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Extremal Orders of Certain Functions Associated with Regular Integers (mod n)

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Abstract

Let $V(n)$ denote the number of positive regular integers (mod n) that are $\leq n$, and let $V_k(n)$ be a multidimensional generalization of the arithmetic function $V(n)$. We find the Dirichlet series of $V_k(n)$ and give the extremal orders of some totients involving arithmetic functions which generalize the sum-of-divisors function and the Dedekind function. We also give the extremal orders of other totients regarding arithmetic functions which generalize the sum of the unitary divisors of n and the unitary function corresponding to $\phi(n)$, the Euler function. Finally, we study extremal orders of some compositions, involving the functions mentioned previously.

1 Introduction

Let $n > 1$ be an integer. An integer a is called regular (mod n) if there exists an integer x such that $a^2x \equiv a \pmod{n}$ (sequence A143869 in Sloane's *Encyclopedia of Integer Sequences*). Several authors investigated properties of regular integers (mod n). Alkam and Osba [1], using ring-theoretic considerations, rediscovered some of the statements proved by Morgado [6, 7], who showed that $a > 1$ is regular (mod n) if and only if $\gcd(a, n)$ is a unitary divisor of n. Tóth [15] gave direct proofs of some properties, because the proofs of [6, 7] were lengthy and those of [1] were ring-theoretical.

Let $\mathrm{Reg}_n = \{a : 1 \le a \le n \text{ and } a \text{ is regular (mod } n)\}$, and $V(n) = \#\mathrm{Reg}_n$. The function V is multiplicative and $V(p^\alpha) = \phi(p^\alpha) + 1 = p^\alpha - p^{\alpha-1} + 1$, where ϕ is the Euler function. Consequently, $V(n) = \sum_{d \| n} \phi(d)$, for every $n \ge 1$, where $d \| n$ means that d is a unitary divisor of n, that is, $d \mid n$ and $\gcd(d, \frac{n}{d}) = 1$. Also $\phi(n) < V(n) \le n$, for every $n > 1$, and $V(n) = n$ if and only if n is a squarefree; see [7, 15, 1]. Thus, the function $V(n)$ is an analogue of the Euler function $\phi(n)$. The function $\phi(n)$ is the sequence A000010 in Sloane's On-Line Encyclopedia of Integer Sequences. Also, the function $V(n)$ is the sequence A055653; see [12].

Apostol and Tóth [4] considered the multidimensional generalization of the function $V(n)$, $V_k(n)$, where $k \ge 1$ is a fixed integer. The function $V_k(n)$ is multiplicative and $V_k(p^\alpha) = \phi_k(p^\alpha) + 1 = p^{\alpha k} - p^{(\alpha-1)k} + 1$, where ϕ_k is the Jordan function of order k. Consequently, $V_k(n) = \sum_{d \| n} \phi_k(d)$, for every $n \ge 1$. Also $\phi_k(n) < V_k(n) \le n^k$, for every $n > 1$ and $V_k(n) = n^k$ if and only if n is squarefree; see [4].

Tóth [15] proved results concerning the minimal and maximal orders of the functions $V(n)$ and $V(n)/\phi(n)$. Alkam and Osba [1] investigated the minimal order of $V(n)$. Sándor and Tóth [10] and Apostol [2] studied the extremal orders of compositions of certain functions.

In Section 2 we present some notation and results involving arithmetical functions. Section 3 is devoted to the study of the Dirichlet series of $V_k(n)$. Extremal orders of the function $V_k(n)$ in connection with $\sigma_k(n)$ and $\psi_k(n)$ are given in Section 4.

In Section 5 we prove some results regarding $V_k(n)$ and unitary analogues of the functions $\sigma_k(n)$ and $\phi_k(n)$.

In Section 6 we give the extremal orders of some compositions of functions from above.

Section 7 provides other limits of compositions of arithmetical functions. We also present some open problems regarding extremal orders of these compositions.

