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Dismal Arithmetic

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To the memory of Martin Gardner (October 21, 1914 – May 22, 2010).

Abstract

Dismal arithmetic is just like the arithmetic you learned in school, only simpler: there are no carries, when you add digits you just take the largest, and when you multiply digits you take the smallest. This paper studies basic number theory in this world, including analogues of the primes, number of divisors, sum of divisors, and the partition function.

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1 Introduction

To remedy the dismal state of arithmetic skills possessed by today's children, we propose a "dismal arithmetic" that will be easier to learn than the usual version. It is easier because there are no carry digits and there is no need to add or multiply digits, or to do anything harder than comparing. In dismal arithmetic, for each pair of digits,

to Add, take the lArger, but

to Multiply, take the sMaller.

That's it! For example: $2 \boldsymbol{+} 5 = 5$, $2 \boldsymbol{\times} 5 = 2$.

Addition or multiplication of larger numbers uses the same rules, always with the proviso that there are no carries. For example, the dismal sum of 169 and 248 is 269 and their dismal product is 12468 (Figure 1).

$$
\begin{array}{rccc}
 & 1 & 6 & 9 \\
\boldsymbol{+} & 2 & 4 & 8 \\
\hline
 & 2 & 6 & 9 \\
\hline
\end{array}
$$

Fig. 1(a) Dismal addition.

$$
\begin{array}{rccccc}
 & & & 1 & 6 & 9 \\
\boldsymbol{\times} & & & 2 & 4 & 8 \\
\hline
 & & & 1 & 6 & 8 \\
 & & 1 & 4 & 4 & \\
 & 1 & 2 & 2 & & \\
\hline
 & 1 & 2 & 4 & 6 & 8 \\
\hline
\end{array}
$$

Fig. 1(b) Dismal multiplication.

One might expect that nothing interesting could arise from such simple rules. However, developing the dismal analogue of ordinary elementary number theory will lead us to some surprisingly difficult questions.

Here are a few dismal analogues of standard sequences. The "even" numbers, $2 \boldsymbol{\times} n$, are

$$0,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 10,\ 11,\ 12,\ 12,\ 12,\ 12,\ 12,\ 12,\ 12,\ 12,\ 20,\ 21,\ 22,\ 22,\ \ldots \tag{1}$$

(entry A171818 in [17]). Note that $n \boldsymbol{+} n$ (which is simply n) is a different sequence. For another, less obvious, analogue of the even numbers, see (13) in §3. The squares, $n \boldsymbol{\times} n$, are

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 100,\ 111,\ 112,\ 113,\ 114,\ 115,\ 116,\ 117,\ 118,\ 119,\ 200,\ \ldots \tag{2}$$

(A087019),[2] the dismal triangular numbers, $0 \boldsymbol{+} 1 \boldsymbol{+} 2 \boldsymbol{+} \cdots \boldsymbol{+} n$, are

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 19,\ 19,\ 19,\ 19,\ 19,\ 19,\ 19,\ 19,\ 19,\ 19,\ 29,\ 29,\ 29,\ 29,\ 29,\ \ldots \tag{3}$$

(A087052), and the dismal factorials, $1 \boldsymbol{\times} 2 \boldsymbol{\times} \cdots \boldsymbol{\times} n$, $n \geq 1$, are

$$\begin{gathered}1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 10,\ 110,\ 1110,\ 11110,\ 111110,\ 1111110,\ 11111110,\ 111111110, \\ 1111111110,\ 11111111110,\ 111111111100,\ 1111111111100,\ 11111111111100,\ \ldots\end{gathered} \tag{4}$$

(A189788).

[2]See also the sums of two squares, A171120. The numbers $10, 11, \ldots, 99$ are not the sum of any number of squares, so there is no dismal analogue of the four-squares theorem.

A formal definition of dismal arithmetic is given in §2, valid for any base b, not just base 10, and it shown there that the commutative, associative, and distributive laws hold (Theorem 1). In that section we also introduce the notion of a "digit map," in order to study how changing individual digits in a dismal calculation affects the answer (Theorem 3, Corollary 4).

The dismal primes are the subject of §3. A necessary condition for a number to be a prime is that it contain a digit equal to $b-1$. The data suggest that if k is large, almost all numbers of length k containing $b-1$ as a digit and not ending with zero are prime, and so the number of primes of length k appears to approach $(b-1)^2\, b^{k-2}$ as $k \to \infty$ (Conjecture 10). In any case, any number with a digit equal to $b-1$ is a product of primes (Theorem 7), and *every* number can be written as r times a product of primes, for some $r \in \{0, 1, \ldots, b-1\}$ (Corollary 8). These factorizations are in general not unique. There is a useful process using digit maps for "promoting" a prime from a lower base to a higher base, which enables us to replace the list of all primes by a shorter list of prime "templates" (Table 3).

Dismal squares are briefly discussed in §4.

In §5 we investigate the different ways to order the dismal numbers, and in particular the partially ordered set defined by the divisibility relation (see Table 1). We will see that greatest common divisors and least common multiples need not exist, so this poset fails to be a lattice. On the other hand, we do have the notion of "relatively prime" and we can define an analogue of the Euler totient function.

In §6 we study the number-of-divisors function $d_b(n)$, and investigate which numbers have the most divisors. It appears that in any base $b \geq 3$, the number $n = (b^k - 1)/(b-1) = 111\ldots1|_b$ has more divisors than any other number of length k. The binary case is slightly different. Here it appears that among all k-digit numbers n, the maximal value of $d_2(n)$ occurs at $n = 2^k - 2 = 111\ldots10|_2$, and this is the unique maximum for $n \neq 2, 4$. Among all *odd* k-digit numbers n, $d_2(n)$ has a unique maximum at $n = 2^k - 1 = 111\ldots111|_2$, and if $k \geq 3$ and $k \neq 5$, the next largest value occurs at $n = 2^k - 3 = 111\ldots101|_2$, its reversal $2^k - 2^{k-2} - 1 = 101\ldots111|_2$, and possibly other values of n (see Conjectures 12-14). Although we cannot prove these conjectures, we are able to determine the exact values of $d_b(111\ldots111|_b)$ and $d_2(111\ldots101|_2)$ (Theorem 17, which extends earlier work of Richard Schroeppel and the second author, and Theorem 18).

The sequence of the number of divisors of $11\ldots11|_2$ (with k 1's) turns out to arise in a variety of different problems, involving compositions, trees, polyominoes, Dyck paths, etc.—see Remark (iii) following Theorem 16. The initial terms can be seen in Table 8. This sequence appears in two entries in [17], A007059 and A079500, and is the subject of a survey article by Frosini and Rinaldi [6]. The asymptotic behavior of this sequence was determined by Kemp [10] and by Knopfmacher and Robbins [12], the latter using the method of Mellin transforms—see (23). This is an example of an asymptotic expansion where the leading term has an oscillating component which, though small, does not go to zero. It is amusing to note that one of the first problems in which the asymptotic behavior was shown to involve a nonvanishing oscillating term was the analysis of the average number of *carries* when two k-digit numbers are added (Knuth [13], answering a question of von Neumann; see also Pippenger [18]). Here we see a similar phenomenon when there are no carries. In studying Conjectures 13 and 14, we observed that the numbers of divisors for the runners-up,

$2^k - 3$ and $2^k - 2^{k-2} - 1$, appeared to be converging to one-fifth of the number of divisors of $11\ldots11|_2$. This is proved in Theorem 23. Our proof is modeled on Knopfmacher and Robbins's proof [12] of (23), and we present the proof in such a way that it yields both results simultaneously.

The sum-of-divisors function $\sigma_b(n)$ is the subject of §7. There are analogues of the perfect numbers, although they seem not to be as interesting as in the classical case. Section 8 discusses the dismal analogue of the partition function. Theorems 27 and 28 give explicit formulas for the number of partitions of n into distinct parts.

This is the second of a series of articles dealing with various kinds of carryless arithmetic, and contains a report of our investigations into dismal arithmetic carried out during the period 2000–2011. This work had its origin in a study by the second author into the results of performing binary arithmetic calculations with the usual addition and multiplication of binary digits replaced by other operations. If addition and multiplication are replaced by the logical operations OR and AND, respectively, we get base 2 dismal arithmetic. (If instead we use XOR and AND, the results are very different, the squares for example now forming the Moser-de Bruijn sequence A000695.) Generalizing from base 2 to base 10 and then to an arbitrary base led to the present work.

In the first article in the series, [1], addition and multiplication were carried out "mod 10", with no carries. A planned third part will discuss even more exotic arithmetics.

Although dismal arithmetic superficially resembles "tropical mathematics" [20], where addition and multiplication are defined by $x \oplus y := \min\{x, y\}$, $x \odot y := x + y$, there is no real connection, since tropical mathematics is defined over $\mathbb{R} \cup \{\infty\}$, uses carries, and is not base-dependent.

Notation

The base will be denoted by b and the largest digit in a base b expansion by $\beta := b - 1$. We write $n = n_{k-1}n_{k-2}\ldots n_1n_0|_b$ to denote the base b representation of the number $\sum_{i=0}^{k-1} n_i b^i$, and we define $\mathrm{len}_b(n) := k$. The components n_i will be called the digits of n, even if $b \neq 10$. In the examples in this paper b will be at most 10, so the notation $n_{k-1}\ldots n_1n_0|_b$ (without commas or spaces) is unambiguous. The symbols $\boldsymbol{+}_b$ and $\boldsymbol{\times}_b$ denote dismal addition and multiplication, and we omit the base b if it is clear from the context. All non-bold operators ($+$, $\times$, $<$, etc.) refer to ordinary arithmetic operations, as do unqualified terms like "smallest," "largest," etc. We usually omit ordinary multiplication signs, but never dismal multiplication signs. We say that p divides n in base b (written $p \prec_b n$) if $p \boldsymbol{\times}_b q = n$ for some q, and that $p = p_{k-1}\ldots p_1p_0|_b$ is dominated by $n = n_{k-1}\ldots n_1n_0|_b$ (written $p \ll_b n$) if $p_i \leq n_i$ for all i. The symbol "$|_b$" always marks the end of a base b expansion of a number, and is never used for "divides in base b."

2 Basic definitions and properties

We began, as we all did, in base 10, but from now on we will allow the base b to be an arbitrary integer ≥ 2.

Let $\mathcal{A}$ denote the set of base b "digits" $\{0, 1, 2, \ldots, b-1\}$, equipped with the two binary operations

$$m \,\boldsymbol{+}_b\, n \ := \ \max\{m, n\}, \quad m \,\boldsymbol{\times}_b\, n \ := \ \min\{m, n\}, \quad \text{for } m, n \in \mathcal{A}\,. \tag{5}$$

A *dismal number* is an element of the semiring $\mathcal{A}[X]$ of polynomials $\sum_{i=0}^{k-1} n_i X^i$, $n_i \in \mathcal{A}$. If $M[X] := \sum_{i=0}^{k-1} m_i X^i$ and $N[X] := \sum_{i=0}^{l-1} n_i X^i$ are dismal numbers then their *dismal sum* is formed by taking the dismal sum of corresponding pairs of digits, analogously to ordinary addition of polynomials:

$$M[X] \,\boldsymbol{+}_b\, N[X] \ := \ \sum_{i=0}^{\max\{k,l\}-1} p_i X^i\,, \tag{6}$$

where $p_i := m_i \,\boldsymbol{+}_b\, n_i$, and their *dismal product* is similarly formed by using dismal arithmetic to convolve the digits, analogously to ordinary multiplication of polynomials:

$$M[X] \,\boldsymbol{\times}_b\, N[X] \ := \ \sum_{i=0}^{k+l-2} q_i X^i\,, \tag{7}$$

where

$$\begin{aligned}
q_0 &:= m_0 \,\boldsymbol{\times}_b\, n_0\,, \\
q_1 &:= (m_0 \,\boldsymbol{\times}_b\, n_1) \ \boldsymbol{+}_b \ (m_1 \,\boldsymbol{\times}_b\, n_0)\,, \\
q_2 &:= (m_0 \,\boldsymbol{\times}_b\, n_2) \ \boldsymbol{+}_b \ (m_1 \,\boldsymbol{\times}_b\, n_1) \ \boldsymbol{+}_b \ (m_2 \,\boldsymbol{\times}_b\, n_0)\,, \\
&\cdots
\end{aligned}$$

We will identify a dismal number $N(X) = \sum_{i=0}^{k-1} n_i X^i$ with the integer n whose base b expansion is $n = \sum_{i=0}^{k-1} n_i b^i$ (n is obtained by evaluating the polynomial $N(X)$ at $X = b$), and we define $\mathrm{len}_b(n) := k$. The rules (5)-(7) then translate into the rules for dismal addition and multiplication stated in §1: there are no carries, and digits are combined according to the rules in (5). The b-ary number $\sum_{i=0}^{k-1} n_i b^i$ will also be written as $n_{k-1} n_{k-2} \ldots n_1 n_0|_b$. Dismal numbers are, by definition, always identified with nonnegative integers. Note that $\mathrm{len}_b(m \,\boldsymbol{+}_b\, n) = \max\{\mathrm{len}_b(m), \mathrm{len}_b(n)\}$ and $\mathrm{len}_b(m \,\boldsymbol{\times}_b\, n) = \mathrm{len}_b(m) + \mathrm{len}_b(n) - 1$.

