankel transform h_n^* of the shifted sequence s_{n+1} is given by*

$$h_n^* = (1-a)^{n+1}(a+b)^{n(n+1)}.$$

Proof. The generating function of s_{n+1} is given by

$$\frac{s(x)-1}{x} = \frac{-(a-1)c(-(a+b)x)}{1+(a-1)xc(-(a+b)x)}.$$

By the theory of Hankel transforms, s_{n+1} then has the same Hankel transform as

$$[x^n](-(a-1)c(-(a+b)x)) = [x^n](1-a)c(-(a+b)x)$$

(this is an example of invariance under the INVERT transform [25]). The result follows from this. □

When $b=0$, the production array of the inverse matrix is given by

$$\left(\begin{array}{ccccccc} -a & 1 & 0 & 0 & 0 & 0 & \cdots \\ a^2 & -2a & 1 & 0 & 0 & 0 & \cdots \\ 0 & a^2 & -2a & 1 & 0 & 0 & \cdots \\ 0 & 0 & a^2 & -2a & 1 & 0 & \cdots \\ 0 & 0 & 0 & a^2 & -2a & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This corresponds to the original matrix, $\left(\frac{1}{1-ax}, \frac{x}{(1-ax)^2}\right)$, being the coefficient array of the family of orthogonal polynomials $Q_n(x)$ defined by

$$Q_n(x) = (x+2a)Q_{n-1}(x) - a^2 Q_{n-2}(x), \quad Q_0(x) = 1, Q_1(x) = x + a.$$

The equation (3) tells us that the first column elements of the inverse matrix $M^{(2)}(a,b)^{-1}$ are the moments for a family of orthogonal polynomials. The production matrix of the inverse of the coefficient array of those polynomials is then given by

$$\left(\begin{array}{ccccccc} -a & 1 & 0 & 0 & 0 & 0 & \cdots \\ a(a+b) & -2(a+b) & 1 & 0 & 0 & 0 & \cdots \\ 0 & (a+b)^2 & -2(a+b) & 1 & 0 & 0 & \cdots \\ 0 & 0 & (a+b)^2 & -2(a+b) & 1 & 0 & \cdots \\ 0 & 0 & 0 & (a+b)^2 & -2(a+b) & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The form of this production matrix implies that the matrix of coefficients of the sought-for orthogonal polynomials is given by

$$\left(\frac{1-bx}{1-(a+b)x}, \frac{x}{(1-(a+b)x)^2}\right).$$

This gives us

Proposition 24. *The first column elements of the inverse Riordan array*

$$\left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^2}\right)^{-1}$$

are the moments of the family of orthogonal polynomials whose coefficient array is given by the Riordan array

$$\left(\frac{1-bx}{1-(a+b)x}, \frac{x}{(1-(a+b)x)^2}\right).$$

Corollary 25. *The family of orthogonal polynomials $R_n(x)$ for which the first column elements g_n of $\left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^2}\right)^{-1}$ are the moments is defined by*

$$R_n(x) = (x + 2(a + b))R_{n-1}(x) - (a + b)^2 R_{n-2}(x), \quad R_0(x) = 1, R_1(x) = x + a,$$

with coefficient array $\left(\frac{1-bx}{1-(a+b)x}, \frac{x}{(1-(a+b)x)^2}\right)$.

Using equation (4) in a similar way, we have the following result.

Proposition 26. *The row sum elements of the inverse Riordan array*

$$\left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^2}\right)^{-1}$$

are the moments of the family of orthogonal polynomials whose coefficient array is given by the Riordan array

$$\left(\frac{1-(b+1)x}{1-(a+b)x}, \frac{x}{(1-(a+b)x)^2}\right).$$

Corollary 27. *The family of orthogonal polynomials $S_n(x)$ for which the row sum elements s_n of $\left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^2}\right)^{-1}$ are the moments is defined by*

$$S_n(x) = (x + 2(a + b))S_{n-1}(x) - (a + b)^2 S_{n-2}(x), \quad S_0(x) = 1, S_1(x) = x + a - 1,$$

with coefficient array $\left(\frac{1-(b+1)x}{1-(a+b)x}, \frac{x}{(1-(a+b)x)^2}\right)$.

We close this section with the following observation. The Hankel transform of the first column elements of the matrix product

$$M^{(2)}(a,b)^{-1} \cdot M^{(1)}(\tilde{a}, \tilde{b})$$

is given by

$$h_n = (a - \tilde{a})^n (a + b)^{n^2}.$$

The generating function of the first column elements of the matrix product is then given by

$$\cfrac{1}{1 + (a-\tilde{a})x - \cfrac{(a-\tilde{a})(a+b)x^2}{1 + 2(a+b)x - \cfrac{(a+b)^2x^2}{1 + 2(a+b)x - \cfrac{(a+b)^2x^2}{1 - \cdots}}}}.$$

An equivalent and simpler form is

$$\cfrac{1}{1 + \cfrac{(a-\tilde{a})x}{1 + \cfrac{(a+b)x}{1 + \cfrac{(a+b)x}{1 + \cdots}}}}.$$

5 Images of $M^{(2)}(a,b)$

In this section we look at the image of $M^{(2)}(a,b)$ under the c-fold binomial transformation. Thus we let

$$B_c = \left(\frac{1}{1-cx}, \frac{x}{1-cx}\right).$$

Then

$$B_c = B^c$$

where $B = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ is the standard binomial array. In particular $B_1 = B$.