2 Preliminaries

In what follows let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} > 1$ be a positive integer and let $k \ge 1$ be an integer. Throughout the paper we will use the following notation:

- $p_1, p_2, \ldots$ — the sequence of the primes;
- $\sigma_k(n)$ — the generalization of $\sigma(n)$, defined by $\sigma_k(n) = \prod_{i=1}^r \frac{p_i^{(\alpha_i+1)k} - 1}{p_i^k - 1}$;

- $\psi_k(n)$ — the generalization of $\psi(n)$, defined by $\psi_k(n) = n^k \prod_{p|n}(1 + \frac{1}{p^k})$;

- $\zeta(s)$ — the Riemann zeta function, $\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$, $s = \sigma + it \in \mathbb{C}$ and $\sigma > 1$;

- $\phi(n)$ — the Euler function, $\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$;

- $\phi_k(n)$ — the Jordan function of order k, $\phi_k(n) = n^k \prod_{p|n}(1 - \frac{1}{p^k})$;

- γ — the Euler constant, $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n)$;

- $\phi^*(n)$ — the unitary analogue of $\phi(n)$, $\phi^*(n) = \prod_{i=1}^{k}(p_i^{\alpha_i} - 1)$;

- $\sigma^*(n)$ — the unitary analogue of $\sigma(n)$, $\sigma^*(n) = \prod_{i=1}^{k}(p_i^{\alpha_i} + 1)$.

For other arithmetic functions defined by regular integers modulo n we refer to the papers [5, 14].

Let $f(n)$ be a nonnegative real-valued multiplicative arithmetic function. Let

$$L = L(f) := \limsup_{n\to\infty} \frac{f(n)}{\log\log n}$$

and

$$\rho(p) = \rho(p, f) := \sup_{\alpha \geq 0} f(p^\alpha)$$

for primes p, and consider the product $R = R(f) := \prod_p (1 - \frac{1}{p})\rho(p)$.

In order to prove the properties below we apply the following results:

Lemma 1. (Tóth and Wirsing [16, Corollary 1]). *If f is a nonnegative real-valued multiplicative arithmetic function such that for each prime p,*

(i) $\rho(p) = \sup_{\alpha\geq 0}(f(p^\alpha)) \leq \left(1 - \frac{1}{p}\right)^{-1}$, *and*

(ii) there is an exponent $e_p = p^{o(1)} \in \mathbb{N}$ *satisfying* $f(p^{e_p}) \geq 1 + \frac{1}{p}$,

then $\limsup_{n\to\infty} \frac{f(n)}{\log\log n} = e^\gamma \prod_p \left(1 - \frac{1}{p}\right)\rho(p)$.

Lemma 2. (Tóth and Wirsing [16, Theorem 1]). *Suppose that $\rho(p) < \infty$ for all primes p and the product R converges unconditionally (i.e., irrespectively of order), improper limits being allowed, then $L \leq e^\gamma R$.*

Lemma 3. (Tóth and Wirsing [16, Theorem 3]). *Suppose that $\rho(p) < \infty$ for all primes p, that for each prime p there is an exponent $e_p = p^{o(1)} \in \mathbb{N}$ such that $\prod_p f(p^{e_p})\rho(p)^{-1} > 0$ and that the product R converges, improper limits being allowed. Then $L \geq e^\gamma R$.*

3 Dirichlet series of $V_k(n)$

Apostol and Petrescu [3] studied the Dirichlet series of $V_1(n) := V(n)$. In what follows we give the Dirichlet series of $V_k(n)$ for $k \geq 2$ and some results involving the Möbius function.

Proposition 4. *For every $s = \sigma + it \in \mathbb{C}$ with $\sigma > k+1$,*

$$\sum_{n\geq 1} \frac{V_k(n)}{n^s} = \zeta(s-k)\zeta(s)\prod_p \left(1 - \frac{p^{2s-k} + p^s - p^{s-k}}{p^{3s-k}}\right).$$

Proof. Let $f(n) = \frac{V_k(n)}{n^s}$. We have

$$\sum_{n\geq 1} |f(n)| \leq \sum_{n\geq 1} \frac{1}{n^{\sigma-k}} < \infty,$$

so the series $\sum_{n\geq 1} \frac{V_k(n)}{n^s}$ converges absolutely for $\sigma > k+1$. Since V_k is multiplicative,

$$\sum_{n\geq 1} \frac{V_k(n)}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^{s-k}}} \cdot \prod_p \frac{1}{1 - \frac{1}{p^s}} \cdot \prod_p \left(1 - \frac{p^{2s-k} + p^s - p^{s-k}}{p^{3s-k}}\right),$$

and the claim follows. □

Corollary 5. *Let $s = \sigma + it \in \mathbb{C}$, $\sigma > k+1$. Then*

$$\sum_{n\geq 1} \frac{\mu(n)V_k(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^{s-k}}\right) = \frac{1}{\zeta(s-k)}.$$