Theorem 1. *The dismal operations $\boldsymbol{+}_b$ and $\boldsymbol{\times}_b$ on $\mathcal{A}[X]$ satisfy the commutative and associative laws, and $\boldsymbol{\times}_b$ distributes over $\boldsymbol{+}_b$.*

Proof. (Sketch.) Each law requires us to show the identity of two polynomials, and so reduces to showing that certain identities hold for the coefficients of each individual degree in the two polynomials. These identities are assertions about min and max in the set $\mathcal{A}$, which hold since $(\mathcal{A}, \leq)$ is a totally ordered set, and $(\mathcal{A}, \min, \max)$ is a distributive lattice (cf. [9]). $\square$

If R denotes the operation of reversing the order of digits and m and n have the same length, then $R(m \,\boldsymbol{+}_b\, n) = R(m) \,\boldsymbol{+}_b\, R(n)$ and $R(m \,\boldsymbol{\times}_b\, n) = R(m) \,\boldsymbol{\times}_b\, R(n)$.

Individual digits in a dismal sum or product can often be varied without affecting the result, so dismal subtraction and division will not be defined. Example: $16 \,\boldsymbol{+}_{10}\, 75 = 26 \,\boldsymbol{+}_{10}\, 75 =$

76, $16 \mathbin{\pmb{\times}}_{10} 75 = 16 \mathbin{\pmb{\times}}_{10} 85 = 165$. This is why dismal numbers form only a semiring. On the other hand, this semiring does have a multiplicative identity (see the next section), and there are no zero divisors.

In certain situations we can give a more precise statement about how changing digits in a dismal sum or product affects the result. We begin with a lemma about ordinary functions of real variables.

Lemma 2. *Let f be a single-valued function of real variables $x_1, \ldots, x_k$, $k \geq 2$, formed by repeatedly composing the functions $(x, y) \mapsto \min\{x, y\}$ and $(x, y) \mapsto \max\{x, y\}$. If g is a nondecreasing function of x, meaning that*

$$x \leq y \quad \Rightarrow \quad g(x) \leq g(y)\,, \tag{8}$$

then

$$f(g(x_1), \ldots, g(x_k)) \;=\; g(f(x_1, \ldots, x_k))\,, \tag{9}$$

for all real $x_1, \ldots, x_k$.

We omit the easy inductive proof.

We define a base b *digit map* to be a nondecreasing function g mapping $\{0, 1, \ldots, b-1\}$ into itself. The map g need not be one-to-one or onto. If g is a digit map and $n = n_{k-1} \ldots n_1 n_0|_b$ then we set $g(n) := g(n_{k-1}) \ldots g(n_1) g(n_0)|_b$.

Theorem 3. *If m and n are dismal numbers and g is a base b digit map, then*

$$\begin{aligned} g(m \mathbin{\pmb{+}}_b n) \;&=\; g(m) \mathbin{\pmb{+}}_b g(n)\,, \\ g(m \mathbin{\pmb{\times}}_b n) \;&=\; g(m) \mathbin{\pmb{\times}}_b g(n)\,. \end{aligned} \tag{10}$$

Proof. This follows from Lemma 2, since the individual digits of $m \mathbin{\pmb{+}}_b n$ and $m \mathbin{\pmb{\times}}_b n$ are functions of the digits of m and n of the type considered in that lemma. □

Corollary 4. *If $p = m \mathbin{\pmb{\times}}_b n$ then p can also be written as $m' \mathbin{\pmb{\times}}_b n'$, where m' and n' use only digits that are digits of p.*

Proof. (Sketch.) Arrange all distinct digits occurring in p, m, and n in increasing order. Then construct a digit map g by increasing or decreasing the digits in m and n that are not in p until they coincide with digits of p, leaving the digits of p fixed. □

For example, consider the product $165 = 16 \mathbin{\pmb{\times}}_{10} 85$ mentioned above. The digits involved are $1, 5, 6, 8$, and the digit map described in the proof fixes 1, 5, and 6 and maps 8 to 6. The resulting factorization is $165 = 16 \mathbin{\pmb{\times}}_{10} 65$. (The additive analogue of Corollary 4 is true, but trivial.)

We will see other applications of Theorem 3 in the next section.

Note that when we are computing the base b dismal sum or product of two numbers p and q, once we have expressed p and q in base b, the value of b plays no further role in the calculation. Of course we need to know b when we convert the result back to an integer, but otherwise b is not used. So we have:

Lemma 5. *If the largest digit that is mentioned in a base b dismal sum or product is d (where $0 \le d \le b-1$), then the same calculation is valid in any base that exceeds d.*

For example, here is the calculation of the base 2 dismal product of $13 = 1101|_2$ and $5 = 101|_2$:

$$\begin{array}{rcccccc} & & & 1 & 1 & 0 & 1 \\ \boldsymbol{\times}_2 & & & & 1 & 0 & 1 \\ \hline & & & 1 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 1 & & \\ \hline & 1 & 1 & 1 & 1 & 0 & 1 \\ \hline \end{array}$$

This tells us that $13 \boldsymbol{\times}_2 5 = 61$, but the same tableau can be read in base 3, giving $37 \boldsymbol{\times}_3 10 = 361$, or in base 10, giving $1101 \boldsymbol{\times}_{10} 101 = 111101$.

Recall that we say that p *divides* n in base b (written $p \prec_b n$) if $p \boldsymbol{\times}_b q = n$ for some q. Since $\mathrm{len}_b(p) \le \mathrm{len}_b(n)$, nonzero numbers have only finitely many divisors. We also say that $p := p_{k-1} \dots p_1 p_0|_b$ is *dominated by* $n := n_{k-1} \dots n_1 n_0|_b$ (written $p \ll_b n$) if $p_i \le n_i$ for all i. Then $p \ll_b n$ if and only if $p \boldsymbol{+}_b n = n$. Another consequence of Lemma 2 is:

Lemma 6. *If $p \ll_b m$ and $q \ll_b n$ then $p \boldsymbol{+}_b q \ll_b m \boldsymbol{+}_b n$ and $p \boldsymbol{\times}_b q \ll_b m \boldsymbol{\times}_b n$.*

Finally, we remark without giving any details that, in any base, the sets of numbers with digits in nondecreasing order, or in nonincreasing order (see A009994 and A009996 for base 10) are closed under dismal addition and multiplication.

3 Dismal primes

In dismal arithmetic in base b, for bases $b > 2$, the multiplicative identity is no longer 1 (for example, $1 \boldsymbol{\times}_{10} 23 = 11$, not 23). In fact, it follows from the definition of multiplication that the multiplicative identity is the largest single-digit base b number, $\beta := b - 1$. For base $b = 10$ we have $\beta = 9$, and indeed the reader will easily check that $9 \boldsymbol{\times}_{10} n = n$ for all n. An empty dismal product is defined to be β, by convention.

If $p \boldsymbol{\times}_b q = \beta$, then $p = q = \beta$, so β is the only unit. We therefore define a *prime* in base b dismal arithmetic to be a number, different from β, whose only factorization is β times itself.

If p is prime, then at least one digit of p must equal β (for if the largest digit were $r < \beta$, then $p = r \boldsymbol{\times}_b p$, and r would be a divisor of p). The base b expansions of the first few primes are 1β (this is the smallest prime), $2\beta, 3\beta \dots \beta - 1\,\beta$, $\beta 0, \beta 1, \dots, \beta\beta, 10\beta, \dots$. In base 10, the primes are

$$\begin{aligned} &19,\ 29,\ 39,\ 49,\ 59,\ 69,\ 79,\ 89,\ 90,\ 91,\ 92,\ 93,\ 94,\ 95,\ 96,\ 97,\ 98,\ 99,\ 109,\ 209,\ 219,\\ &309,\ 319,\ 329,\ 409,\ 419,\ 429,\ 439,\ 509,\ 519,\ 529,\ 539,\ 549,\ 609,\ 619,\ 629,\ 639,\ \dots \end{aligned} \tag{11}$$

(A087097). Notice that the presence of a digit equal to β is a necessary but not sufficient condition for a number to be a prime: $11\beta|_b = 1\beta|_b \boldsymbol{\times}_b 1\beta|_b$ is not prime (see A087984 for these exceptions in the case $b = 10$). In base 2, the primes (written in base 2) are

$$10,\ 11,\ 101,\ 1001,\ 1011,\ 1101,\ 10001,\ 10011,\ 10111,\ 11001,\ 11101,\ 100001,\ \dots \tag{12}$$

(A171000). In view of the interpretation of base 2 dismal arithmetic in terms of Boolean operations mentioned in §1, the corresponding polynomials

$$X,\ X+1,\ X^2+1,\ X^3+1,\ X^3+X+1,\ \ X^3+X^2+1,\ X^4+1,\ X^4+X+1,\ \ldots,$$

together with 1, might be called the OR-irreducible Boolean polynomials. Their decimal equivalents,

$$1,\ 2,\ 3,\ 5,\ 9,\ 11,\ 13,\ 17,\ 19,\ 23,\ 25,\ 29,\ 33,\ 35,\ 37,\ \ldots$$

form sequence A067139 in [17], contributed by Jens Voß in 2002.

All numbers of the form $100\ldots00\beta|_b$ (with zero or more internal zeros) are base b primes, since there is no way that $p \times_b q$ can have the form $100\ldots00\beta|_b$ unless p or q is a single-digit number. So there are certainly infinitely many primes in any base.

Since $1\beta|_b$ is the smallest prime, the numbers $1\beta|_b \times_b n$ are another analogue of the even numbers. Whereas the first version of the even numbers, given in (1) for base 10, was simply "replace all digits in n that are bigger than 2 with 2's," this version is more interesting. In base 10 we get

$$0,\ 11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 17,\ 18,\ 19,\ 110,\ 111,\ 112,\ 113,\ 114,\ 115,\ 116,\ 117,\ 118,\ \ldots \tag{13}$$

(A162672, which contains repetitions and is not monotonic).

If $b > 2$, there are numbers which cannot be written as a product of primes (e.g., 1).

