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On the Base-b Expansion of the Number of Trailing Zeros of $b^k!$

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Abstract

Let $Z_b(n)$ denote the number of trailing zeroes in the base-b expansion of $n!$. In this paper we study the connection between the expression of $\vartheta(b) := \lim_{n\to\infty} Z_b(n)/n$ in base b, and that of $Z_b(b^k)$.

In particular, if b is a prime power, we will show the equality between the k digits of $Z_b(b^k)$ and the first k digits in the fractional part of $\vartheta(b)$. In the general case we will see that this equality still holds except for, at most, the last $\lfloor \log_b(k) + 3 \rfloor$ digits. We finally show that this bound can be improved if b is square-free and present some conjectures about this bound.

1 Introduction

In what follows we let $Z_b(n)$ denote the number of trailing zeroes in the base-b expansion of $n!$. It is a classic topic in elementary number theory to compute the number of trailing zeroes of the base-10 expansion of the factorial of an integer; i.e., $Z_{10}(n)$ (see [1, 2, 5] for instance and A027868 of *The On-Line Encyclopedia of Integer Sequences*). In fact, this question can be extended to arbitrary bases and the first author [6] studied in detail the behavior of the function Z_b (see A054861 for $b = 3$ and A011371 for $b = 2$).

This is an old topic which has recently drawn attention due to the increasing computing possibilities and to the popular of computer algebra systems. In fact, we can find many sequences in the *OEIS* which are related to this topic. For instance: A000966 (number of zeroes that $n!$ will never end in), A173558 (the smallest number whose factorial has 10^n trailing zeroes), A173292 (numbers whose factorial has exactly 10^n trailing zeroes), A181582 (smallest prime p such that $p!$ ends with exactly 10^n trailing zeroes) or A173345 (the number of trailing zeroes of n superfactorial) just to name a few.

In recent work (Hart et al. [3], motivated by Schmuland [7]), the following particularity about the number of trailing zeroes in the factorial of the powers of 10 (see A173228) was shown:

$$\begin{aligned}
Z_{10}(10) &= 2,\\
Z_{10}(10^2) &= 24,\\
Z_{10}(10^3) &= 249,\\
Z_{10}(10^4) &= 2499,\\
Z_{10}(10^5) &= 24999,\\
Z_{10}(10^6) &= 249998,\\
Z_{10}(10^7) &= 2499999,\\
Z_{10}(10^8) &= 24999999,\\
Z_{10}(10^9) &= 249999998,\\
Z_{10}(10^{10}) &= 2499999997.
\end{aligned}$$

Although the sequence of final 9's is broken for some values of the exponent, it seems clear that the number of 9's grows indefinitely. For instance,

$$\begin{aligned}
Z_{10}(10^{50}) &= 24999999999999999999999999999999999999999999999989,\\
Z_{10}(10^{100}) &= 2499999999999999999999999999999999999999999999999999999999\\
&\quad 9999999999999999999999999999999999999982.
\end{aligned}$$

This behavior and the fact that the number of digits of $Z_{10}(10^m)$ is m, motivated the study of the asymptotic behavior of $\dfrac{Z_{10}(10^k)}{10^k}$ by Treuden [8], where it was shown (as suggested by the previous examples) that

$$\lim_{n\to\infty} \frac{Z_{10}(n)}{n} = \frac{1}{4}.$$

Also, in [8] it was suggested (and was proved in [3]) that if $b = p_1^{r_1} \cdots p_s^{r_s}$, then

$$\lim_{n\to\infty} \frac{Z_b(n)}{n} = \min_{1\le i\le s} \frac{1}{r_i(p_i - 1)}.$$

Now, it is natural to wonder about what will be the behavior of the digits of $Z_b(b^k)$ for other values of b. Let us have a look at the case $k = 20$ for various values of b.

$$\begin{aligned}
Z_2(2^{20}) &= 1048575,\\
Z_3(3^{20}) &= 1743392200,\\
Z_4(4^{20}) &= 549755813887,\\
Z_5(5^{20}) &= 23841857910156,\\
Z_6(6^{20}) &= 1828079220031481,\\
Z_7(7^{20}) &= 13298711049602000,\\
Z_8(8^{20}) &= 384307168202282325,\\
Z_9(9^{20}) &= 3039416364764232200,\\
Z_{10}(10^{20}) &= 24999999999999999996,\\
Z_{11}(11^{20}) &= 67274999493256000920,\\
Z_{12}(12^{20}) &= 1916879996223737561074,\\
Z_{13}(13^{20}) &= 1583746981240066619900,\\
Z_{14}(14^{20}) &= 1394470923754746692675 9,\\
Z_{15}(15^{20}) &= 8313141825199127197265 2,\\
Z_{16}(16^{20}) &= 302231454903657293676543,\\
Z_{17}(17^{20}) &= 254014462915473282650100,\\
Z_{18}(18^{20}) &= 3187059054099019543609340,\\
Z_{19}(19^{20}) &= 2088331858752553232964200,\\
Z_{20}(20^{20}) &= 26214399999999999999999991,\\
Z_{21}(21^{20}) &= 46369738241158591439532728,\\
Z_{30}(30^{20}) &= 871696100249999999999999999987.
\end{aligned}$$

In the light of this data, it may seem that the only interesting behavior takes place at the multiples of 10. Nevertheless, this is not the case, as can be seen having a look at the base-b expansion of the considered number:

$$\begin{aligned}
Z_2(2^{20}) &= \{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1\}_2,\\
Z_3(3^{20}) &= \{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1\}_3,\\
Z_4(4^{20}) &= \{1,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3\}_4,\\
Z_5(5^{20}) &= \{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1\}_5,\\
Z_6(6^{20}) &= \{2,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,4,5\}_6,\\
Z_7(7^{20}) &= \{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1\}_7,\\
Z_8(8^{20}) &= \{2,5,2,5,2,5,2,5,2,5,2,5,2,5,2,5,2,5,2,5\}_8,\\
Z_9(9^{20}) &= \{2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\}_9,\\
Z_{10}(10^{20}) &= \{2,4,9,9,9,9,9,9,9,9,9,9,9,9,9,9,9,9,9,6\}_{10},
\end{aligned}$$