Also,

$$\sum_{n\geq 1} \frac{|\mu(n)|V_k(n)}{n^s} = \prod_p \left(1 + \frac{1}{p^{s-k}}\right) = \frac{\zeta(s-k)}{\zeta(2s-2k)}.$$

Proof. For$f(n) = \frac{\mu(n)V_k(n)}{n^s}$ the series $\sum_{n\geq 1}|f(n)|$ converges absolutely, so

$$\sum_{n\geq 1} \frac{\mu(n)V_k(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^{s-k}}\right) = \frac{1}{\zeta(s-k)}.$$

For the second assertion take $f(n) = \frac{|\mu(n)|V_k(n)}{n^s}$. □

4 Extremal orders concerning classical generalized arithmetic functions

For the quotient $\frac{\sigma_k(n)}{V_k(n)}$, we notice that $\frac{\sigma_k(n)}{V_k(n)} \geq 1$ for every $n \geq 1$. Since

$$\lim_{\substack{p\to\infty \\ p \text{ prime}}} \frac{\sigma_k(p)}{V_k(p)} = \lim_{\substack{p\to\infty \\ p \text{ prime}}} \frac{p^k+1}{p^k} = 1,$$

we get

$$\liminf_{n\to\infty} \frac{\sigma_k(n)}{V_k(n)} = 1;$$

hence the minimal order of $\frac{\sigma_k(n)}{V_k(n)}$ is 1. Now consider the quotient

$$\frac{\psi_k(n)}{V_k(n)}.$$

Since

$$\frac{\psi_k(n)}{V_k(n)} \geq 1$$

for every $n \geq 1$ and

$$\frac{\psi_k(p)}{V_k(p)} = \frac{p^k+1}{p^k}$$

for every prime p, it is immediate that

$$\liminf_{n\to\infty} \frac{\psi_k(n)}{V_k(n)} = 1.$$

Thus, the minimal order of $\frac{\psi_k(n)}{V_k(n)}$ is 1. It is known that

$$\limsup_{n\to\infty} \frac{\sigma(n)}{V(n)(\log\log n)^2} = e^{2\gamma}$$

and

$$\limsup_{n\to\infty} \frac{\psi(n)}{V(n)(\log\log n)^2} = \frac{6}{\pi^2}e^{2\gamma};$$

see [2]. Proposition 6 shows that the maximal order of $\frac{\sigma_k(n)}{V_k(n)}$ and $\frac{\psi_k(n)}{V_k(n)}$ is $\frac{6}{\pi^2}e^{2\gamma}(\log\log n)^2$.

Proposition 6. *For $k \geq 2$,*

$$\limsup_{n\to\infty} \frac{\sigma_k(n)}{V_k(n)(\log\log n)^2} = \limsup_{n\to\infty} \frac{\psi_k(n)}{V_k(n)(\log\log n)^2} = \frac{6}{\pi^2}e^{2\gamma}.$$

Proof. Take $f(n) = \sqrt{\frac{\sigma_k(n)}{V_k(n)}}$. Then

$$f(p^\alpha) = \sqrt{\frac{p^{(\alpha+1)k} - 1}{(p^k - 1)(p^{\alpha k} - p^{(\alpha-1)k} + 1)}} \leq \sqrt{\frac{p+1}{p-1}} = \rho(p) < \left(1 - \frac{1}{p}\right)^{-1}$$

and

$$f(p^2) = \sqrt{\frac{p^{3k} - 1}{(p^k - 1)(p^{2k} - p^k + 1)}} \geq 1 + \frac{1}{p}$$

for every prime p, so (ii) in Lemma 1 is satisfied. We obtain

$$\limsup_{n\to\infty} \frac{\sqrt{\sigma_k(n)}}{\sqrt{V_k(n)} \log\log n} = \prod_p \sqrt{1 - \frac{1}{p^2}} e^\gamma = \sqrt{\frac{6}{\pi^2}} e^\gamma,$$

so

$$\limsup_{n\to\infty} \frac{\sigma_k(n)}{V_k(n)(\log\log n)^2} = \frac{6}{\pi^2} e^{2\gamma}.$$

Since $\psi_k(n) \leq \sigma_k(n)$ and for the primes p we have $\psi_k(p) = \sigma_k(p) = p^k + 1$, the result for $\frac{\psi_k(n)}{V_k(n)(\log\log n)^2}$ follows from the previous one. □