Theorem 7. *Any base b number with a digit equal to β is a (possibly empty) dismal product of dismal primes.*

Proof. The number β itself is the empty product of primes. Every two-digit number with β as a digit is already a prime. If there are more than two digits, either the number is a prime, or it factorizes into the product of two numbers, both of which must have β as a digit. The result follows by induction. □

Corollary 8. *Every base b number can be written as r times a dismal product of dismal primes, for some $r \in \{0, 1, 2, \ldots, b-1\}$.*

Proof. Let r be the largest digit of n. If $r = \beta$ the result follows from the theorem. Otherwise, let m be obtained by changing all occurrences of r in the b-ary expansion of n to β's, so that $n = r \times_b m$, and apply the theorem to m. □

Even when it exists, the factorization into a dismal product of dismal primes is in general not unique. In base 10, for example, the list of numbers with at least two different factorizations into a product of primes is

$$1119,\ 1129,\ 1139,\ 1149,\ 1159,\ 1169,\ 1179,\ 1189,\ 1191,\ 1192,\ 1193,\ 1194,\ 1195,\ \ldots, \tag{14}$$

where for instance $1119 = 19 \times_{10} 19 \times_{10} 19 = 19 \times_{10} 109$ (A171004).

We can, of course, study the primes dividing n, even if n does not contain a digit equal to β. Without giving any details, we mention that [17] contains the following sequences: the number of distinct prime divisors of n (A088469), their dismal sum (A088470), and dismal

k	base 2	base 10	k	base 2	k	base 2	k	base 2
1	0	0	11	323	21	442313	31	510471015
2	2	18	12	682	22	902921	32	1027067090
3	1	81	13	1424	23	1833029	33	2065390101
4	3	1539	14	2902	24	3719745	34	4151081457
5	5	20457	15	5956	25	7548521	35	8336751732
6	9	242217	16	12368	26	15264350	36	16734781946
7	19	2894799	17	25329	27	30859444	37	33583213577
8	39	33535839	18	51866	28	62355854	38	67357328359
9	77	381591711	19	106427	29	125773168	39	135056786787
10	168	?	20	217216	30	253461052	40	?

Table 1: Numbers of dismal primes with k digits in bases 2 and 10 (A169912, A087636).

product (A088471);[3] also the lists of numbers n such that the dismal sum of the distinct prime divisors of n is $< n$ (A088472), $\leq n$ (A088473), $\geq n$ (A088475), $> n$ (A088476); as well as the numbers n such that the dismal product of the distinct prime divisors of n is $< n$ (A088477), $\leq n$ (A088478), $= n$ (A088479), $\geq n$ (A088480), and $> n$ (A088481). There is no analogue of A088476 or A088481 in ordinary arithmetic.

One omission from the above list is explained by the following theorem.

Theorem 9. *In base b dismal arithmetic, n is prime if and only if the dismal sum of its distinct dismal prime divisors is equal to n.*

Proof. If $n = p$ is prime then the sum of the primes dividing it is p. Suppose n is not prime and let m be the sum of the distinct dismal primes diving n. If n is divisible by a prime p with $\mathrm{len}_b(p) = \mathrm{len}_b(n)$, then $n = r \times_b p$, $r < \beta$, the largest digit in n is r, and so $m \neq n$ since $n \ll_b p \ll_b m$. If $\mathrm{len}_b(p) < \mathrm{len}_b(n)$ for all prime divisors p, then $\mathrm{len}_b(m) < \mathrm{len}_b(n)$, and again $m \neq n$. ∎

We now consider how many primes there are. Let $\pi_b(k)$ denote the number of base b dismal primes with k digits. Table 1 shows the initial values of $\pi_2(k)$ and $\pi_{10}(k)$. Necessary conditions for a number n to be prime are that it contain β as a digit and (if $k > 2$) does not end with 0. There are

$$(b-1)^2\, b^{k-2} \ - \ (b-2)\,(b-1)^{k-2} \tag{15}$$

such numbers. It seems likely that, as k increases, almost all of these numbers will be prime, and the data in Table 1 is consistent with this. We therefore make the following conjecture.

Conjecture 10.

$$\pi_k(b) \ \sim \ (b-1)^2\, b^{k-2} \quad \text{as } k \to \infty\,. \tag{16}$$

[3] A088471 has an unusual beginning: $9, 9, 9, 9, 9, 9, 9, 9, 9, 90, 123456789987654321, 19, 19, 19, \ldots$.

We can get a lower bound on $\pi_b(k)$ by producing large numbers of primes, using the process of "promotion." We call a base b number with at least two digits a *pseudoprime* if its only factorizations are of the form $n = p \,\text{✖}_b\, q$ where at least one of p and q has length 1. In base b, n is a prime if and only if it is a pseudoprime and contains a digit β. If $n_{k-1}n_{k-2}...n_0|_b$ is a pseudoprime and r is its maximal digit, then $n_{k-1}n_{k-2}...n_0|_{r+1}$ is a base $r+1$ prime and furthermore $n_{k-1}n_{k-2}...n_0|_c$ is a pseudoprime in any base $c \geq r+1$. In base 2 there is no difference between primes and pseudoprimes. As long as we exclude numbers ending with 0, reversing the digits of a number does not change its status as a prime or pseudoprime.

The advantage of working with pseudoprimes rather than primes is that the inverse image of a pseudoprime under a digit map (see §2) is again a pseudoprime.

Theorem 11. *For a base b digit map g, if $g(n_{k-1}n_{k-2}...n_0|_b)$ is a pseudoprime and $g(n_{k-1})$ is not 0, then $n_{k-1}n_{k-2}...n_0|_b$ is a pseudoprime.*

Proof. This follows immediately from Theorem 3. □

So if p is a pseudoprime, then any number n with the property that there is a digit map sending n to p is also a pseudoprime; we think of n as being obtained by "promoting" p, and call p the "template" for n.

Here is an equivalent way to describe the promotion process. Suppose the distinct digits in p, the template, or number to be promoted, are $d_1 < d_2 < d_3 < \cdots$. For each d_i, choose a set of digits $S(d_i)$ such that all the digits in $S(d_i)$ are strictly less than all those in $S(d_j)$, for $i < j$. Replace any digit d_i in p by any digit in $S(d_i)$. All numbers n obtained in this way are promoted versions of p (the required digit map g being defined by $g(c) = d_i$ for all $c \in S(d_i)$).

Any pseudoprime (in any base) with at most four digits can be obtained by promoting a base 2 prime. At length 2, $11|_2$ is a prime (see (12)), so every 2-digit number $rs|_b$ is a pseudoprime, using the digit map g that sends r and s to 1. This is valid even if $s = 0$, since (8) still holds. At length 3, $101|_2$ is a base 2 prime, so any three-digit number $rst|_b$ with $r > s$, $t > s$ is a pseudoprime (take $g(r) = g(t) = 1, g(s) = 0$), and this captures all three-digit pseudoprimes. There are three templates of length 4, $1001|_2$, $1011|_2$, and $1101|_2$. These can be promoted to capture all four-digit primes, which are the numbers $rstu|_b$ for which one of the following holds:

$$
\begin{aligned}
&r \text{ and } u \text{ are both strictly greater than } s \text{ and } t\,,\\
&r, s, \text{ and } u \text{ are strictly greater than } t\,,\\
&r, t, \text{ and } u \text{ are strictly greater than } s\,.
\end{aligned}
$$

For lengths greater than four, we must use some nonbinary templates to capture all pseudoprimes, and as the length k increases so does the fraction of nonbinary templates required, as shown in Table 2. The columns labeled (a) and (b) give the number of binary templates and the total number of templates, respectively, and the columns (c) and (d) give the number of base 10 primes obtained by promoting the templates in columns (a) and (b).

We can reduce the list of templates by omitting those that are reversals of others. Table 3 shows the reduced list of templates of lengths ≤ 6.

k	(a)	(b)	(c)	(d)
2	1	1	18	18
3	1	1	81	81
4	3	3	1539	1539
5	5	8	17661	20457
6	9	51	135489	242217

Table 2: For lengths $k = 2$ through 6, the numbers of (a) binary templates, (b) all templates, (c) base 10 primes obtained by promoting the binary templates, and (d) base 10 primes obtained by promoting all the templates.

11	100001	102212	120212
101	100011	102221	120221
1001	100101	103223	120222
1011	100111	103233	121022
10001	101011	110212	121102
10011	101221	112021	122102
10111	101222	112022	122202
12021	102201	120021	132023
12022	102202	120022	133023

Table 3: Reduced list of prime templates: every pseudoprime of length ≤ 6 can be obtained by promoting one of these 36 pseudoprimes or its reversal (A191420).

Since $1\underbrace{00\ldots0}_{k-2}1|_2$ is prime, the promotion process tells us for example that the numbers $r\underbrace{s_1s_2\ldots s_{k-2}}_{k-2}\beta|_b$ are prime, for $1 \le r \le \beta$, provided each of the digits s_i is in the range 0 through $r-1$. This gives $(b-1)^{k-2} + 2(b-2)^{k-2} + \cdots = O(b^{k-2})$ primes of length k, which for $b > 2$ is of exponential growth but smaller than (16).

4 Dismal squares

One might expect that it would be easier to find the number of dismal squares of a given length than the number of dismal primes, but we have not investigated squares as thoroughly, and we do not even have a precise conjecture about their asymptotic behavior. In base 2, the first few dismal squares, written in base 10, are

$$0,\ 1,\ 4,\ 7,\ 16,\ 21,\ 28,\ 31,\ 64,\ 73,\ 84,\ 95,\ 112,\ 125,\ 124,\ 127,\ 256,\ 273,\ \ldots, \tag{17}$$

(A067398, also contributed by Jens Voß in 2002), and the numbers of squares of lengths 1 (including 0), $3, 5, 7, \ldots$ are

$$2,\ 2,\ 4,\ 8,\ 15,\ 29,\ 55,\ 105,\ 197,\ 367,\ 678,\ 1261,\ 2326,\ 4293,\ 7902,\ 14431,\ \ldots \tag{18}$$

(A190820). In base 10, the first few squares were given in (2), and the numbers of squares of lengths 1 (including 0), 3, 5, $7, \ldots$ are

$$10,\ 90,\ 900,\ 9000,\ 74667,\ 608673, \ldots$$

(A172199). The sequence of base 10 squares is not monotonic (for example $1011 < 1020$ yet $1011 \times_{10} 1011 = 1011111 > 1020 \times_{10} 1020 = 1010200$), and contains repetitions. The numbers which are squares in more than one way are

$$111111111,\ 111111112,\ 111111113,\ 111111114,\ 111111115,\ 111111116,\ 111111117,\ \ldots,$$

e.g., $111111111 = 11011 \times_{10} 11011 = 11111 \times_{10} 11111$ (A180513, A181319).

We briefly mention two other questions about squares to which we do not know the answer: (i) In base 2, how many square roots does $2^{2k+1} - 1$ have? This is a kind of combinatorial covering problem. For $k = 0, 1, \ldots$ the counts are

$$\begin{aligned}&1,\ 1,\ 1,\ 1,\ 2,\ 3,\ 5,\ 9,\ 15,\ 28,\ 50,\ 95,\ 174,\ 337,\ 637,\ 1231,\ 2373,\ 4618,\ 8974,\ 17567,\ 34387,\\ &67561,\ 132945,\ 262096,\ 517373,\ 1023366,\ 2025627,\ 4014861,\ 7964971,\ 15814414,\\ &31424805,\ 62490481,\ 124330234,\ 247514283,\ 492990898,\ 982307460,\ 1958093809,\\ &3904594162,\ 7788271542,\ 15539347702,\ 31012331211,\ \ldots\end{aligned} \tag{19}$$

(A191701). Is there a formula or recurrence for this sequence? (ii) In base b, if we consider all p such that $p \times_b p = n$, does one of them dominate all the others (in the $\gg_b$ sense)? If so the dominating one could be called the "principal" square root.

5 The divisibility poset

One drawback to dismal arithmetic is that there is more than one way to order the dismal numbers, and no ordering is fully satisfactory.

The usual order on the nonnegative integers ($<$ or $\leq$) is unsatisfactory, since (working in base 10) we have $18 < 25$ yet $18 \mathbin{\boldsymbol{+}} 32 = 38 > 25 \mathbin{\boldsymbol{+}} 32 = 35$, $32 < 41$ yet $32 \mathbin{\boldsymbol{\times}} 3 = 32 > 41 \mathbin{\boldsymbol{\times}} 3 = 31$.