Proposition 28. *The image of $M^{(2)}(a,b)$ under the c-fold binomial transformation B_c is $M^{(2)}(a+c, b-c)$:*

$$B_c \cdot M^{(2)}(a,b) = M^{(2)}(a+c, b-c). \tag{5}$$

Proof. We have

$$\begin{aligned}
\left(\frac{1}{1-cx}, \frac{x}{1-cx}\right) \cdot \left(\frac{1}{1-ax}, \frac{x(1+bx)}{(1-ax)^2}\right) &= \left(\frac{1}{1-cx}\frac{1}{1-\frac{ax}{1-cx}}, \frac{\frac{x}{1-cx}(1+b\frac{x}{1-cx})}{(1-a\frac{x}{1-cx})^2}\right)\\
&= \left(\frac{1}{1-(a+c)x}, \frac{x(1+(b-c)x)}{(1-(a+c)x)^2}\right).
\end{aligned}$$

$\square$

Example 29. The image of the Riordan array $\left(\frac{1}{1-x}, \frac{x(1+(b-1)x)}{1-x}\right)$ under the inverse binomial transformation is the Riordan array $(1, x(1+bx))$. This latter matrix has inverse $(1, xc(-bx))$. Thus we have

$$(1, xc(-bx)) \cdot B^{-1} = \left(\frac{1}{1-x}, \frac{x(1+(b-1)x)}{1-x}\right)^{-1},$$

that is,

$$(1, xc(-bx)) \cdot B^{-1} = M^{(2)}(-1, b-1)^{-1}.$$

Example 30. We have

$$B^{-1} \cdot M^{(2)}(a,-1) = M^{(2)}(a-1, 0) = \left(\frac{1}{1-(a-1)x}, \frac{x}{(1-(a-1)x)^2}\right).$$

Thus the image of $M^{(2)}(a,-1)$ under B^{-1} is the coefficient array of the family of orthogonal polynomials $P_n(x;a)$ defined by

$$P_n(x;a) = (x - 2(1-a))P_{n-1}(x;a) - (a-1)^2 P_{n-2}(x;a), \quad P_0(x;a) = 1, P_1(x;a) = x + a - 1.$$

More generally, we see that

$$B_b \cdot M^{(2)}(a,b) = M^{(2)}(a+b, 0),$$

where $M^{(2)}(a+b,0)$ is the coefficient array of the family of orthogonal polynomials $P_n(x)$ defined by

$$P_n(x;a) = (x+2(a+b))P_{n-1}(x;a) - (a+b)^2 P_{n-2}(x;a), \quad P_0(x;a) = 1, P_1(x;a) = x+a+b.$$

Equivalently

$$M^{(2)}(a,b) = B_b^{-1} M^{(2)}(a+b,0) = B^{-b} M^{(2)}(a+b,0).$$

We finish this section by remarking that the property (5) can be generalized. For instance, we define

$$\tilde{M}^{(3)}(a,b,c) = \left(\frac{1}{1-ax}, \frac{x(1+bx+cx^2)}{(1-ax)^3}\right).$$

We can then show that

$$B_d \tilde{M}^{(3)}(a,b,c) = \tilde{M}^{(3)}(a+d, b-2d, c-bd+d^2).$$

Similarly, we get

$$B_e \tilde{M}^{(4)}(a,b,c,d) = \tilde{M}^{(4)}(a+e, b-3e, -c+2be-3e^2, d-ce+be^2-e^3).$$

6 A Christoffel-Darboux conjecture

We can regard the matrix $M^{(m)}(a,b)$ as the coefficient array of a family of polynomials $P_n(x;m,a,b)$. We have

$$P_n(x;m,a,b) = \sum_{k=0}^{n} T_{n,k}^{(m)}(a,b) x^k.$$

For each n, the Christoffel-Darboux quotient

$$\frac{P_{n+1}(x;m,a,b)P_n(y;m,a,b) - P_n(x;m,a,b)P_{n+1}(y;m,a,b)}{x-y}$$

is the bi-variate generator of a $(n+1)\times(n+1)$ matrix $D_n(m,a,b)$. We then have the following conjecture.

Conjecture 31. We have

$$|D_n(2,a,b)| = a^{\binom{n+1}{2}} (a+b)^n (a+2b)^{\binom{n}{2}}.$$

It is interesting to compare the matrices defined by the quotient above with those defined by

$$\sum_{j=0}^{n} P_j(x;m,a,b) P_j(y;m,a,b).$$

In the case of monic orthogonal polynomials, such expressions lead to the same matrices. However, $M^{(m)}(a,b)$ is not in general the coefficient array of a family of monic orthogonal polynomials, and hence the two family of matrices will in general differ. In the case $m=2$, $a=b=1$, we can conjecture the following.

Conjecture 32. The matrix $\tilde{D}_n(2,1,1)$ with generating function

$$\sum_{k=0}^{n} P_j(x;2,1,1)P_j(y;2,1,1)$$

is given by

$$M_n^{(2)}(a,b)^t M_n^{(2)}(a,b).$$

We then have

$$|\tilde{D}_n(2,1,1)| = 1 \quad \text{for } n \geq 0.$$

In the above the notation M_n denotes the matrix formed from the first $(n+1)$ rows and columns of M.

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