5 Extremal orders concerning unitary analogues of σ_k and ϕ_k

In what follows we consider the functions $\sigma_k^*(n)$ and $\phi_k^*(n)$, representing the generalizations for the sum of the unitary divisors of n and the unitary analogue Euler function, respectively. Let $k \geq 1$ be a fixed integer. We have $\sigma_k^*(n) = \sum_{d \| n} d^k$ and $\sigma_k^*(p^\alpha) = p^{\alpha k} + 1$. Also,

$$\phi_k^*(n) := \sum_{\substack{\gcd(a_1,\ldots,a_k) \in \{1,2,\ldots,n\}^k \\ \gcd(\gcd(a_1,a_2,\ldots,a_k),n)_* = 1}} 1 = \sum_{d \| n} d^k \mu^*(\frac{n}{d}),$$

and hence $\phi_k^*(p^\alpha) = p^{\alpha k} - 1$. Note that

$$\gcd(a,b)_* = \max\{d : d | a, d \parallel b\}$$

and $\mu^*(n)$ is the unitary analogue of the Möbius function, given by $\mu^*(n) = (-1)^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of n. The functions $\sigma_k^*(n)$ and $\phi_k^*(n)$ are multiplicative. Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime factorisation of $n > 1$. We obtain

$$\phi_k^*(n) = (p_1^{\alpha_1 k} - 1) \cdots (p_r^{\alpha_r k} - 1) \quad \text{and} \quad \sigma_k^*(n) = (p_1^{\alpha_1 k} + 1) \cdots (p_r^{\alpha_r k} + 1).$$

Observe that $\sigma_k^*(n) = \sigma_k(n)$ and $\phi_k^*(n) = \phi_k(n)$ for all squarefree n. Furthermore, for every $n \geq 1$,

$$\phi_k(n) \leq \phi_k^*(n) \leq n^k \leq \sigma_k^*(n) \leq \sigma_k(n).$$

Recall that an integer $n > 1$ is called powerful if it is divisible by the square of each of its prime factors. A powerful integer is also called a squarefull integer. We give extremal orders for the quotients $\frac{\sigma_k^*(n)}{V_k(n)}$ and $\frac{\phi_k^*(n)}{V_k(n)}$, the minimal order of $\frac{\phi_k^*(n)}{V_k(n)}$ being studied for powerful numbers. Since $\frac{\sigma_k^*(n)}{V_k(n)} \geq 1$ for every $n \geq 1$ and $\lim_{p\to\infty} \frac{\sigma_k^*(p)}{V_k(p)} = \lim_{p\to\infty} \frac{p^k+1}{p^k} = 1$ for prime numbers p, it follows that $\liminf_{n\to\infty} \frac{\sigma_k^*(n)}{V_k(n)} = 1$.

If n is powerful, it is easy to see that $\frac{\phi_k^*(n)}{V_k(n)} \geq 1$, taking into account that $\frac{\phi_k^*(p^\alpha)}{V_k(p^\alpha)} \geq 1$ for $\alpha \geq 2$. For prime numbers p, we notice that

$$\lim_{p\to\infty} \frac{\phi_k^*(p^2)}{V_k(p^2)} = \lim_{p\to\infty} \frac{p^{2k}-1}{p^{2k}-p^k+1} = 1,$$

which implies that

$$\liminf_{n\to\infty} \frac{\phi_k^*(n)}{V_k(n)} = 1,$$

so the minimal order of $\frac{\phi_k^*(n)}{V_k(n)}$ is 1.

For the maximal orders of these quotients we have

Proposition 7. *For $k \geq 1$,*

$$\limsup_{n\to\infty} \frac{\sigma_k^*(n)}{V_k(n)\log\log n} = \limsup_{n\to\infty} \frac{\phi_k^*(n)}{V_k(n)\log\log n} = e^\gamma.$$