The dominance order ($\ll_b$) is more satisfactory, in view of Lemma 6 and the distributive law of Theorem 1, but has the drawback that m divides n does not imply that $m \ll_b n$ (e.g., $12|_{10}$ divides $11|_{10}$, yet $11|_{10} \ll_{10} 12|_{10}$).

The partial order induced by divisibility ($\prec_b$) is worth discussing, as it has some interesting properties and is the best way to look at dismal numbers as long as we are considering only questions of factorization and divisibility. For simplicity we will restrict the discussion to base 10.

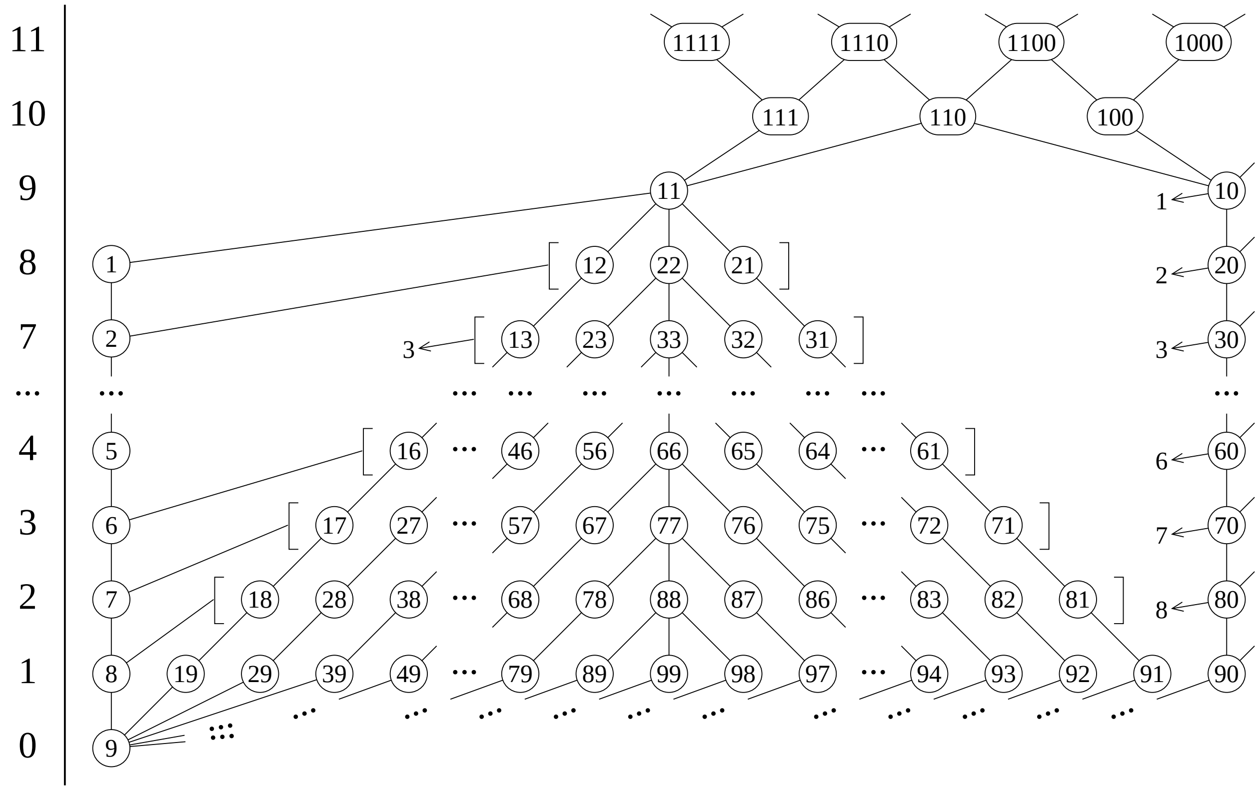

Figure 1: Beginning of the divisibility poset (see text for description). The left-hand column gives the rank (A161813).

Figure 1 displays the beginning of the Hasse diagram ([21, p. 99]) of this partially ordered set (or poset), and shows all positive numbers with one or two digits, and a few larger numbers. There are too many edges to draw in the diagram, so we will describe them in words.

The multiplicative identity 9, the "zero element" in the poset, is at the base. The other single-digit numbers $8 \prec 7 \prec 6 \prec \cdots \prec 1$ are above it in the left-hand column, and 8 is joined to 9. The numbers are arranged in rows according to their rank (shown at the extreme left of the diagram). The numbers of rank 1 consist of 8 and the (infinitely many) primes: $19, 29, \ldots, 91, 90, 109, 209, 219, 309, \ldots$ (A144171). All of these are joined to 9. The numbers of rank 2 are $7, 18, 28, \ldots, 81, 80, 108, 119, \ldots$ (A144175), and so on. Every two-digit number of rank h ($1 \leq h \leq 9$) is joined to the single-digit number $h - 1$ on the left of the diagram., as indicated by the square brackets. A two-digit number to the left of the central column of the pyramid is joined to the number diagonally below it to the left (e.g., 56 is joined to 57), and a two-digit number to the right of the central column is joined to the number diagonally below it to the right (e.g., 65 is joined to 75). A number in the central column is joined to the three numbers immediately below it (e.g., 66 is joined to $67, 77, 76$). A number $r0$ ($1 \leq r \leq 9$) in the right-hand column is joined to the number immediately below it and to the single-digit number r.

Only a few numbers with more than two digits are shown, but a more complete diagram would show for example that 1 is joined to, besides 10 and 11, many other decimal numbers whose digits are 0's and 1's, such as 101, 1001, 1011, ... all of rank 9. The figure is complete in the sense that all downward joins are shown for all the numbers in the diagram.

One perhaps surprising property of the divisibility poset is that the greatest lower bound (or greatest common divisor) $m \wedge n$ and the least upper bound (or least common multiple) $m \vee n$ of two dismal numbers m and n need not exist, and so this poset fails to be a lattice [9, Chap. 1]. For example, again working in base 10, the rank 2 numbers 8989 and 9898 are each divisible by (and joined to) the nine primes $909, 919, \ldots, 989$. However, these nine primes are incomparable in the $\prec_{10}$ order, so neither $8989 \wedge 9898$ nor $909 \vee 919$ exist.

Although greatest common divisors need not exist, we can still define two dismal numbers to be *relatively prime* if their only common divisor is the unit β.

In the next section we will study the number of divisors function $d_b(n)$. Candidates for divisors of n are all numbers m with $\text{len}_b(m) \leq \text{len}_b(n)$. It is therefore appropriate to define the dismal analogue of the Euler totient function, $\phi_b(n)$, to be the number of numbers m with $\text{len}_b(m) \leq \text{len}_b(n)$ which are relatively prime to n. The initial values of $\phi_2(n)$ and $\phi_{10}(n)$ are shown in Table 4.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\phi_2(n)$	1	2	2	4	6	2	4	8	14	6	14	5	14	5	7	16	30	14	30	12
$\phi_{10}(n)$	1	1	1	1	1	1	1	1	9	18	2	18	18	18	18	18	18	18	90	18

Table 4: Values of totient functions $\phi_2(n)$ and $\phi_{10}(n)$ (A191674, A191675).

6 The number of dismal divisors

Let $d_b(n)$ denote the number of dismal divisors of n in base b, and let $\sigma_b(n)$ denote the dismal sum of the dismal divisors of n. These functions are more irregular than their classical analogues, as can be seen from the examples in Table 5, and there are no simple formulas for them. In this section we study some of the properties of $d_b(n)$. Note that if r is the largest digit in n, then the smallest divisor of n is r, and the largest divisor is the number obtained by changing all the r's in n to β's.

n	divisors (base 10)	$d_{10}(n)$	$\sigma_{10}(n)$
1	$1, 2, 3, 4, 5, 6, 7, 8, 9$	9	9
2	$2, 3, 4, 5, 6, 7, 8, 9$	8	9
3	$3, 4, 5, 6, 7, 8, 9$	7	9
4	$4, 5, 6, 7, 8, 9$	6	9
5	$5, 6, 7, 8, 9$	5	9
6	$6, 7, 8, 9$	4	9
7	$7, 8, 9$	3	9
8	$8, 9$	2	9
9	9	1	9
10	$1, \ldots, 9, 10, 20, \ldots, 90$	18	99
11	$1, \ldots, 9, rs$ with $1 \le r, s \le 9$	90	99
12	$2, \ldots, 9, 12, 13, \ldots, 19$	16	19

Table 5: In base 10, the dismal divisors of the numbers 1 through 12 and the corresponding values of $d_{10}(n)$ and $\sigma_{10}(n)$ (A189506, A087029, A087416).

A base b dismal prime p has two divisors, $b-1$ and p, so $d_b(p) = 2$. In the other direction, a divisor of a k-digit number n has at most k digits, so

$$2 \le d_b(n) \le b^k - 1 \,. \tag{20}$$

Base b numbers of the form $111\ldots1|_b$ (that is, in which all the base b digits are 1) come close to meeting this upper bound—see Remark (iv) following Theorem 17. We make the following conjectures.

Conjecture 12. In any base $b \ge 3$, among all k-digit numbers n, $d_b(n)$ has a unique maximum at $n = (b^k - 1)/(b-1) = 111\ldots1|_b$.

Conjecture 13. In base 2, among all k-digit numbers n, the maximal value of $d_2(n)$ occurs at $n = 2^k - 2 = 111\ldots10|_2$, and this is the unique maximum for $n \ne 2, 4$.

Conjecture 14. In base 2, among all *odd* k-digit numbers n, $d_2(n)$ has a unique maximum at $n = 2^k - 1 = 111\ldots111|_2$, and if $k \ge 3$ and $k \ne 5$, the second-largest value of $d_2(n)$ occurs at $n = 2^k - 3 = 111\ldots101|_2$, $n = 2^k - 2^{k-2} - 1 = 101\ldots111|_2$, and possibly other values of n.

The numerical evidence supporting these conjectures is compelling. For example, in base 10, if we study the sequence $d_{10}(n)$, $n \geq 1$ (A087029) for $n \leq 10^6$, and write down $d_{10}(n)$ each time it exceeds $d_{10}(m)$ for all $m < n$, we obtain the values

$$9, 18, 90, 180, 819, 1638, 7461, 14922, 67968\,,$$

(see A186443) at these (decimal) values of n:

$$1, 10, 11, 110, 111, 1110, 1111, 11110, 11111\,.$$

If n is a 5-digit decimal number, the eight largest values of $d_{10}(n)$ are, in decreasing order,

$$67968, 39624, 21812, 14922, 11202, 9616, 6732, 6570,$$

at these values of n:

$$11111, 22222, 33333, 11110, 44444, 12222 \text{ or } 22220 \text{ or } 22221, 11011, 10111 \text{ or } 11101.$$

The number $n = (10^k - 1)/9 = 111\ldots1|_{10}$ is a clear winner among all k-digit decimal numbers for $k \leq 5$. The data for bases 3 through 9 is equally supportive of Conjecture 12. Likewise, the binary data strongly supports Conjectures 13 and 14—see Table 6 for the initial values of $d_2(n)$. Table 6 also shows why $k = 5$ is mentioned as an exception in Conjecture 14: among 5-digit odd numbers, $d_2(11011|_2) = 4$ is the runner-up, ahead of $d_2(10111|_2) = d_2(11101|_2) = 2$.

n (base 2)	$d_2(n)$	n (base 2)	$d_2(n)$	n (base 2)	$d_2(n)$	n (base 2)	$d_2(n)$
–	–	1000	4	10000	5	11000	8
1	1	1001	2	10001	2	11001	2
10	2	1010	4	10010	4	11010	4
11	2	1011	2	10011	2	11011	4
100	3	1100	6	10100	6	11100	9
101	2	1101	2	10101	3	11101	2
110	4	1110	6	10110	4	11110	10
111	3	1111	5	10111	2	11111	8

Table 6: In base 2, the number of dismal divisors of the numbers 1 through 31 (A067399).