$Z_{11}(11^{20}) = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}_{11}$,
$Z_{12}(12^{20}) = \{5, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 10, 10\}_{12}$,
$Z_{13}(13^{20}) = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}_{13}$,
$Z_{14}(14^{20}) = \{2, 4, 9, 4, 9, 4, 9, 4, 9, 4, 9, 4, 9, 4, 9, 4, 9, 4, 9, 1\}_{14}$,
$Z_{15}(15^{20}) = \{3, 11, 3, 11, 3, 11, 3, 11, 3, 11, 3, 11, 3, 11, 3, 11, 3, 11, 3, 7\}_{15}$,
$Z_{16}(16^{20}) = \{3, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15\}_{16}$,
$Z_{17}(17^{20}) = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}_{17}$,
$Z_{18}(18^{20}) = \{4, 8, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 14\}_{18}$,
$Z_{19}(19^{20}) = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}_{19}$,
$Z_{20}(20^{20}) = \{4, 19, 19, 19, 19, 19, 19, 19, 19, 19, 19, 19, 19, 19, 19, 19, 19, 19, 19, 11\}_{20}$,
$Z_{21}(21^{20}) = \{3, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 5\}_{21}$,
$Z_{22}(22^{20}) = \{2, 4, 8, 17, 13, 4, 8, 17, 13, 4, 8, 17, 13, 4, 8, 17, 13, 4, 8, 14\}_{22}$,
$Z_{23}(23^{20}) = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}_{23}$,
$Z_{24}(24^{20}) = \{7, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 18\}_{24}$,
$Z_{25}(25^{20}) = \{3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3\}_{25}$.

Of course, the origin of these cyclic repetitions is closely related to the periodic expression of $\vartheta(b) = \lim_{n\to\infty} \dfrac{Z_b(n)}{n}$ when expressed in the base b. In this paper we study the relation between $\vartheta(b)$ and the base-b expansion of $Z_b(b^k)$. If b is a prime power, we will show the equality between the k digits of $Z_b(b^k)$ and the first k digits in the fractional part of $\vartheta(b)$ (i.e., $\lfloor \vartheta(b) b^k \rfloor$).

In the general case this equality does not hold. We are then interested in investigating the number of final digits of $Z_b(b^k)$ which break the previous coincidence. To do so, let us consider $\eta(b,k) := \lfloor \log_b(\lfloor \vartheta(b) b^k \rfloor - Z_b(b^k)) + 1 \rfloor$. This represents the number of digits of the base-b expansion of $\lfloor \vartheta(b) b^k \rfloor - Z_b(b^k)$ and observe that the number of unequal final digits between $\lfloor \vartheta(b) b^k \rfloor$ and $Z_b(b^k)$ is at most $\eta(b,k)+1$. We will show that

$$\eta(b,k) \le \lfloor \log_b k + 2 \rfloor$$

is the best possible upper bound for $\eta(b,k)$. Nevertheless, we will improve this upper bound in the case when b is square-free and present some conjectures about this bound.

Observe that if $\vartheta(b)$ admits a finite expansion in base b (we will say that $\vartheta(b)$ is *exact* in base b), then it admits two different b-adic expansions: the finite natural one and the infinite one. Throughout the paper we will always consider the latter and we will understand the floor function as the truncation of the fractional part. Consequently, if $\vartheta(b)$ is exact, then $\lfloor \vartheta(b) b^k \rfloor$ is to be interpreted as $\vartheta(b) b^k - 1$. For instance, if $b = 10$ and $k = 6$ then $\vartheta(10) = \frac{1}{4}$ and we will consider $\frac{1}{4} = 0.24999\cdots$ Thus $\lfloor \vartheta(10) 10^6 \rfloor$ will be interpreted as $249999 = 250000 - 1 = \vartheta(10) 10^6 - 1$.

The paper is organized as follows. In Section 2 we present the basic facts and technical results that will be used in the rest of the paper. In Section 3 we study the prime-power case, establishing the equality between the k digits of $Z_b(b^k)$ and the first k digits of the base-b expansion of $\vartheta(b)$. Finally, in Section 4, we study the general case, where the latter equality does not hold and give a bound to the number of unequal digits.

2 Technical results

In this section we present some technical results which will be very useful in the sequel. Some of them are well-known and are presented without proof. The section is divided into three parts, the first is devoted to results related to $Z_b(b^k)$, the second mostly deals with the base-b expansion of $\vartheta(b)$ when b is a prime-power and the third one is devoted to present the main lemma which will be crucial in the paper.

2.1 Some results about $Z_b(n)$

We start this subsection with the following well-known lemma, which was first proved by Legendre [4], that we present without proof.

Lemma 1.

1. $Z_p(n) = \sum_{i\geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor = \frac{n-\sigma_p(n)}{p-1}$, *where* $\sigma_p(n)$ *is the sum of the digits of the base-*p *expansion of* n.
2. $Z_{p^r}(n) = \left\lfloor \frac{Z_p(n)}{r} \right\rfloor$ *for every* $r \geq 1$.
3. *If* $b = p_1^{r_1} \cdots p_s^{r_s}$, *then* $Z_b(n) = \min_{1\leq i\leq s} Z_{p_i^{r_i}}(n)$.

As a straightforward consequence of the preceding lemma we obtain the following corollary.