Proof. Take $f(n) = \frac{\sigma_k^*(n)}{V_k(n)}$ in Lemma 2, which is a nonnegative real-valued multiplicative arithmetic function. We have

$$f(p^\alpha) = \frac{p^{\alpha k}+1}{p^{\alpha k}-p^{(\alpha-1)k}+1} \leq \left(1-\frac{1}{p}\right)^{-1} = \rho(p) < \infty$$

and $R = 1$, so

$$\limsup_{n\to\infty} \frac{\sigma_k^*(n)}{V_k(n)\log\log n} \leq e^\gamma.$$

Now let $g(n) = \frac{\phi_k^*(n)}{V_k(n)}$. Here

$$g(p^\alpha) = \frac{p^{\alpha k}-1}{p^{\alpha k}-p^{(\alpha-1)k}+1} \leq \left(1-\frac{1}{p}\right)^{-1} = \rho(p),$$

and

$$R = \prod_p g(p^1)(\rho(p))^{-1} = \prod_p (p^k+1)\cdot\frac{p-1}{p} > 0.$$

Hence, by Lemma 3 we have

$$\limsup_{n\to\infty} \frac{\phi_k^*(n)}{V_k(n)\log\log n} \geq e^\gamma.$$

It is obvious that $\phi_k^*(n) \leq \sigma_k^*(n)$ for every $n \geq 1$. We obtain

$$e^\gamma \leq \limsup_{n\to\infty} \frac{\phi_k^*(n)}{V_k(n)\log\log n} \leq \limsup_{n\to\infty} \frac{\sigma_k^*(n)}{V_k(n)\log\log n} \leq e^\gamma,$$

which shows that

$$\limsup_{n\to\infty} \frac{\sigma_k^*(n)}{V_k(n)\log\log n} = \limsup_{n\to\infty} \frac{\phi_k^*(n)}{V_k(n)\log\log n} = e^\gamma,$$

as desired. $\square$

Corollary 8. *The maximal order of both* $\frac{\sigma_k^*(n)}{V_k(n)}$ *and* $\frac{\phi_k^*(n)}{V_k(n)}$ *is* $e^\gamma \log\log n$.

6 Extremal orders regarding compositions of arithmetical functions

We now move to the study of extremal orders of some composite arithmetic functions. We start with $V_k(V_k(n))$ and $\phi_k(V_k(n))$.

We know that $V_k(n) \leq n^k$ for every $n \geq 1$, so

$$\frac{V_k(V_k(n))}{n^{k^2}} \leq \frac{(V_k(n))^k}{n^{k^2}} \leq \frac{(n^k)^k}{n^{k^2}} = 1$$

and

$$\lim_{\substack{p\to\infty \\ p \text{ prime}}} \frac{V_k(V_k(p))}{p^{k^2}} = \lim_{\substack{p\to\infty \\ p \text{ prime}}} \frac{V_k(p^k)}{p^{k^2}} = \lim_{\substack{p\to\infty \\ p \text{ prime}}} \frac{p^{k^2} - p^{(k-1)k} + 1}{p^{k^2}} = 1,$$

so the maximal order of $V_k(V_k(n))$ is n^{k^2}. Since $\phi_k(n) \leq n^k$ and $V_k(n) \leq n^k$ for any $n \geq 1$, we have $\frac{\phi_k(V_k(n))}{n^{k^2}} \leq \frac{(V_k(n))^k}{n^{k^2}} \leq 1$. But $\lim_{\substack{p\to\infty \\ p \text{ prime}}} \frac{\phi_k(V_k(p))}{p^{k^2}} = \lim_{p\to\infty} \frac{p^{k^2} - p^{(k-1)k}}{p^{k^2}} = 1$, so the maximal order of $\phi_k(V_k(n))$ is n^{k^2}.

The maximal order of $V(\phi(n))$ was investigated in [2]. Using the general idea of that proof, we show

Proposition 9. *The maximal order of* $V_k(\phi_k(n))$ *is* n^{k^2}.

Proof. We will use Linnik's theorem which states that if $\gcd(t, \ell) = 1$, then there exists a prime p such that $p \equiv \ell \pmod{t}$ and $p \ll t^c$, where c is a constant (one can take $c \leq 11$).