For a more dramatic illustration of Conjectures 12 and 13, see the graphs of sequences A087029 and A067399 in [17]. Although the evidence is convincing, we have not, unfortunately, succeeded in proving these conjectures.

We are able to determine the exact values of $d_b(111\ldots111|_b)$ for all b (the conjectural winner for $b \geq 3$ and the conjectural winner among odd numbers in the binary case) and $d_2(111\ldots101|_2) = d_2(101\ldots111)$ (the conjectural runners-up among odd binary numbers of length $k > 5$).

We begin with a lemma that describes the effect of trailing zeros.

Lemma 15. *If the base b expansion of n ends with exactly $r \geq 0$ zeros, so that $n = mb^r$, with $b \nmid m$, then*

$$d_b(n) = (r+1)d_b(m)\,.$$

Proof. If $p \,\boldsymbol{\times}_b\, q = m$ then $b \nmid p$, $b \nmid q$, and the $r+1$ numbers pb^i $(0 \leq i \leq r)$ dismally divide n, since

$$(pb^i) \,\boldsymbol{\times}_b\, (qb^{r-i}) = n\,.$$

Conversely, if $p' \,\boldsymbol{\times}_b\, q' = n$ then $p' = pb^i$, $q' = qb^{r-i}$, $b \nmid p$, $b \nmid q$, for some i with $0 \leq i \leq r$. So each dismal divisor of m corresponds to exactly $r+1$ dismal divisors of n. □

For example, in base 10 the dismal divisors of 7 are $7, 8, 9$ and the dismal divisors of 700 are $7, 8, 9, 70, 80, 90, 700, 800, 900$.

One reason Conjectures 12-14 seem hard to prove is the erratic behavior of $d_b(n)$. In contrast to the above lemma, the effect of internal zeros is hard to analyze. Suppose the i-th digit in the b-ary expansion of n is zero. This implies that if $n = p \,\boldsymbol{\times}_b\, q$, then all entries in the i-th column of the long multiplication tableau must be zero, which imposes many constraints on the b-ary expansions of p and q. One would expect, therefore, that changing the zero digit to a larger number—thus weakening the constraints—would always increase the number of divisors of n. Roughly speaking, this is true, but there are many cases where it fails. For example, $d_2(11111|_2) = 8$, but $d_2(11110|_2) = 10$ (see Table 6, Lemma 15 and Theorem 16). Again, $d_2(10101|_2) = 3$, but $10111|_2$ is prime, so $d_2(10111|_2) = 2$. In any base b, $11\underbrace{00\ldots0}_{r}|_b$ has $(r+1)((b-1)^2+(b-1))$ divisors, whereas $11\underbrace{00\ldots0}_{r-1}1|_b$ has $(b-1)^3+(b-1)$ divisors, a smaller number if r is large.

The next result was conjectured by the second author and proved by Richard Schroeppel in 2001 [15]. An alternative proof (via a bijection with a certain class of polyominoes) was given by Frosini and Rinaldi in 2006 [6]. We give a version of Schroeppel's elegant direct proof, partly because it has never been published, and partly because we will use similar arguments later.

Theorem 16. *In base 2, the number of dismal divisors of $111\ldots1|_2$ (with k 1's) is equal to the number of compositions of k into parts of which the first is at least as great as all the other parts.*

Proof. Suppose $p \,\boldsymbol{\times}_2\, q = 111\ldots1|_2$ (with k 1's) where $\mathrm{len}_2(p) = r$, $\mathrm{len}_2(q) = k+1-r$. By examining the long multiplication tableau for $p \,\boldsymbol{\times}_2\, q$, we see that it is also true that $p \,\boldsymbol{\times}_2\, q' = 111\ldots1|_2$ where $q' = 111\ldots1|_2$, with $k+1-r$ 1's (for if there is a 1 in each column of the tableau for $p \,\boldsymbol{\times}_2\, q$, that is still true for $p \,\boldsymbol{\times}_2\, q'$). So in order to find all the divisors p of $111\ldots1|_2$ (with k 1's) we may assume that the cofactor q has the form $111\ldots1|_2$ (with s 1's, for some s, $1 \leq s \leq k$).

We establish the desired result by exhibiting a bijection between the two sets. Let $k = c_1 + c_2 + \cdots + c_t$ be a composition of k in which $c_1 \geq c_i \geq 1$ $(2 \leq i \leq t)$. Let $\psi(c_i)$ denote the binary vector $000\ldots01$ with $c_i - 1$ 0's and a single 1. The divisor p corresponding to this composition has binary representation given by the concatenation

$$1\,\psi(c_2)\,\psi(c_3)\,\ldots\,\psi(c_r)|_2\,, \tag{21}$$

of length $k+1-c_1$. If we set $q = 111\ldots1|_2$, of length c_1, then $p \times_2 q = 111\ldots1|_2$ (with k 1's). This follows from the fact that if the binary representation of p contains a string of exactly s 0's:

$$\ldots 1 \underbrace{000\ldots0}_{s} 1 \ldots,$$

then, when we form the product $p \times_2 q$, the 1 immediately to the right of these 0's will propagate leftward to cover the 0's if and only if $\mathrm{len}_2(q) \geq s+1$, which is exactly the condition that $c_1 \geq c_i$ for all $i \geq 2$. □

For example, the eight compositions of 5 in which no part exceeds the first and the corresponding factorizations $p \times_2 q$ of $11111|_2$ are shown in Table 7. Dots have been inserted in p to indicate the division into the pieces $\psi(c_i)$.

Composition of 5	divisor p	cofactor q
5	$1\|_2$	$11111\|_2$
41	$1.1\|_2$	$1111\|_2$
32	$1.01\|_2$	$111\|_2$
311	$1.1.1\|_2$	$111\|_2$
221	$1.01.1\|_2$	$11\|_2$
212	$1.1.01\|_2$	$11\|_2$
2111	$1.1.1.1\|_2$	$11\|_2$
11111	$1.1.1.1.1\|_2$	$1\|_2$

Table 7: Illustrating the bijection used to prove Theorem 16.

Remarks.

(i) Using the bijection defined by (21), the number of 1's in the binary expansion of the divisor p is equal to the number of parts in the corresponding composition.

(ii) It follows immediately from the interpretation in terms of compositions that the numbers $d_2(2^k-1)$ have generating function

$$\begin{aligned} \sum_{k=1}^{\infty} d_2(2^k-1)\, z^k &= \sum_{l=1}^{\infty} \frac{z^l}{1-(z+z^2+\cdots+z^l)} \\ &= \sum_{l=1}^{\infty} \frac{(1-z)z^l}{1-2z+z^{l+1}} \end{aligned} \tag{22}$$

(the index of summation, l, corresponds to the first part in the composition).

(iii) The initial values of this sequence are shown in Table 8. This sequence appears in entries A007059 and A079500 in [17], although the indexing is different in each case. The sequence also occurs in at least four other contexts besides the two mentioned in Theorem 16, namely in the enumeration of balanced ordered trees (Kemp [10]), of polyominoes that tile the plane by translation (Beauquier and Nivat [2], Brlek et al. [3]), of Dyck paths (see A007059), and in counting solutions to the postage stamp problem (again see A007059). The article by Frosini and Rinaldi [6] gives bijections between four of these six enumerations.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_2(2^k-1)$	1	2	3	5	8	14	24	43	77	140	256	472	874	1628	3045	5719

Table 8: Values of $d_2(2^k - 1)$.

In this context we should also mention the recent article of Rawlings and Tiefenbruck [19], which, although not directly related to the problems we consider, discusses other connections between the enumeration of compositions, permutations, polyominoes, and binary words.

(iv) The asymptotic behavior of this sequence is quite subtle. From the work of Kemp [10] and Knopfmacher and Robbins [12] it follows that

$$d_2(2^k-1) \;\sim\; \frac{2^k}{k\log 2}\,(1+\Theta_k),\ \text{ as } k\to\infty\,, \tag{23}$$

where Θ_k is a bounded oscillating function with $|\Theta_k| < 10^{-5}$ (see the proof of Theorem 23 below).

In order to determine $d_b(111\ldots1|_b)$ for bases $b > 2$, we first classify compositions in which no part exceeds the first according to the number of parts. Let $T(k,t)$ denote the number of compositions of k into exactly t parts (with $1 \le t \le k$) such that no part exceeds the first. Table 9 shows the initial values. This is entry A184957 in [17]. [4]

$k\backslash t$	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	1	1	1					
4	1	2	1	1				
5	1	2	3	1	1			
6	1	3	4	4	1	1		
7	1	3	6	7	5	1	1	
8	1	4	8	11	11	6	1	1

Table 9: Initial values of of $T(k,t)$, the number of compositions of k into exactly t parts such that no part exceeds the first.

The values of $T(k,t)$ are easily computed via the auxiliary variables $\gamma(k,t,m)$, which we define to be the number of compositions of k into t parts of which the first part, m, is the greatest (for $1 \le t \le k$, $1 \le m \le k$). We have the recurrence

$$\gamma(k,t,m) \;=\; \sum_{j=1}^{\min\{m,k+2-t-m\}} \gamma(k-j,t-1,m) \tag{24}$$

[4] An array equivalent to this, A156041, was contributed to [17] by J. Grahl in 2009 and later studied by A. P. Heinz and R. H. Hardin.

(classifying compositions according to the *last* part, j), for $m > 1$, $t > 1$, $t+m < k-1$, with initial conditions

$$\begin{aligned}\gamma(k,t,1) &= \delta_{t,k}\,,\\ \gamma(k,1,m) &= \delta_{m,k}\,,\end{aligned}$$

where $\delta_{i,j} = 1$ if $i = j$ or 0 if $i \neq j$. Then

$$T(k,t) \;=\; \sum_{m=1}^{k+1-t} \gamma(k,t,m)\,. \tag{25}$$

Since $\gamma(k,t,m)$ is the coefficient of z^k in $z^m(z+z^2+\cdots+z^m)^{t-1}$, it follows that column t of Table 9 has generating function

$$\sum_{k=1}^{\infty} T(k,t) z^k \;=\; \frac{z^{t-1}}{(1-z)^{t-1}} \sum_{r=0}^{t-1} (-1)^r \binom{t-1}{r} \frac{z^{r+1}}{1-z^{r+1}}\,. \tag{26}$$

Since the total number of compositions of k into t parts is $\binom{k-1}{t-1}$, and in at least a fraction $\frac{1}{t}$ of them the first part is the greatest, we have the bounds

$$\frac{1}{t}\binom{k-1}{t-1} \;\le\; T(k,t) \;\le\; \binom{k-1}{t-1}. \tag{27}$$

Theorem 17.

$$d_b\Big(\frac{b^k-1}{b-1}\Big) \;=\; d_b(\underbrace{11\ldots1}_{k}|_b) \;=\; \sum_{t=1}^{k} T(k,t)(b-1)^t\,. \tag{28}$$

Proof. Suppose $p \mathbin{\pmb{\times}}_b q = \underbrace{11\ldots1}_{k}|_b$. At least one of p and q, say q, must contain only digits 0 and 1 (for if p contains a digit $i > 1$ and q contains a digit $j > 1$, then $i \mathbin{\pmb{\times}}_b j = \min\{i,j\} > 1$ will appear somewhere in $p \mathbin{\pmb{\times}}_b q$). As in the proof of Theorem 16 we may assume that this q has the form $\underbrace{11\ldots1}_{s}|_b$ for some s with $1 \le s \le k$. Suppose $p = \sum_{i=0}^{r-1} p_i 2^i$ with $p_i \in \{0,1\}$ is a divisor of $\underbrace{11\ldots1}_{k}|_2$, so that

$$p \mathbin{\pmb{\times}}_2 (2^{k+1-r}-1) \;=\; 2^k - 1\,. \tag{29}$$

By Lemma 5, $p' := \sum_{i=0}^{r-1} p_i b^i$ is a base b dismal divisor of $\frac{b^k-1}{b-1}$:

$$p' \mathbin{\pmb{\times}}_b \frac{b^{k+1-r}-1}{b-1} \;=\; \frac{b^k-1}{b-1}\,. \tag{30}$$

Furthermore, (30) still holds if any of the p_i that are 1 are changed to any digit in the range $\{1,2,\ldots,b-1\}$. Conversely, any base b dismal divisor p' of $\frac{b^k-1}{b-1}$ remains a divisor if all the nonzero digits in the base b expansion of p' are replaced by 1's. So each divisor of $\underbrace{11\ldots1}_{k}|_2$ with t 1's corresponds to $(b-1)^t$ divisors of $\frac{b^k-1}{b-1}$. Since there are $T(k,t)$ divisors of 2^k-1 with t 1's, the result follows. $\square$

$k\backslash b$	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9
2	2	6	12	20	30	42	56	72	90
3	3	14	39	84	155	258	399	584	819
4	5	34	129	356	805	1590	2849	4744	7461
5	8	82	426	1508	4180	9798	20342	38536	67968
6	14	206	1434	6452	21830	60594	145586	313544	619902
7	24	526	4890	27828	114580	375954	1044246	2555080	5660208

Table 10: Table of $d_b(11\dots1|_b)$ (with k 1's), the number of base b dismal divisors of $\frac{b^k-1}{b-1}$ (rows: A002378, A027444, A186636; columns A079500, A186523).