Corollary 2. *Let* l *be any integer and* p *be a prime. Then*

$$Z_p(lp^n) = lZ_p(p^n) + Z_p(l).$$

Recall that $\vartheta(b) = \lim_{n\to\infty} \frac{Z_b(n)}{n}$. In previous work [3, 8] an explicit expression for $\vartheta(b)$ was given. Namely,

Proposition 3. *If* $b = p_1^{r_1} \cdots p_s^{r_s}$, *then*

$$\vartheta(b) = \min_{1\leq i\leq s} \frac{1}{r_i(p_i-1)}.$$

Remark 4. The sequence $S_n = \frac{1}{\vartheta(n)}$ appears in *The On-Line Encyclopedia of Integer Sequences* as sequence A090624. It is interesting to observe that it was included in the Encyclopedia 5 years before the formula for $\vartheta(n)$ was found.

The following lemma will be of great importance in sections 3 and 4.

Lemma 5. *Let $k \geq 0$ be an integer.*

1. *If $b > 1$ is an integer, then:*

$$0 < Z_b(b^{k+1}) - bZ_b(b^k).$$

2. *If b is a prime power, then:*

$$0 < Z_b(b^{k+1}) - bZ_b(b^k) < b.$$

Proof. 1. Fist of all observe that $k\lfloor x \rfloor \leq \lfloor kx \rfloor$ for all $k \in \mathbb{Z}$, $x \in \mathbb{R}$ and that if $x \notin \mathbb{Z}$ and $kx \in \mathbb{Z}$ the inequality is strict. Now, for some prime divisor of b (with exponent r in the decomposition of b) we have that:

$$bZ_b(b^k) = b\left\lfloor \frac{1}{r}\sum_{i\geq 1}\left\lfloor \frac{b^k}{p^i}\right\rfloor\right\rfloor < \left\lfloor \frac{1}{r}\sum_{i\geq 1}\left\lfloor \frac{b^{k+1}}{p^i}\right\rfloor\right\rfloor = Z_b(b^{k+1}),$$

since $\dfrac{b^k}{p^{r(k+1)}}$ is not an integer, while $b\dfrac{b^k}{p^{r(k+1)}}$ is.

2. Put $b = p^n$. Then, Corollary 2 implies that $Z_p(p^{(k+1)n}) = p^n Z_p(p^{kn}) + Z_p(p^n)$. Now, if r is the reminder of the division between $Z_p(p^{(k+1)n})$ and n and s is the reminder of the division between $Z_p(p^{kn})$ and n it follows that

$$Z_{p^n}(p^{(k+1)n}) = \frac{Z_p(p^{(k+1)n}) - r}{n},$$

$$Z_{p^n}(p^{kn}) = \frac{Z_p(p^{kn}) - s}{n}.$$

Thus, since $0 \leq r, s \leq n-1$, we have

$$\begin{aligned} Z_{p^n}(p^{(k+1)n}) - p^n Z_{p^n}(p^{kn}) &= \frac{Z_p(p^n) + p^n s - r}{n} \leq \frac{Z_p(p^n) + p^n s}{n} \\ &= \frac{\frac{p^n-1}{p-1} + p^n s}{n} < \frac{p^n + p^n s}{n} \\ &\leq \frac{p^n + p^n(n-1)}{n} = p^n. \end{aligned}$$

$\square$

2.2 The base-p^n expansion of $\vartheta(p^n)$

Let us start by introducing some notation. With $q = \{d.d_1d_2\cdots d_t\,\overbrace{d_{t+1}\cdots d_{t+s}}\}_b$ we mean that the fractional part of q in base b consists of t digits followed by a periodic sequence of s digits ($d_{t+i} = d_{t+i+s}$ for all $i > 0$). Clearly t can be arbitrarily large and the length of the period can be any multiple of s, so we will usually assume that t and s are minimal. We will say that q is exact in base b if there exists $k \geq 1$ such that $d_i = 0$ for every $i \geq k$ or $d_i = b-1$ for every $i \geq k$; i.e.,

$$q = \{d.d_1d_2\cdots d_t\,\overbrace{b-1}\}_b = \{d.d_1d_2\cdots d_t+1\,\overbrace{\ 0\ }\}_b$$

Lemma 6. *Let p be a prime and $1 \leq r \in \mathbb{Z}$. Then $\dfrac{1}{r(p-1)}$ is exact in base p^r if and only if $p = 2$ and r is a power of 2.*

Proof. This is a straightforward consequence from the fact that $\dfrac{1}{n}$ is exact in base p^r if and only if $\mathrm{rad}(n) = p$; i.e., if and only if n is a power of p. $\square$

Now we will present some results about the base-b expansion of $\vartheta(b)$ when $b = p^n$ is a prime-power.

Lemma 7. *Let p be a prime and $b = p^n$ with $n \in \mathbb{N}$. Then*

$$\vartheta(p^n) = \{0.d_1d_2\cdots d_t\,\overbrace{d_{t+1}\cdots d_{t+s}}\}_b \Longleftrightarrow \frac{p^{nt}}{n}\sum_{i=0}^{sn-1} p^i \in \mathbb{Z}.$$

Proof. First of all observe that

$$\frac{p^{nt}}{n}\sum_{i=0}^{sn-1} p^i = \frac{p^{nt}(p^{sn}-1)}{n(p-1)} = b^t\vartheta(p^n)(b^s-1).$$

Let us suppose that $\vartheta(p^n) = \{0.d_1d_2\cdots d_t\,\overbrace{d_{t+1}\cdots d_{t+s}}\}_b$. Then we have that

$$\begin{aligned} b^t\vartheta(p^n) &= \{d_1d_2\cdots d_t.\,\overbrace{d_{t+1}\cdots d_{t+s}}\}_b, \\ b^{t+s}\vartheta(p^n) &= \{d_1d_2\cdots d_td_{t+1}\cdots d_{t+s}.\,\overbrace{d_{t+1}\cdots d_{t+s}}\}_b. \end{aligned}$$

and it is enough to subtract both expressions to obtain that $b^t\vartheta(p^n)(b^s-1) \in \mathbb{Z}$.