Let $A = \prod_{\substack{k<p\le x \\ p \text{ prime}}} p$. Since $\gcd(A^2, A+1) = 1$, by Linnik's theorem there is a prime number q such that $q \equiv A+1 \pmod{A^2}$ and $q \ll (A^2)^c = A^{2c}$, where c satisfies $c \le 11$. Also, $q^k \equiv kA+1 \pmod{A^2}$. Let q be the least prime satisfying the above condition. We have $\phi_k(q) = q^k - 1 = AB$, where $B = k + sA$, for some s. Thus $\gcd(A, B) = 1$, so B is free of prime factors $\le x$ and $> k$. Since $V_k(n)$ is multiplicative, we have

$$\frac{V_k(\phi_k(q))}{q^{k^2}} = \frac{V_k(AB)}{(AB+1)^k} = \frac{V_k(A)}{A^k} \cdot \frac{V_k(B)}{B^k} \cdot \frac{(AB)^k}{(AB+1)^k}. \tag{1}$$

Here $\frac{(AB)^k}{(AB+1)^k} \to 1$ as $x \to \infty$, so it is sufficient to study $\frac{V_k(A)}{A^k}$ and $\frac{V_k(B)}{B^k}$. Clearly,

$$\frac{V_k(A)}{A^k} = 1. \tag{2}$$

We have $A = \prod_{k<p\le x} p \le \prod_{p \le x} p = e^{O(x)}$. Since $B \ll A^{10}$ we obtain $B \ll e^{O(x)}$, so

$$\log B \ll x. \tag{3}$$

If $B = \prod_{i=1}^r q_i^{b_i}$ is the prime factorization of B, we obtain, taking into account that $k \ge 1$ is a fixed integer, that $\log B = \sum_{i=1}^r b_i \log q_i > (\log x) \sum_{i=1}^r b_i$ for sufficiently large x. But $\sum_{i=1}^r b_i \ge r$, so $\log B > k \log x$, implying that $r < \frac{\log B}{\log x} \ll \frac{x}{\log x}$ (by (3)). Since

$$\frac{V_k(B)}{B^k} > \prod_{i=1}^r \left(1 - \frac{1}{q_i^k}\right) \ge \prod_{i=1}^r \left(1 - \frac{1}{q_i}\right) > \left(1 - \frac{1}{x}\right)^r \ge \left(1 - \frac{1}{x}\right)^{O(\frac{x}{\log x})},$$

We obtain

$$\frac{V_k(B)}{B^k} > 1 + O\left(\frac{1}{\log x}\right). \tag{4}$$

By (1), (2), (4) and $\frac{(AB)^k}{(AB+1)^k} \to 1$ as $x \to \infty$, we obtain

$$\frac{V_k(\phi_k(q))}{q^{k^2}} > 1 + O\left(\frac{1}{\log x}\right). \tag{5}$$

By relation (5), and since $\frac{V_k(\phi_k(n))}{n^{k^2}} \le \frac{(\phi_k(n))^k}{n^{k^2}} \le 1$, the claim follows. $\square$

The maximal order of $V(\phi^*(n))$ is n (see [2]). For the maximal order of $V_k(\phi^*(n))$ we give

Proposition 10.

$$\limsup_{n\to\infty} \frac{V_k(\phi^*(n))}{n^k} = 1.$$

Proof. We apply the following lemma:

If a is an integer, $a > 1$, p is a prime number and $f(n)$ is an arithmetical function satisfying $\phi(n) \le f(n) \le \sigma(n)$, one has

$$\lim_{p\to\infty} \frac{f(N(a,p))}{N(a,p)} = 1, \tag{6}$$

where $N(a,p) = \frac{a^p-1}{a-1}$ (see, e.g., Suryanarayana [13]).

Since $\phi^*(n) \le n$, it follows that $V_k(\phi^*(n)) \le (\phi^*(n))^k \le n^k$, so

$$\frac{\sqrt[k]{V_k(\phi^*(n))}}{n} \le 1. \tag{7}$$

Obviously, $\sqrt[k]{V_k(n)}$ meets the conditions of the lemma. We have

$$\lim_{\substack{p\to\infty\\ p \text{ prime}}} \frac{\sqrt[k]{V_k(\phi^*(2^p))}}{2^p} = \lim_{\substack{p\to\infty\\ p \text{ prime}}} \frac{\sqrt[k]{V_k(2^p-1)}}{2^p-1} = \lim_{\substack{p\to\infty\\ p \text{ prime}}} \frac{\sqrt[k]{V_k(N(2,p))}}{N(2,p)} = 1. \tag{8}$$

Now (7) and (8) imply $\limsup_{n\to\infty} \frac{\sqrt[k]{V_k(\phi^*(n))}}{n} = 1$, and we are done. □