Remarks.

(i) Table 10 shows the initial values of $d_b(\underbrace{11\dots1}_{k}|_b)$.

(ii) Theorem 17 reduces to Theorem 16 in the case $b=2$.

(iii) From (27) and (28) we have

$$\frac{b^k-1}{k} \;\le\; d_b(\underbrace{11\dots1}_{k}|_b) \;\le\; (b-1)b^{k-1}\,. \tag{31}$$

For $b=2$ we also have the asymptotic estimate (23).

We now study the runners-up in the binary case (among odd numbers of length greater than 5), namely the numbers $111\dots101|_2$ and $101\dots111|_2$. The simplest way to state the result is to give the generating function.

Theorem 18.

$$\begin{aligned}\sum_{k=3}^{\infty} d_2(2^k-3)z^k \;&=\; z \;+\; \frac{z^3}{1-z} \;+\; \sum_{l=3}^{\infty} \frac{(1-z)^2 z^l}{1-2z+z^{l-1}-z^l+z^{l+2}} \\ &=\; z+2z^3+2z^4+2z^5+4z^6+6z^7+10z^8+\cdots\,. \quad (\textit{A188288})\end{aligned} \tag{32}$$

We will deduce Theorem 18 from Theorem 21 below.

Suppose $p \,\text{✖}_2\, q = 2^k-3$, $k \ge 3$, where $\text{len}_2(p)=h$, $\text{len}_2(q)=l$, with $h+l=k+1$. In order to find all choices for p, we note that the binary expansions of p and q must end with $\dots01$, and that, as in the proofs of Theorems 16 and 17, we may assume that $q=2^l-3$. Our approach is to fix l and allow h to vary. Let $M_h^{(l)}$ denote the number of binary numbers p with $\text{len}_2(p)=h$ such that

$$p \,\text{✖}_2\, \underbrace{111\dots101}_{l}|_2 \;=\; \underbrace{111\dots101}_{h+l-1}|_2\,. \tag{33}$$

Suppose the binary expansion of p is

$$1\,x_v\,x_{v-1}\,\dots\,x_3\,x_2\,x_1\,0\,1|_2\,,$$

where $v := h-3$ and the x_i are 0 or 1. The long multiplication tableau for (33) implies that the x_i must satisfy certain Boolean equations (remember that $\boldsymbol{+}_2$ is the logical OR; in what follows we will write $\boldsymbol{+}$ rather than $\boldsymbol{+}_2$). For example, the tableau for $l = 4$ and $h = 9$, $v = 6$ is shown in Figure 2.

$$
\begin{array}{cccccccccccc}
 & & & 1 & x_6 & x_5 & x_4 & x_3 & x_2 & x_1 & 0 & 1 \\
\boldsymbol{\times}_2 & & & & & & & & 1 & 1 & 0 & 1 \\
\hline
 & & & 1 & x_6 & x_5 & x_4 & x_3 & x_2 & x_1 & 0 & 1 \\
 & 1 & x_6 & x_5 & x_4 & x_3 & x_2 & x_1 & 0 & 1 & & \\
1 & x_6 & x_5 & x_4 & x_3 & x_2 & x_1 & 0 & 1 & & & \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
\hline
\end{array}
$$

Fig. 2.

By reading down the columns, we obtain the equations

$$
\begin{aligned}
x_1 \boldsymbol{+} x_3 &= 1\,, \\
x_1 \boldsymbol{+} x_2 \boldsymbol{+} x_4 &= 1\,, \\
x_2 \boldsymbol{+} x_3 \boldsymbol{+} x_5 &= 1\,, \\
x_3 \boldsymbol{+} x_4 \boldsymbol{+} x_6 &= 1\,, \\
x_5 \boldsymbol{+} x_6 &= 1\,.
\end{aligned}
$$

There are $M_9^{(4)} = 29$ solutions $(x_1, \ldots, x_6)$ to these equations. Table 11 shows the initial values of $M_h^{(l)}$, as found by computer.

$h \backslash l$	1	2	3	4	5	6	7	8
1	1	0	1	1	1	1	1	1
2	0	0	0	0	0	0	0	0
3	1	0	0	1	1	1	1	1
4	1	0	1	1	2	2	2	2
5	1	0	2	3	3	4	4	4
6	1	0	2	5	7	7	8	8
7	1	0	3	9	13	15	15	16
8	1	0	6	16	24	29	31	31
9	1	0	10	29	47	56	61	63
10	1	0	15	53	89	110	120	125
11	1	0	24	96	170	216	238	248
12	1	0	40	174	326	422	471	494

Table 11: Table of $M_h^{(l)}$ (columns 3 and 4 are A070550 and A188223).

Inspection of the table suggests that the l-th column satisfies the recurrence

$$
M_h^{(l)} \;=\; M_{h-1}^{(l)} + M_{h-2}^{(l)} + \cdots + M_{h-l+2}^{(l)} + M_{h-l}^{(l)} + M_{h-l-1}^{(l)}\,, \tag{34}
$$

for $l \ge 3$. This will be established in Corollary 22.

We consider the cases $h \le l+1$ and $h \ge l+2$ separately. For $h \le l+1$, it is straightforward to show the following:

$$M_2^{(1)} \;=\; 0\,, \quad M_h^{(1)} \;=\; 1\ (h \ne 2)\,, \quad M_h^{(2)} \;=\; 0\,, \tag{35}$$

and, for $l \ge 3$, $h \le l+1$,

$$M_h^{(l)} \;=\; \begin{cases} 1, & \text{if } h = 1; \\ 0, & \text{if } h = 2; \\ 2^{h-3}, & \text{if } 3 \le h \le l-1; \\ 2^{h-3} - 1, & \text{if } h = l \text{ or } l+1. \end{cases} \tag{36}$$

This accounts for the entries in Table 11 that are on or above the line $h - l = 1$.

We now consider the case $3 \le l \le h-2 = v+1$. The multiplication tableau leads to two special equations,

$$\begin{cases} x_2 \;=\; 1, & \text{if } l = 3; \\ x_1 \boldsymbol{+} x_2 \boldsymbol{+} \cdots \boldsymbol{+} x_{l-3} \boldsymbol{+} x_{l-1} \;=\; 1; & \text{if } l \ge 4, \end{cases} \tag{37}$$

and

$$x_{v-l+3} \boldsymbol{+} x_{v-l+4} \boldsymbol{+} \cdots \boldsymbol{+} x_{v-1} \boldsymbol{+} x_v \;=\; 1\,, \tag{38}$$

together with a family of $v - l + 1$ further equations, which, if $l = 3$, are

$$x_1 \boldsymbol{+} x_3 \;=\; x_2 \boldsymbol{+} x_4 \;=\; \cdots \;=\; x_{v-3} \boldsymbol{+} x_{v-1} \;=\; x_{v-2} \boldsymbol{+} x_v \;=\; 1\,, \tag{39}$$

or, if $l \ge 4$, are

$$\begin{aligned} x_1 \boldsymbol{+} x_2 \boldsymbol{+} x_3 \boldsymbol{+} \cdots \boldsymbol{+} x_{l-2} \boldsymbol{+} x_l \;&=\; 1\,, \\ x_2 \boldsymbol{+} x_3 \boldsymbol{+} x_4 \boldsymbol{+} \cdots \boldsymbol{+} x_{l-1} \boldsymbol{+} x_{l+1} \;&=\; 1\,, \\ \cdots\cdots\cdots\cdots\cdots \\ x_{v-l+1} \boldsymbol{+} x_{v-l+2} \boldsymbol{+} x_{v-l+3} \boldsymbol{+} \cdots \boldsymbol{+} x_{v-2} \boldsymbol{+} x_v \;&=\; 1\,. \end{aligned} \tag{40}$$

The two special equations (37) and (38) involve variables with both low and high indices, which makes induction difficult. We therefore define a simpler system of Boolean equations in which the special constraints apply only to the high-indexed variables.

For $l \ge 3$ and $n \ge 1$, let $D_n^{(l)}$ denote the number of binary vectors $x_1 x_2 \ldots x_n$ of length n that end with $x_n = 1$, do not contain any substring

$$\underbrace{00\ldots000}_{l} \text{ or } \underbrace{00\ldots010}_{l}$$

and do not end with

$$\underbrace{00\ldots01}_{l-1}\,.$$

Equivalently, $D_n^{(l)}$ is the number of solutions to the Boolean equations

$$\begin{aligned} x_1 \boldsymbol{+} x_2 \boldsymbol{+} x_3 \boldsymbol{+} \cdots \boldsymbol{+} x_{l-2} \boldsymbol{+} x_l &= 1\,, \\ x_2 \boldsymbol{+} x_3 \boldsymbol{+} x_4 \boldsymbol{+} \cdots \boldsymbol{+} x_{l-1} \boldsymbol{+} x_{l+1} &= 1\,, \\ \ldots\ldots\ldots\ldots\ldots \\ x_{n-l} \boldsymbol{+} x_{n-l+1} \boldsymbol{+} x_{n-l+2} \boldsymbol{+} \cdots \boldsymbol{+} x_{n-3} \boldsymbol{+} x_{n-1} &= 1\,, \end{aligned} \tag{41}$$

and

$$x_{n-l+2} \boldsymbol{+} x_{n-l+3} \boldsymbol{+} \cdots \boldsymbol{+} x_{n-2} \boldsymbol{+} x_{n-1} = 1\,, \quad x_n = 1\,. \tag{42}$$

We also set $D_0^{(l)} := 1$.

Theorem 19. *For $l \geq 3$, $n \geq 1$, there is a one-to-one correspondence between binary vectors of length n satisfying the $D_n^{(l)}$ equations and compositions of n into parts taken from the set*

$$\{1, 2, \ldots, l-2, l, l+1\}\,. \tag{43}$$

Proof. We exhibit a bijection between the two sets. Let c be a composition $n = c_1+c_2+\cdots+c_r$ into parts from (43). For $1 \leq i \leq l-2$, let $\psi(i) := 00\ldots01$, of length i and ending with a single 1, let $\psi(l) := 00\ldots011$, of length l, let $\psi(l+1) := 00\ldots0011$, of length $l+1$, and let

$$\psi(c) \ := \ \psi(c_1)\,\psi(c_2)\,\ldots\,\psi(c_r)\,, \tag{44}$$

a binary vector of length n. Note that the $\psi(i)$ for $1 \leq i \leq l-2$ contain runs of at most $l-3$ zeros. Runs of $l-2$ or $l-1$ zeros in $\psi(c)$ are therefore followed by two ones. So conditions (41) and (42) are satisfied. Conversely, given a binary vector satisfying the $D_n^{(l)}$ equations, we can decompose it into substrings $\psi(i)$ by reading it from left to right. □

$n \backslash l$	3	4	5	6
0	1	1	1	1
1	1	1	1	1
2	1	2	2	2
3	2	3	4	4
4	4	6	7	8
5	6	11	14	15
6	9	20	27	30
7	15	36	51	59
8	25	65	98	115

Table 12: Table of $D_n^{(l)}$ (the columns are A006498, A079976, A079968, A189101).