Conversely, assume that $b^t\vartheta(p^n)(b^s-1) \in \mathbb{Z}$. It is easy to see that there exists a sequence $\{d_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$, not eventually null, such that

$$\vartheta(p^n) = \sum_{i=1}^{\infty} d_ip^{-ni} \text{ with } 0 \leq d_i < b,\ \forall i \in \mathbb{N}.$$

Consequently,

$$b^t\vartheta(p^n) = z_1 + \sum_{i=1}^{\infty} d_{t+i}p^{-ni} \text{ with } z_1 \in \mathbb{Z},$$
$$b^{(t+s)}\vartheta(p^n) = z_2 + \sum_{i=1}^{\infty} d_{t+s+i}p^{-ni} \text{ with } z_2 \in \mathbb{Z},$$

where

$$0 < \sum_{i=1}^{\infty} d_{t+i}p^{-ni} \leq 1,$$
$$0 < \sum_{i=1}^{\infty} d_{t+s+i}p^{-ni} \leq 1.$$

Since $b^{(t+s)}\vartheta(p^n) - b^t\vartheta(p^n) = b^t\vartheta(p^n)(b^s - 1) \in \mathbb{Z}$ we have that

$$\sum_{i=1}^{\infty} d_{t+s+i}p^{-ni} = \sum_{i=1}^{\infty} d_{t+i}p^{-ni}.$$

From this fact it readily follows that $d_{t+i} = d_{t+s+i}$ and the proof is complete. □

Lemma 8. *Let p be a prime, $b = p^n$, and $(n,s,k) \in \mathbb{N}^3$. Then*

$$\frac{b^k}{n}\sum_{i=0}^{sn-1} p^i \in \mathbb{Z} \Longleftrightarrow \frac{b}{n}\sum_{i=0}^{sn-1} p^i \in \mathbb{Z}.$$

Or, in other words,

$$\frac{p^{nk}(p^{sn}-1)}{n(p-1)} \in \mathbb{Z} \Longleftrightarrow \frac{p^n(p^{sn}-1)}{n(p-1)} \in \mathbb{Z}.$$

Proof. We can write $n = p^r n'$ with $r \geq 0$ and $\gcd(p,n') = 1$. Then $b^k = p^{kn} = p^{p^r n' k}$ and observe that $r < p^r < p^r n' < p^r n' k$.

Assume that $\frac{b^k}{n}\sum_{i=0}^{sn-1} p^i \in \mathbb{Z}$, then $\frac{p^{p^r n' k - r}}{n'}\sum_{i=0}^{sn-1} p^i \in \mathbb{Z}$ and, since $\gcd(p,n') = 1$ it follows that $\frac{1}{n'}\sum_{i=0}^{sn-1} p^i \in \mathbb{Z}$ and $\frac{p^{p^r n' - r}}{n'}\sum_{i=0}^{sn-1} p^i \in \mathbb{Z}$. But $\frac{p^{p^r n' - r}}{n'} = \frac{b}{n}$ and we are done.

The converse is obvious. □

Corollary 9. *The base-p^n expansion of $\vartheta(p^n)$ is pure periodic or mixed periodic with only one non-periodic figure; i.e., either $\vartheta(p^n) = \{0.d_1\overbrace{d_2\cdots d_{s+1}}\}_{p^n}$ with $d_{s+1} \neq d_1$ or $\vartheta(p^n) = \{0.\overbrace{d_1\cdots d_s}\}_{p^n}$*

Proof. If $\vartheta(p^n) = \{0.d_1d_2\cdots d_t\overbrace{d_{t+1}\cdots d_{t+s}}\}_{p^n}$, then we have that $\frac{p^{nt}}{n}\sum_{i=0}^{sn-1} p^i \in \mathbb{Z}$. By the previous lemma this implies that $\frac{p^n}{n}\sum_{i=0}^{sn-1} p^i \in \mathbb{Z}$ and, consequently, $\vartheta(p^n) = \{0.d_1\overbrace{d_2\cdots d_{s+1}}\}_{p^n}$. Finally, if $d_{s+1} = d_1$, then $\vartheta(p^n)$ is pure periodic and this completes the proof. □

2.3 The main lemma

The following lemma will be crucial in the next section.

Lemma 10. *Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of integers and define $\Delta_1 := S_1$ and $\Delta_n := S_{n+1}-bS_n$. If $0 < \Delta_n < b$, for all $n \in \mathbb{N}$, then the following items hold:*

1. $S_n = \displaystyle\sum_{i=1}^{n} b^{n-i}\Delta_i$.
2. $\lfloor \log_b S_n \rfloor + 1 = n$.
3. $\ell := \displaystyle\lim_{n\to\infty} \frac{S_n}{b^n} = \sum_{i=1}^{\infty} b^{-i}\Delta_i$.
4. *If $\Delta_i = (b-1)$ for all $i > 1$, then $S_k = \ell b^k - 1$ for every $k > 1$.*
5. *If ℓ is not exact in base b, then $S_k = \lfloor \ell b^k \rfloor$ for every k.*
6. *If $\ell = \sum_{i=1}^{\infty} b^{-i}\Delta_n \in \mathbb{Q}$, then there exists $(t,s) \in \mathbb{N}^2$ with $s > 0$ such that $\Delta_{n+s} = \Delta_n$ (and $S_{n+s} = S_n$) for all $n > t$.*

Proof.

1. It follows from inductive arguments, since $S_1 = \Delta_1$.
2. Consequence of (1).
3. Observe that $\displaystyle\frac{S_n}{b^n} = \sum_{i=1}^{n} b^{-i}\Delta_i$ and take limits.
4. We must consider two cases.