Apostol [2] proved that

$$\limsup_{n\to\infty} \frac{\sigma(\phi^*(n))}{V(\phi^*(n))(\log\log n)^2} = \limsup_{n\to\infty} \frac{\sigma(\phi^*(n))}{V(\phi^*(n))(\log\log \phi^*(n))^2} = e^{2\gamma}$$

and

$$\limsup_{n\to\infty} \frac{\psi(\phi^*(n))}{V(\phi^*(n))(\log\log n)^2} \limsup_{n\to\infty} \frac{\psi(\phi^*(n))}{V(\phi^*(n))(\log\log \phi^*(n))^2} = \frac{6}{\pi^2}e^{2\gamma}.$$

The maximal orders of $\frac{\sigma_k(\phi^*(n))}{V_k(\phi^*(n))}$ and $\frac{\psi_k(\phi^*(n))}{V_k(\phi^*(n))}$ are given by

Proposition 11. *For $k \ge 2$ we have*

(i) $\limsup_{n\to\infty} \frac{\sigma_k(\phi^*(n))}{V_k(\phi^*(n))(\log\log n)^2} = \limsup_{n\to\infty} \frac{\sigma_k(\phi^*(n))}{V_k(\phi^*(n))(\log\log \phi^*(n))^2} = \frac{6}{\pi^2}e^{2\gamma}$,

(ii) $\limsup_{n\to\infty} \frac{\psi_k(\phi^*(n))}{V_k(\phi^*(n))(\log\log n)^2} = \limsup_{n\to\infty} \frac{\psi_k(\phi^*(n))}{V_k(\phi^*(n))(\log\log \phi^*(n))^2} = \frac{6}{\pi^2}e^{2\gamma}$.

Proof. (i) Let

$$l_1 := \limsup_{n\to\infty} \frac{\sigma_k(\phi^*(n))}{V_k(\phi^*(n))(\log\log n)^2} \text{ and } l_2 := \limsup_{n\to\infty} \frac{\sigma_k(\phi^*(n))}{V_k(\phi^*(n))(\log\log \phi^*(n))^2}.$$

Since $\phi^*(n) \le n$ for every $n \ge 1$, we have

$$\begin{aligned} l_1 &= \limsup_{n\to\infty} \frac{\sigma_k(\phi^*(n))}{V_k(\phi^*(n))(\log\log n)^2} \\ &\le l_2 = \limsup_{n\to\infty} \frac{\sigma_k(\phi^*(n))}{V_k(\phi^*(n))(\log\log \phi^*(n))^2} \\ &\le \limsup_{m\to\infty} \frac{\sigma_k(m)}{V_k(m)(\log\log m)^2} = \frac{6}{\pi^2}e^{2\gamma}, \end{aligned}$$

by Proposition 6. Since $\gcd(n,1)=1$, by Linnik's theorem, there exists a prime number p such that $p \equiv 1 \pmod n$ and $p \ll n^c$. Let p_n be the least prime such that $p_n \equiv 1 \pmod n$, for every n. Then $n \mid p_n - 1$ and $p_n \ll n^c$, so $\log\log p_n \sim \log\log n$.

Observe that $a \mid b$ implies $\frac{\sigma_k(a)}{V_k(a)} \leq \frac{\sigma_k(b)}{V_k(b)}$. If $p^\beta \mid p^\alpha$ ($\beta \leq \alpha$), it is easy to see that $\frac{\sigma_k(p^\beta)}{V_k(p^\beta)} \leq \frac{\sigma_k(p^\alpha)}{V_k(p^\alpha)}$. The general case follows, taking into account that $\frac{\sigma_k(n)}{V_k(n)}$ is multiplicative. So,

$$\frac{\sigma_k(\phi^*(p_n))}{V_k(\phi^*(p_n))(\log\log p_n)^2} = \frac{\sigma_k(p_n-1)}{V_k(p_n-1)(\log\log p_n)^2} \sim \frac{\sigma_k(p_n-1)}{V_k(p_n-1)(\log\log n)^2}.$$

On the other hand,

$$\frac{\sigma_k(p_n-1)}{V_k(p_n-1)(\log\log n)^2} \geq \frac{\sigma_k(n)}{V_k(n)(\log\log n)^2}.$$

Therefore,

$$\limsup_{n\to\infty} \frac{\sigma_k(\phi^*(n))}{V_k(\phi^*(n))(\log\log n)^2} \geq \limsup_{n\to\infty} \frac{\sigma_k(\phi^*(p_n))}{V_k(\phi^*(p_n))(\log\log p_n)^2}$$
$$\geq \limsup_{n\to\infty} \frac{\sigma_k(n)}{V_k(n)(\log\log n)^2} = \frac{6}{\pi^2}e^{2\gamma}.$$

We obtain $\frac{6}{\pi^2}e^{2\gamma} \leq l_1 \leq l_2 \leq \frac{6}{\pi^2}e^{2\gamma}$, and hence $l_1 = l_2 = \frac{6}{\pi^2}e^{2\gamma}$.