The generating function for the $D_n^{(l)}$ follows immediately from the theorem:

Corollary 20. *For $l \geq 3$, the numbers $D_n^{(l)}$ have generating function*

$$\mathcal{D}^{(l)}(z) := \sum_{n=0}^{\infty} D_n^{(l)} z^n = \frac{1}{1-(z+z^2+\cdots+z^{l-2}+z^l+z^{l+1})}$$
$$= \frac{1-z}{1-2z+z^{l-1}-z^l+z^{l+2}} . \quad (45)$$

Table 12 shows the initial values of $D_n^{(l)}$, computed using the generating function. The $l = 4$ and $l = 5$ columns are in [17] as entries A079976 and A079968, taken from a paper by D. H. Lehmer on enumerating permutations $(\pi_1, \ldots, \pi_n)$ with restrictions on the displacements $\pi_i - i$ ([14]; see also [11]).

We now express the numbers $M_h^{(l)}$ in terms of the $D_n^{(l)}$. We consider the l values $x_1, \ldots, x_l$ in a solution to the $M_h^{(l)}$ equations, and classify them according to the number of leading zeros. There are just $l-1$ possibilities, as shown in Table 13, and in each case the $M_h^{(l)}$ equations reduce to an instance of the $D_n^{(l)}$ equations. For example, if $x_1 = 1$ the $M_h^{(l)}$ equations reduce to an instance of the $D_{h-3}^{(l)}$ equations. (E.g., if $l = 5$ and we set $x_1 = 1$, equations (37), (38), (40) become $x_{v-2} \boldsymbol{+} x_{v-1} \boldsymbol{+} x_v = 1$, $x_2 \boldsymbol{+} x_3 \boldsymbol{+} x_4 \boldsymbol{+} x_6 = 1$, …, $x_{v-4} \boldsymbol{+} x_{v-3} \boldsymbol{+} x_{v-2} \boldsymbol{+} x_v = 1$, which, if we subtract 1 from each subscript, are the equations for $D_v^{(5)}$, that is, $D_{h-3}^{(5)}$.)

Setting, in $M_h^{(l)}$,	leads to
$x_1 = 1$	$D_{h-3}^{(l)}$
$x_1 = 0, x_2 = 1$	$D_{h-4}^{(l)}$
$x_1 = x_2 = 0, x_3 = 1$	$D_{h-5}^{(l)}$
…	…
$x_1 = \cdots = x_{l-4} = 0, x_{l-3} = 1$	$D_{h-l+1}^{(l)}$
$x_1 = \cdots = x_{l-3} = 0, x_{l-2} = x_{l-1} = 1$	$D_{h-l-1}^{(l)}$
$x_1 = \cdots = x_{l-2} = 0, x_{l-1} = x_l = 1$	$D_{h-l-2}^{(l)}$

Table 13: Expressing $M_h^{(l)}$ in terms of $D_n^{(l)}$.

We have therefore shown that for $l \geq 3$, $h \geq 3$,

$$M_h^{(l)} = D_{h-3}^{(l)} + D_{h-4}^{(l)} + D_{h-5}^{(l)} + \cdots + D_{h-l+1}^{(l)} + D_{h-l-1}^{(l)} + D_{h-l-2}^{(l)} . \quad (46)$$

Table 14 illustrates (46) in the case $l = 4$.

From (46) and Corollary 20, and taking into account the values of $M_h^{(l)}$ for $h < 3$, we obtain:

h	$M_h^{(4)}$	$D_{h-3}^{(4)}$	$D_{h-5}^{(4)}$	$D_{h-6}^{(4)}$
3	1	1	–	–
4	1	1	–	–
5	3	2	1	–
6	5	3	1	1
7	9	6	2	1
8	16	11	3	2
9	29	20	6	3

Table 14: Illustrating $M_h^{(4)} = D_{h-3}^{(4)} + D_{h-5}^{(4)} + D_{h-6}^{(4)}$.

Theorem 21. *For $l \geq 3$,*

$$\begin{aligned}\mathcal{M}^{(l)}(z) \;:=\; \sum_{h=1}^{\infty} M_h^{(l)} z^h \;&=\; z + (z^3 + z^4 + \cdots + z^{l-1} + z^{l+1} + z^{l+2})\,\mathcal{D}^{(l)}(z) \\ &=\; \frac{z(1-z)}{1-(z+z^2+\cdots+z^{l-2}+z^l+z^{l+1})} \\ &=\; \frac{z(1-z)^2}{1-2z+z^{l-1}-z^l+z^{l+2}}\,. \end{aligned} \tag{47}$$

It is now straightforward to obtain the recurrence for $M_h^{(l)}$ from the generating function in the second line of the display. We omit the proof.

Corollary 22. *For $l \geq 3$, $M_h^{(l)}$ satisfies the recurrence (34) with initial conditions (35), (36).*

We can now give the proof of Theorem 18. From the definition of $M_h^{(l)}$, we have

$$d_2(2^k - 3) \;=\; \sum_{l=1}^{k} M_{k-l+1}^{(l)}\,.$$

That is, $d_2(2^k-3)$ is the sum of the coefficient of z^k in $\mathcal{M}^{(1)}(z)$, the coefficient of z^{k-1} in $\mathcal{M}^{(2)}(z)$, ..., and the coefficient of z^1 in $\mathcal{M}^{(k)}(z)$. In other words, $d_2(2^k-3)$ is the coefficient of z^k in

$$\mathcal{M}^{(1)}(z) + z\mathcal{M}^{(2)}(z) + z^2\mathcal{M}^{(3)}(z) + \cdots + z^{k-1}\mathcal{M}^{(k)}(z)\,,$$

and now (32) follows from $\mathcal{M}^{(1)}(z) = z + z^3/(1-z)$, $\mathcal{M}^{(2)}(z) = 0$, and Theorem 21. This completes the proof of Theorem 18.

Remark. The $l = 3$ column of the $M_h^{(l)}$ table (Table 11) is an interesting sequence in its own right.[5] To analyze it directly, first consider the system of simultaneous Boolean equations

$$x_1 \boldsymbol{+} x_2 = x_2 \boldsymbol{+} x_3 \boldsymbol{+} \cdots \boldsymbol{+} x_{n-1} \boldsymbol{+} x_n = 1\,, \tag{48}$$

[5] It is entry A070550 in [17], which contains a comment by Ed Pegg, Jr., that it arises in the analysis of Penney's game.

for $n \geq 2$, involving a chain of linked pairs of variables. An easy induction shows that the number of solutions is the Fibonacci number F_{n+2} (cf. A000045).[6] Second, the equations for $M_h^{(3)}$, (38) and (39), break up into two disjoint chains like (48), and we find that

$$M_h^{(3)} = \begin{cases} F_{(n-2)/2}F_{n/2}, & \text{if } n \text{ is even;} \\ F_{(n-3)/2}F_{(n+1)/2}, & \text{if } n \text{ is odd.} \end{cases} \tag{49}$$

From (49) we can derive the recurrence $M_h^{(3)} = M_{h-1}^{(3)} + M_{h-3}^{(3)} + M_{h-4}^{(3)}$ and the generating function

$$\mathcal{M}^{(3)}(z) = \frac{z(1-z)}{1-z-z^3-z^4}, \tag{50}$$

in agreement with (34) and (47).

The final result in this section will describe the asymptotic behavior of the sequence $d_2(2^k-3)$. When investigating Conjectures 13 and 14, we observed that among all numbers n with k binary digits, the number 2^k-1 was the clear winner, with values close to the estimate (23). The runners-up, a long way behind, were 2^k-3 and $2^k-2^{k-2}-1$ (and sometimes other values of n), all with the same number of dismal divisors, for which the number of dismal divisors appeared to be converging to one-fifth of the number of divisors of the winner, or in other words it appeared that

$$\frac{d_2(2^k-3)}{d_2(2^k-1)} \rightarrow \frac{1}{5}, \text{ as } k \rightarrow \infty. \tag{51}$$

We will now establish this from the generating function (32).

Theorem 23.

$$d_2(2^k-3) \sim \frac{2^k}{5k\log 2}(1+\bar{\Theta}_k), \text{ as } k \rightarrow \infty, \tag{52}$$

where $\bar{\Theta}_k$ is a bounded oscillating function with $|\bar{\Theta}_k| < 10^{-5}$.

Proof. Our proof is modeled on Knopfmacher and Robbins's proof [12] of (23), which uses the method of Mellin transforms as presented by Flajolet, Gourdon, and Dumas [5]. We will indicate how the Knopfmacher-Robbins proof can be reworded so as to establish (23) and (52) simultaneously.

Knopfmacher and Robbins work, not with (22), but with

$$f(z) := \sum_{l=1}^{\infty} \frac{(1-z)z^l}{1-2z+z^l}, \tag{53}$$

which is the generating function for the number of compositions of n into parts of which the first is *strictly* greater than all the other parts (A007059). Equations (22) and (53) basically differ just by a factor of z. Then [12] shows that the coefficient of z^n in $f(z)$ is

$$\frac{2^{n-1}}{n\log 2}(1+\Theta), \tag{54}$$

[6]This result could also be obtained by the Goulden-Jackson cluster method, as implemented by Noonan and Zeilberger [8], [16].

for some small oscillating function Θ, which implies (23).

So as to have a function with the same form as (53), we consider, not (32), but

$$f(z) \;:=\; \sum_{l=2}^{\infty} \frac{(1-z)^2 z^l}{1-2z+z^l-z^{l+1}+z^{l+3}}\,, \tag{55}$$

and we will show that the coefficient of z^n is

$$\frac{2^{n+1}}{5n\log 2}(1+\Theta)\,, \tag{56}$$

for some (different) small oscillating function Θ, which implies (52). We can change the lower index of summation in (55) from 2 to 1, since the $l=1$ term is the generating function for the Padovan sequence (A000931), which grows at a much slower rate than (56).

In what follows, we simply record how the expressions in Knopfmacher and Robbins's proof [12] need to be modified so as to apply simultaneously to (53), which we refer to as case I, and (55), which we call case II. We follow Knopfmacher and Robbins's notation, except that we use j and m as local variables, rather than k, to avoid confusion with the k in the statement of the theorem. Some typographical errors in [12] have been silently corrected.

Let ρ_j denote the smallest root of the denominator of the j-th summand in $f(z)$ that lies between 0 and 1. Then

$$\rho_j \;=\; \frac{1}{2}\left(1+\tau 2^{-j}+O(j2^{-2j})\right), \tag{57}$$

where $\tau=1$ (case I) or 5/8 (case II).