   If $\Delta_1 < (b-1)$, then $\ell = \frac{(\Delta_1+1)}{b}$, $\ell b^k - 1 = (\Delta_1+1)b^{k-1} - 1$ and also
   $$S_k = \sum_{i=1}^{k} b^{k-i}\Delta_i = \Delta_1 b^{k-1} + \sum_{i=2}^{k} b^{k-i}(b-1) = \Delta_1 b^{k-1} + b^{k-1} - 1.$$
   Now, if $\Delta_1 = b-1$, then $\ell = 1$ and
   $$S_k = \sum_{i=1}^{k} b^{k-i}(b-1) = \sum_{i=1}^{k} b^{k+1-i} - \sum_{i=1}^{k} b^{k-i} = b^k - 1.$$
5. Observe that
   $$\ell b^k = \sum_{i=1}^{\infty} b^{k-i}\Delta_i = \sum_{i=1}^{k} b^{k-i}\Delta_i + \sum_{i=1}^{\infty} b^{-i}\Delta_{i+k}.$$
   Now, if ℓ is not exact, it follows that $\displaystyle\sum_{i=1}^{\infty} b^{-i}\Delta_{i+k} < 1$ and consequently,
   $$\lfloor \ell b^k \rfloor = \left\lfloor \sum_{i=1}^{k} b^{k-i}\Delta_i + \sum_{i=1}^{\infty} b^{-i}\Delta_i \right\rfloor = \sum_{i=1}^{k} b^{k-i}\Delta_i = S_k.$$

6. It is clear since the base-b expansion of any rational number is periodic.

$\square$

3 The prime power case

The next theorem establishes the equality between the digits of the base-b expansion of $Z_b(b^k)$ and the first k digits of the base-b expansion of $\vartheta(b)$ if b is a prime power. In passing we will also prove some other interesting properties.

Theorem 11. *Let p be a prime and $b = p^n$. Consider the sequence $a_1 = Z_b(b)$, $a_k := Z_b(b^{(k+1)}) - bZ_b(b^k)$ and let s be the smallest integer such that $\theta := \frac{b}{n}\sum_{i=0}^{sn-1} p^i \in \mathbb{Z}$. Then, the following items hold:*

1. $Z_b(b^k) = \sum_{i=1}^{k} a_i b^{k-i}$.
2. $\vartheta(b) = \lim_{n\to\infty} \frac{Z_b(b^n)}{b^n} = \sum_{i=1}^{\infty} b^{-i} a_i$.
3. *The base-b expansion of $\vartheta(b)$ is*
$$\vartheta(b) = \begin{cases} \{0.a_1\,\overbrace{a_2\cdots a_{s+1}}\}_b, & \text{if } \frac{\theta}{b} \notin \mathbb{Z}\ ; \\ \{0.\,\overbrace{a_1 a_2\cdots a_s}\}_b & \text{if } \frac{\theta}{b} \in \mathbb{Z}\ . \end{cases}$$
4. $a_k = a_{k+s}$ *for all* $k > 1$. *Moreover, if* $\frac{\theta}{b} \in \mathbb{Z}$, *then* $a_k = a_{k+s}$ *for all* $k > 0$.
5. $$\#\{a_k\}_{i\in\mathbb{N}} = \begin{cases} s, & \text{if } \frac{\theta}{b} \in \mathbb{Z}; \\ s+1, & \text{otherwise.} \end{cases}$$
6. *If* $\frac{\theta}{b} \in \mathbb{Z}$, *then* $Z_b(b^s) = \frac{\theta}{b}$. *Otherwise,* $Z_b(b^{s+1}) = \theta + Z_b(b)$.
7. *If* $b+1$ *is not a Fermat number, then* $Z_b(b^k) = \lfloor b^k \vartheta(b) \rfloor$.
8. *If* $b+1$ *is a Fermat number, then* $Z_b(b^k) = b^k\vartheta(b) - 1$.

Proof. First of all observe that, due to Corollary 9, there exists an integer s such that $\theta = \frac{b}{n}\sum_{i=0}^{sn-1} p^i \in \mathbb{Z}$ or, equivalently, such that $\vartheta(p^n) = \{0.d_1\,\overbrace{d_2\cdots d_{s+1}}\}_b$. Moreover, if $\frac{1}{n}\sum_{i=0}^{sn-1} p^i \in \mathbb{Z}$, then $\vartheta(p^n) = \{0.\,\overbrace{d_1\cdots d_s}\}_b$. Also, by Lemma 2, we have that $0 < a_k < b$ so we are in the conditions of Lemma 10.

After these considerations we can proceed to the proof of the theorem.

1. Apply Lemma 10(1).

2. Apply Lemma 10(3).

3. Due to Corollary 9.

4. Idem.

5. Obvious by the minimality of s.

6. If $\frac{\theta}{b} \in \mathbb{Z}$, then $\frac{1}{n}\sum_{i=0}^{sn-1} p^i = \frac{\theta}{b} \in \mathbb{Z}$. This implies that $\vartheta(b) = \{0.\overbrace{a_1 a_2 \cdots a_s}\}_b$, so $b^s\vartheta(b) = \{a_1 a_2 \cdots a_s.\overbrace{a_1 a_2 \cdots a_s}\}_b$ and, consequently,

$$\frac{\theta}{b} = \vartheta(b)(b^s - 1) = \sum_{i=1}^{s} a_i b^{k-i} = Z_b(b^s).$$

   Now, if $\frac{\theta}{b} \notin \mathbb{Z}$ then $\frac{b}{n}\sum_{i=0}^{sn-1} p^i = \theta \in \mathbb{Z}$ so $\vartheta(b) = \{0.a_1\overbrace{a_2 a_3 \ldots a_{s+1}}\}_b$ and it follows that

$$\begin{aligned} b^{s+1}\vartheta(b) &= \{a_1 a_2 \cdots a_{s+1}.\overbrace{a_2 a_3 \cdots a_{s+1}}\}_b, \\ b\vartheta(b) &= \{a_1.\overbrace{a_2 a_3 \cdots a_{s+1}}\}_b. \end{aligned}$$

   Consequently,

$$\theta = \frac{b}{n}\sum_{i=0}^{sn-1} p^i = b\vartheta(b)(b^s - 1) = \sum_{i=1}^{s+1} a_i b^{k-i} - a_1 = Z_b(b^{s+1}) - Z_b(b).$$

7. If $p^n + 1$ is not a Fermat number, then $\vartheta(p^n)$ is not exact due to Lemma 7. Then, it is enough to apply Lemma 10(5).