(ii) The proof is similar to the proof of (i), taking into account that $a \mid b$ implies $\frac{\psi_k(a)}{V_k(a)} \leq \frac{\psi_k(b)}{V_k(b)}$ and $\limsup_{n\to\infty} \frac{\psi_k(n)}{V_k(n)(\log\log n)^2} = \frac{6}{\pi^2}e^{2\gamma}$, by Proposition 6. $\square$

So, the maximal orders of both $\frac{\sigma_k(\phi^*(n))}{V_k(\phi^*(n))}$ and $\frac{\psi_k(\phi^*(n))}{V_k(\phi^*(n))}$ are $\frac{6}{\pi^2}e^{2\gamma}(\log\log n)^2$.

In a similar manner, since

$$\limsup_{n\to\infty} \frac{\sigma_k^*(n)}{V_k(n)\log\log n} = \limsup_{n\to\infty} \frac{\phi_k^*(n)}{V_k(n)\log\log n} = e^\gamma$$

(using Proposition 7), the fact that $a \mid b$ implies $\frac{\sigma_k^*(a)}{V_k(a)} \leq \frac{\sigma_k^*(b)}{V_k(b)}$ and $\frac{\phi_k^*(a)}{V_k(a)} \leq \frac{\phi_k^*(b)}{V_k(b)}$, respectively, it can be shown that

$$\limsup_{n\to\infty} \frac{\sigma_k^*(\phi^*(n))}{V_k(\phi^*(n))\log\log n} = \limsup_{n\to\infty} \frac{\sigma_k^*(\phi^*(n))}{V_k(\phi^*(n))\log\log \phi^*(n)} = e^\gamma$$

and

$$\limsup_{n\to\infty} \frac{\phi_k^*(\phi^*(n))}{V_k(\phi^*(n))\log\log n} = \limsup_{n\to\infty} \frac{\phi_k^*(\phi^*(n))}{V_k(\phi^*(n))\log\log \phi^*(n)} = e^\gamma.$$

7 Open Problems

Open Problem 12. Note that

$$\liminf_{n\to\infty} \frac{V_k(\phi(n))}{n^k} = \liminf_{n\to\infty} \frac{V_k(\phi^*(n))}{n^k} = \liminf_{n\to\infty} \frac{\phi_k^*(V(n))}{n^k} = 0.$$

For $n_k = p_1 \cdots p_r$ (the product of the first r primes), we have

$$\frac{V_k(\phi(n_r))}{n_r^k} = \frac{V_k((p_1-1)\cdots(p_r-1))}{p_1^k \cdots p_r^k} \le \frac{(p_1-1)^k \cdots (p_r-1)^k}{p_1^k \cdots p_r^k} = \left((1-\frac{1}{p_1})\cdots(1-\frac{1}{p_r})\right)^k,$$

so

$$\lim_{r\to\infty} \frac{V_k(\phi(n_r))}{n_r^k} = \lim_{r\to\infty} \left((1-\frac{1}{p_1})\cdots(1-\frac{1}{p_r})\right)^k = 0,$$

similarly the other relations. What are the minimal orders for the $V_k(\phi(n))$, $V_k(\phi^*(n))$, and $\phi_k^*(V(n))$?

Open Problem 13. Taking $n_r = p_1 \cdots p_r$ (the product of the first r primes),

$$\frac{\sigma_k^*(V(n_r))}{n_r^k} = \frac{\sigma_k^*(p_1\cdots p_r)}{p_1^k \cdots p_r^k} = \frac{(p_1^k+1)\cdots(p_r^k+1)}{p_1^k \cdots p_r^k} = \left((1+\frac{1}{p_1})\cdots(1+\frac{1}{p_r})\right)^k \to \infty$$

as $r \to \infty$, so $\limsup_{n\to\infty} \frac{\sigma_k^*(V(n))}{n^k} = \infty$. What is the maximal order for $\sigma_k^*(V(n))$?

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