Let $q_{n,j}$ denote the coefficient of z^n in the j-th summand in $f(z)$. Then

$$\begin{aligned} q_{n,j} &\approx 2^{n-j-\epsilon}\left(1-\frac{\tau}{2^j}\right)^n \\ &\approx 2^{n-j-\epsilon}\, e^{-\tau n/2^j}\,, \end{aligned} \tag{58}$$

where $\epsilon=1$ in case I or 2 in case II. Next, f_n, the coefficient of z^n in $f(z)$, is

$$2^{n-\epsilon}\left(\sum_{j=2}^{\infty} 2^{-j} e^{-\tau n/2^j} + o(1)\right).$$

Let

$$g(x) \;:=\; \sum_{j=2}^{\infty} 2^{-j} e^{-\tau x/2^j}\,.$$

The Mellin transform of $g(x)$ is

$$g^*(s) \;:=\; \frac{1}{\tau^s}\,\frac{2^{2(s-1)}}{1-2^{s-1}}\,\Gamma(s), \quad 0<\Re(s)<1\,.$$

To compute f_n we use (following Knopfmacher and Robbins) the Mellin inversion formula

$$g(x) \;=\; \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} x^{-s} g^*(s)\,ds\,.$$

Now $g^*(s)x^{-s}$ has a simple pole at $s = 1 + \chi_m$, for each $m \in \mathbb{Z}$, where $\chi_m = 2\pi i m/\log 2$, with residue

$$- \frac{1}{x \log 2} \frac{\Gamma(1+\chi_m)\, e^{-2\pi i m \log_2 x}}{\tau^{1+\chi_m}}.$$

After combining the contributions from all the poles, we have

$$f_n \;=\; 2^{n-\epsilon} \sum_{m=-\infty}^{\infty} \frac{1}{n \log 2} \frac{\Gamma(1+2\pi i m/\log 2)\, e^{-2\pi i m \log_2 n}}{\tau^{1+2\pi i m/\log 2}}. \tag{59}$$

The term for $m = 0$ dominates, and we obtain the desired results (54) and (56). $\square$

7 The sum of dismal divisors

We briefly discuss the dismal sum-of-divisors function $\sigma_b(n)$ (see A188548, A190632, and A087416 for bases 2, 3, and 10).

Theorem 24. *In any base $b \geq 2$, if $\mathrm{len}_b(n) = k$, then*

$$n \leq \sigma_b(n) \leq b^k - 1\,, \tag{60}$$

and $\sigma_b(n) = n$ if and only if $n \equiv b-1 \pmod b$.

Proof. The first assertion follows because n divides itself, and no divisor has length greater than k. If $n \not\equiv b-1 \pmod b$ then since $b-1$ is the multiplicative unit, $\sigma_b(n) \neq n$. Suppose $n \equiv b-1 \pmod b$ and p is a divisor of n, with say $p \times_b q = n$. Both p and q must end with β. From the long multiplication tableau, $p \ll_b n$, so $p +_b n = n$, and therefore $\sigma_b(n) = n$. $\square$

In ordinary arithmetic, a number n is perfect if its sum of divisors is $2n$. In dismal arithmetic, $n +_b n = n$. So the second part of the theorem might, by a stretch, be interpreted as saying that the numbers congruent to $b-1 \pmod b$ are the base b perfect dismal numbers.

In base 2, then, $\sigma_2(n) = n$ if and only if n is odd. Examination of the data shows that, if n is even, with $\mathrm{len}_2(n) = k$, often $\sigma_2(n)$ takes its maximal value, $2^k - 1$. Table 15 shows the first few exceptions, which are characterized in the next theorem.

Theorem 25. *Suppose $n = 2^r m$ with $r \geq 1$, m odd, and $\mathrm{len}_2(n) = k$. Then*

$$\sigma_2(n) \;=\; 2^k - 1 \;=\; \underbrace{11\ldots1}_{k}|_2$$

unless the binary expansion of m contains a run of more than r consecutive zeros.

Proof. Since m is odd, $\sigma_2(m) = m$. Therefore

$$\sigma_2(n) \;=\; m|_2 \;+_2\; m0|_2 \;+_2\; m00|_2 \;+_2\; \cdots \;+_2\; m\underbrace{00\ldots0}_{r}|_2\,,$$

and any string $\underbrace{00\ldots0}_{i}1$ in m will become $\underbrace{11\ldots1}_{i}1$ in $\sigma_2(n)$ unless i exceeds r. $\square$

n	$\sigma_2(n)$	n	$\sigma_2(n)$
10010	11011	1001010	1101111
100010	110011	1001110	1101111
100110	110111	1010010	1111011
110010	111011	1100010	1110011
1000010	1100011	1100110	1110111
1000100	1110111	1110010	1111011
1000110	1100111	......	......

Table 15: Even numbers n such that $\sigma_2(n)$ is not of the form $11\ldots1|_2$ (A190149–A190151). Both n and $\sigma_2(n)$ are written in base 2.

The first entry in Table 15 is explained by the fact that $n = 10010|_2$, $r = 1$, $m = 1001|_2$, and m contains a run of two zeros.

We also considered two other possible definitions of perfect numbers: (i) n is perfect in base $b \geq 3$ if the dismal sum of the dismal divisors of n is equal to $2 \times_b n$. We leave it to the reader to verify that for this to happen, b must be 3, and then n is perfect if and only if $n \equiv 2 \bmod 4$. (ii) n is perfect in base $b \geq 2$ if the dismal sum of the dismal divisors of n different from n is equal to n. But here n cannot be $b-1$, so $b-1$ is a divisor, and n ends with $b-1$. This implies that n has no divisors of length $\mathrm{len}_b(n)$ except n itself, so the sum cannot equal n, and therefore no such n exists.

We end this section with a conjecture (see A186442):

Conjecture 26. For all $n > 1$, $d_{10}(n) < \sigma_{10}(n)$.

8 Dismal partitions

Since $n +_b n = n$, it only makes sense to consider partitions into distinct parts (otherwise every number has infinitely many different partitions). We define $p_b(n)$ to be the number of ways of writing

$$n \;=\; m_1 +_b m_2 +_b \cdots +_b m_l \,, \tag{61}$$

for some $l \geq 1$ and distinct positive integers m_i, without regard to the order of summation. We set $p_b(0) = 1$ by convention.

For example, $p_3(7) = p_3(21|_3) = 22$, since (working in base 3) 21 is equal to 21 $+_3$ any subset of $\{20, 11, 10, 1\}$ (16 solutions), 20 $+_3$ 11 $+_3$ any subset of $\{10, 1\}$ (4 solutions), and 20 $+_3$ 1 $+_3$ any subset of $\{10\}$ (2 solutions), for a total of 22 solutions.

Remarks.

(i) Permuting the digits of n does not change $p_b(n)$.

(ii) Any zero digits in n can be ignored. If n' is the base b number obtained by dropping any n_i's that are zero, $p_b(n') = p_b(n)$.

(iii) Although we will not make any use of it, there is a generating function for the $p_b(n)$ analogous to that for the classical case. If we interpret $z^m \times_b z^n$ to mean $z^{m +_b n}$, then we have the formal power series

$$1+p_b(1)z+p_b(2)z^2+p_b(3)z^3+\cdots \;=\; (1+z)\,\text{✖}_b\,(1+z^2)\,\text{✖}_b\,(1+z^3)\,\text{✖}_b\,\cdots\,.$$

(iv) The sequences $p_2(n)$ and $p_{10}(n)$ form entries A054244 and A087079 in [17], contributed by the second author in 2000 and 2003, respectively.

In the remainder of this section we index the digits of n by $\{1,2,\ldots,n\}$, in order to simplify the discussion of subsets of these indices.

Theorem 27. *If $n=n_1n_2\ldots n_k|_2$, $n_i\in\{0,1\}$, and the binary weight of n is w, then $p_2(n)$ is equal to the number of set-covers of a labeled w-set by nonempty sets (cf. A003465, [4, p. 165]), that is,*

$$p_2(n) \;=\; \frac{1}{2}\sum_{i=0}^{w}(-1)^{w-i}\binom{w}{i}2^{2^i}\,. \tag{62}$$

Proof. From Remark (ii), $p_2(n)=p_2(\underbrace{11\ldots1}_{w}|_2)$. There is an obvious one-to-one correspondence between collections of distinct nonempty subsets of $\{1,\ldots,w\}$ whose union is $\{1,\ldots,w\}$ and sets of distinct nonzero binary vectors whose dismal sum is $\underbrace{11\ldots1}_{w}|_2$. ☐

Theorem 28. *If $n=n_1n_2\ldots n_k|_b$, $0\le n_i\le b-1$, then*

$$p_b(n) \;=\; \frac{1}{2}\sum_{S\subseteq\{1,\ldots,k\}}(-1)^{|S|}\,2^{\prod_i(n_i+\epsilon_i)}\,, \tag{63}$$

where $\epsilon_i=0$ if $i\in S$, $\epsilon_i=1$ if $i\notin S$.

Proof. The set Ω_n of $x\ll_b n$ is a partially ordered set (with respect to the operator $\ll_b$) with Möbius function given by [21, §3.8.4]

$$\mu(x_1\ldots x_k|_b,\,y_1\ldots y_k|_b) \;=\; \begin{cases}(-1)^{\sum_i(y_i-x_i)}, & \text{if } y_i-x_i=0 \text{ or } 1 \text{ for all } i;\\ 0, & \text{otherwise.}\end{cases} \tag{64}$$

Every subset of $\Omega_n\setminus\{0\}$ has dismal sum equal to some number $\ll_b n$, so we have

$$\sum_{x\ll_b n}p_b(x) \;=\; 2^{\prod_i(n_i+1)-1}\,.$$

From the Möbius inversion formula [21, §3.7.1] we get

$$p_b(n) \;=\; \frac{1}{2}\sum_{S\subseteq\{1,\ldots,k\}}(-1)^{|S|}\,2^{\prod_{i\in S}n_i\prod_{j\notin S}(n_j+1)}\,,$$

which implies (63). ☐

Theorem 28 reduces to Theorem 27 if all n_i are 0 or 1.

Corollary 29. *Suppose $n = n_1 n_2 \ldots n_k|_b$, and let y be the ordinary product $n_1 \times n_2 \times \cdots \times n_k$. Then $p_b(n)$ is divisible (in ordinary arithmetic!) by 2^{y-1}.*

Corollary 30. *For a single-digit number $n = n_1|_b$, $p_b(n) = 2^{n_1 - 1}$. For a two-digit number $n = n_1 n_2|_b$,*

$$p_b(n) \;=\; 2^{(n_1+1)(n_2+1)-2} \;-\; 2^{n_1 n_2 - 1}(2^{n_1} - 1)(2^{n_2} - 1)\,. \tag{65}$$

Eq. (65) was found by Wasserman [17, entry A087079].

It follows from the above discussion that in any base b, the only numbers n such that $p_b(n) = 1$ are $0|_b$, $1|_b$, $10|_b$, $100|_b$, $1000|_b, \ldots$, and that all other numbers n have the property that the dismal sum of the numbers $x \ll_b n$ is n. These two classes might be called "additive primes" and "additive perfect numbers."

9 Conclusion and future explorations

We have attempted to show that dismal arithmetic, despite its simple definition, is worth studying for the interesting problems that arise. We have left many questions unanswered: the "prime number theorem" of Conjecture 10, the questions about the numbers of divisors stated in Conjectures 12-14 (in particular, is it true that $11\ldots1|_{10}$ has more divisors than any other base 10 number with the same number of digits?), the base 10 dismal analog of $d(n) \le \sigma(n)$ (Conjecture 26), and the two questions about dismal squares at the end of §4—in particular, is there a recurrence for the sequence (19)? There are numerous other questions that we have not investigated (for example, if $x \ll_b y$, what can be said about $x \boldsymbol{\times}_b y$?).

We have made no mention of the complexity of deciding if a number is a dismal prime, or of finding dismal factorizations. In base 2 such questions reduce to solving a set of simultaneous quadratic Boolean equations, where a typical equation might be

$$(x_0 \boldsymbol{\times}_2 y_4) \;\boldsymbol{+}_2\; (x_1 \boldsymbol{\times}_2 y_3) \;\boldsymbol{+}_2\; \cdots \;\boldsymbol{+}_2\; (x_4 \boldsymbol{\times}_2 y_0) \;=\; 0 \text{ (or 1)}\,.$$

This becomes a question about the satisfiability of a complicated Boolean expression, and is likely to be hard to solve in general [7].

While we have focused on the cases $b = 2$ and $b = 10$, it would be nice to better understand the qualitative differences across a wider range of bases. For example, while $b = 2$ is a kind of Boolean arithmetic, does $b = 3$ correspond to a three-valued logic? Do odd b behave differently from even b? More generally, what other interesting mathematical structures might be modeled by dismal arithmetic?

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