8. If $p^n + 1$ is a Fermat number; then $p = 2$ and n is a power of 2. In this case $\vartheta(p^n)$ is exact and Lemma 10(4) applies.

□

Let us recall that a base-b repunit with k digits, $R_k^{(b)}$, is an integer whose base-b expansion consists exactly of k ones; i.e.,

$$R_k^{(b)} := \{\overbrace{1, 1, \ldots, 1}^{k}\}_b = \sum_{i=0}^{k-1} b^i = \frac{b^k - 1}{b - 1}.$$

In the same way, a base-b repdigit with k digits is a multiple of a base-b repunit with k digits, i.e., an integer of the form $\alpha R_k^{(b)} = \{\overbrace{\alpha, \alpha, \ldots, \alpha}^{k}\}_b$ with $1 \leq \alpha \leq b - 1$.

Proposition 12. *Let p be a prime. If* $\dfrac{1}{n}\sum_{i=0}^{n-1} p^i = \dfrac{R_n^{(p)}}{n} \in \mathbb{Z}$*, then $Z_{p^n}(p^{nk})$ is a base-p^n repdigit with k digits for all $k \in \mathbb{Z}$. Namely,*

$$Z_{p^n}(p^{nk}) = \frac{R_n^{(p)} R_k^{(p^n)}}{n}.$$

Proof. Theorem 11 implies that $\vartheta(p^n) = \{0.\overbrace{a_1}\}_{p^n}$, where $a_1 = \dfrac{R_n^{(p)}}{n}$. Consequently,

$$Z_{p^n}(p^{nk}) = \{\overbrace{a_1, a_1, \ldots, a_1}^{k}\}_{p^n} = \frac{R_n^{(p)}}{n} R_k^{(p^n)}.$$

□

It is interesting to particularize the previous result for $n = 1, 2$.

Corollary 13. *Let p be a prime. Then $Z_p(p^k)$ is a base-p repunit with k digits for every integer k.*

Proof. Follows immediately from the previous proposition, since $\dfrac{R_1^{(p)}}{1} = 1 \in \mathbb{Z}$. □

Corollary 14. *Let p be an odd prime, then $Z_{p^2}(p^{2k})$ is a base-p^2 repdigit.*

Proof. If p is odd, then $\dfrac{R_2^{(p)}}{2} = \dfrac{p+1}{2} \in \mathbb{Z}$. □

If p is odd, the above corollary can be generalized for any power of 2. Namely, we have the following.

Proposition 15. *If p is an odd prime, then $Z_{p^{2^m}}(p^{2^m k})$ is a base-$p^{2^m k}$ repdigit.*

Proof. $\dfrac{R_{2^m}^{(p)}}{2^m} = \dfrac{1}{2^m}\displaystyle\prod_{i=0}^{m-1}(1 + p^{2^i}) \in \mathbb{Z}$. □

Remark 16. We have seen that $\vartheta(p) = \{0.\overbrace{1}\}_p$ for every prime p. Nevertheless, the set of pairs $(b_1, b_2) \in \mathbb{Z}^2$ such that the base-b_1 expansion of $\vartheta(b_1)$ and the base b_2 expansion of $\vartheta(b_2)$ coincide seems to be very small. In fact for $b_i \leq 40000$ there are only two such couples. Namely,

$$\vartheta(81) = \{0.\overbrace{10}\}_{81} \text{ and } \vartheta(361) = \{0.\overbrace{10}\}_{361}.$$
$$\vartheta(343) = \{0.\overbrace{19}\}_{343} \text{ and } \vartheta(1369) = \{0.\overbrace{19}\}_{1369}.$$

4 The general case

If b is not a prime power, there is no equality between the k digits of $Z_b(b^k)$ and the first k digits of the base-b expansion of $\vartheta(b)$. As a consequence, $Z_b(b^k)$ presents certain anomalies in its final digits (see A174807). For instance,

$$\begin{aligned} Z_{10}(10^9) &= 24999999\mathbf{8}, \\ Z_{10}(10^{99}) &= 249999999\cdots 999999\mathbf{80}, \\ Z_{10}(10^{999}) &= 249999999\cdots 99999\mathbf{791}, \\ Z_{10}(10^{9999}) &= 249999999\cdots 99999\mathbf{7859}. \end{aligned}$$

Or, in a different base,

$$\begin{aligned} Z_6(6^5) &= \{2,5,5,5,\mathbf{4}\}_6, \\ Z_6(6^{6^2-1}) &= \{2,5,5,5,5,5,5,5,5,5,5,\ldots,5,5,5,5,5,5,5,5,5,5,5,5,5,5,\mathbf{4},\mathbf{1}\}_6, \\ Z_6(6^{6^3-1}) &= \{2,5,5,5,5,5,5,5,5,5,5,\ldots,5,5,5,5,5,5,5,5,5,5,5,5,5,5,\mathbf{4},\mathbf{1},\mathbf{5}\}_6, \\ Z_6(6^{6^4-1}) &= \{2,5,5,5,5,5,5,5,5,5,5,\ldots,5,5,5,5,5,5,5,5,5,5,5,5,5,5,\mathbf{4},\mathbf{0},\mathbf{3},\mathbf{4}\}_6, \\ Z_6(6^{6^5-1}) &= \{2,5,5,5,5,5,5,5,5,5,5,\ldots,5,5,5,5,5,5,5,5,5,5,5,5,5,5,\mathbf{4},\mathbf{0},\mathbf{3},\mathbf{3},\mathbf{4}\}_6. \end{aligned}$$

If $\vartheta(b)$ is not exact it is clear that any convergent sequence with limit $\vartheta(b)$ will share with this value an increasing number of digits. To prove that this is still true even if $\vartheta(b)$ is exact (like in the previous examples, where $\vartheta(10) = \{0.25\}_{10}$ and $\vartheta(6) = \{0.3\}_6$) we need to prove the following result.

Proposition 17. *The sequence* $\{\gamma_k\}_{k\geq 1} := \left\{\frac{Z_b(b^k)}{b^k}\right\}_{k\geq 1}$ *is strictly increasing. As a consequence,* $\dfrac{Z_b(b^k)}{b^k} < \lim_{n\to\infty} \dfrac{Z_b(n)}{n}$ *for every* $k > 0$.

Proof. $\dfrac{\gamma_{k+1}}{\gamma_k} = \dfrac{Z_b(b^{k+1})}{bZ_b(b^k)} > 1$ due to Lemma 5(1). □

We have already seen in the previous section that if b is a prime power, then the number of digits of the base-b expansion of $Z_b(b^k)$ is exactly k. Now we will see that this is also true for a general b.

Proposition 18. *The number of digits of the base-b expansion of $Z_b(b^k)$ is exactly k; i.e.,*

$$\lfloor \log_b Z_b(b^k) \rfloor + 1 = k.$$

Proof. By Lemma 2 (1), we know that $bZ_b(b^k) < Z_b(b^{(k+1)})$. Taking logarithms we have $1 + \log_b Z_b(b^k) < \log_b Z_b(b^{k+1})$, which clearly implies that $\lfloor \log_b Z_b(b^k) \rfloor < \lfloor \log_b Z_b(b^{k+1}) \rfloor$.

Thus, the number of digits of the base-b expansion of $Z_b(b^{k+1})$ is greater than that of $Z_b(b^k)$. Since $\lfloor \log_b Z_b(b) \rfloor = 0$, it follows that $1 + \lfloor \log_b Z_b(b^k) \rfloor \geq k$.

Let us see now that the equality holds. Assume, on the contrary, that $1 + \lfloor \log_b Z_b(b^{k_0}) \rfloor > k_0$ for certain k_0. Then $1 + \lfloor \log_b Z_b(b^m) \rfloor > m$ for every $m \geq k_0$. This clearly implies that

$\frac{Z_b(b^m)}{b^m} > 1$ for every $m > k_0$ and $\vartheta(b) = \lim_{n\to\infty} \frac{Z_b(b^m)}{b^m} \geq 1$, which is clearly a contradiction since by definition $\vartheta(b) \leq 1$, the equality only holds for $b = 2$ and $\frac{Z_2(2^m)}{2^m} = \frac{2^m - 1}{2^m} < 1$. $\square$

We have seen that in the general case the equality between the k digits of $Z_b(b^k)$ and the first k digits of the base-b expansion of $\vartheta(b)$ does not hold. It is then interesting to study how many digits differ.

To do so, let us introduce some notation:

$$\alpha(b,k) = \lfloor \vartheta(b)b^k \rfloor - Z_b(b^k).$$

$$\eta(b,k) = \lfloor \log_b \alpha(b,k) + 1 \rfloor.$$

Observe that the number of different digits that we are studying is, at most, $\eta(b,k)+1$. Now we can give an upper bound for $\eta(b,k)$.

Theorem 19. *The number of digits of the base-b expansion of $\alpha(b,k)$ is smaller or equal than the number of digits of the base-b expansion of k, plus 1; i.e.,*

$$\eta(b,k) \leq \lfloor \log_b k + 2 \rfloor.$$

Proof. If $k = 1$, then $\lfloor \vartheta(b)b \rfloor - Z_b(b) \leq b$, since $\vartheta(b) \leq 1$ and $Z_b(b) \geq 0$. This implies that $\eta(b,1) = \lfloor \log_b (\lfloor \vartheta(b)b \rfloor - Z_b(b)) + 1 \rfloor \leq \lfloor \log_b b + 1 \rfloor = 2 = \lfloor \log_b k + 2 \rfloor$, as claimed.

Now, let $k \geq 2$. Put $b = p_1^{r_1} \cdots p_s^{r_s}$ and assume, without loss of generality, that p_1 is such that $\min_{1\leq i\leq s} \frac{1}{r_i(p_i - 1)} = \frac{1}{r_1(p_1 - 1)}$. In that case $Z_b(b^k) = \left\lfloor \frac{b^k - \sigma_{p_1}(b^k)}{r_1(p_1 - 1)} \right\rfloor$ and $\vartheta(b) = \frac{1}{r_1(p_1 - 1)}$. Also observe that if $\beta = \frac{b}{p_1^{r_1}}$, then $\sigma_{p_1}(b^k) = \sigma_{p_1}(\beta^k)$. Now

$$\begin{aligned} \lfloor \vartheta(b)b^k \rfloor - Z_b(b^k) &= \lfloor \vartheta(b)b^k \rfloor - \left\lfloor \frac{b^k - \sigma_{p_1}(b^k)}{r_1(p_1 - 1)} \right\rfloor \leq \frac{b^k - (b^k - \sigma_{p_1}(b^k))}{r_1(p_1 - 1)} + 1 \\ &= 1 + \frac{\sigma_{p_1}(\beta^k)}{r_1(p_1 - 1)} \leq 1 + \frac{(p_1 - 1)(1 + \lfloor \log_{p_1} \beta^k \rfloor}{r_1(p_1 - 1)} \\ &= 1 + \frac{1 + \lfloor \log_{p_1} \beta^k \rfloor}{r_1} \leq 2 + \lfloor \log_{p_1} \beta^k \rfloor = \lfloor \log_{p_1} p_1^2 \beta^k \rfloor \\ &\leq \lfloor \log_{p_1} b^k \rfloor. \end{aligned}$$

Consequently we find

$$\begin{aligned} \eta(b,k) &= \left\lfloor \log_b \left(\lfloor \vartheta(b)b^k \rfloor - Z_b(b^k) \right) + 1 \right\rfloor \leq \left\lfloor \log_b \left(\lfloor \log_{p_1} b^k \rfloor \right) \right\rfloor + 1 \leq \lfloor \log_b (k \log_{p_1} b) \rfloor + 1 \\ &= \lfloor \log_b k + \log_b \log_{p_1} b \rfloor + 1 \leq \lfloor \log_b k \rfloor + 2 = \lfloor \log_b k + 2 \rfloor. \end{aligned}$$

$\square$

Remark 20. The bound obtained in the previous theorem is the best possible one. In fact, there exists values of the pair (b,k) such that $\eta(b,k) = \lfloor \log_b k + 2 \rfloor$. Namely, if $k = b-1$ and $b < 1000$ the following values:

$$b = 120,\ 180,\ 240,\ 336,\ 360,\ 378,\ 420,\ 448,\ 504,\ 560,\ 630,\ 672,\ 720,\ 756,\ 840,\ 945$$

satisfy the equation $\eta(b, b-1) = 2 = \lfloor \log_b(b-1) + 2 \rfloor$.

Corollary 21. *The number of unequal digits between $Z_b(b^k)$ and the first k digits of the base-b expansion of $\vartheta(b)$ is smaller or equal than the number of digits of k plus 2.*

Proof. The number of unequal digits is, at most, $\eta(b,k)+1$ which, by the previous theorem, is smaller or equal than $(\lfloor \log_b k \rfloor + 1) + 2$. □

Remark 22. It is interesting to observe that, as far as the authors have been able to test computationally, the inequality given in the preceding corollary is strict. Nevertheless we have not found a proof for this fact, so it remains a conjecture.

Conjecture 23. The number of unequal digits between $Z_b(b^k)$ and the first k digits of the base-b expansion of $\vartheta(b)$ is smaller or equal than the number of digits of k plus 1.

If b is square-free, we can improve the bound given in Theorem 19.

Proposition 24. *Let $b = p_1 \cdots p_s$ be a square-free integer ($s \geq 2$). Then $\vartheta(b)b^k - Z_b(b^k) \leq k(s-1)$ for every k. As a consequence $\eta(b,k) \leq \lfloor \log_b k + \log_b(s-1) + 1 \rfloor \leq \lfloor \log_b k + 1.21 \rfloor$.*

Proof. We can suppose, without loss of generality, that p_1 is the greatest prime in the decomposition of b. Then $\vartheta(b)b^k = \dfrac{b^k}{p_1 - 1}$ and also $Z_b(b^k) = Z_{p_1}(b^k) = \dfrac{b^k - \sigma_{p_1}(b^k)}{p_1 - 1}$. Consequently $\vartheta(b)b^k - Z_b(b^k) = \dfrac{\sigma_{p_1}(b^k)}{p_1 - 1} = \dfrac{\sigma_{p_1}(p_2^k \cdots p_s^k)}{p_1 - 1}$.

Now, since $(p_2 \cdots p_s)^k < p_1^{k(s-1)}$ it follows that $\sigma_{p_1}(p_2^k \cdots p_s^k) \leq (p_1 - 1)k(s-1)$ so we get that $\vartheta(b)b^k - Z_b(b^k) \leq k(s-1)$.

To end the proof it is enough to recall the definition of $\eta(b,k)$ and to observe that $\log_b(s-1) \leq \log_{30}(2) < 0.21$. □

Remark 25. The previous proposition can be refined in some special cases. For instance,

1. If b is the product of 2 distinct primes; i.e., if $s = 2$ in the proposition, then $\eta(b,k) \leq \lfloor \log_b k + 1 \rfloor$.
2. If b is square-free and k is a power of b, then

$$\eta(b,k) = \eta(b, b^m) \leq \lfloor m + 1.21 \rfloor = m + 1 = \lfloor \log_b k + 1 \rfloor.$$

This remark motivates this final conjecture, which remains open, that closes the paper.

Conjecture 26. If b is a square-free integer, then $\eta(b,k) \leq \lfloor \log_b k + 1 \rfloor$.

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References

[1] Harriet Griffin. *Elementary Theory of Numbers*. McGraw-Hill, 1954.

[2] Hansraj Gupta. *Selected Topics in Number Theory*. Abacus Press, 1980.

[3] D. S. Hart, J. E. Marengo, D. A. Narayan, and D. S. Ross. On the number of trailing zeros in n! *College Math. J.*, **39** (2008), 139–145.

[4] A. M. Legendre. *Essai sur la Théorie des Nombres*. Cambridge University Press, 2009.

[5] I. Niven, H. S. Zuckerman, and H. L. Montgomery. *An Introduction to the Theory of Numbers*. John Wiley & Sons Inc., 1991.

[6] A. M. Oller-Marcén. A new look at the trailing zeroes of $n!$. Preprint, June 26 2009. Available at `http://arxiv.org/abs/0906.4868`.

[7] B. Schmuland. Shouting factorials. *Pi in the Sky*, **7** (2003), 10–12.

[8] M. L. Treuden. Frequencies of digits in factorials: an experimental approach. *College Math. J.*, **25** (1994), 48–55.

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