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Realizability of Integer Sequences as Differences of Fixed Point Count Sequences

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Abstract

A sequence of non-negative integers is *exactly realizable* as the fixed point counts sequence of a dynamical system if and only if it gives rise to a sequence of non-negative orbit counts. This provides a simple realizability criterion based on the transformation between fixed point and orbit counts. Here, we extend the concept of exact realizability to realizability of integer sequences as differences of the two fixed point counts sequences originating from a dynamical system and a topological factor. A criterion analogous to the one for exact realizability is given and the structure of the resulting set of integer sequences is outlined.

1 Introduction and general setting

A (topological) dynamical system (X, T) is given by a topological space X and a continuous map $T : X \to X$. Associated to each dynamical system are two sequences $a = (a_n)_{n\geq1}$ and $c = (c_n)_{n\geq1}$, defined by

$$a_n = |\{x \in X \mid T^n(x) = x\}| \text{ and } c_n = |\{\mathcal{O} \mid \mathcal{O} \text{ is a periodic orbit of length } n\}| . \qquad (1)$$

They count the fixed points and periodic orbits, respectively. In interesting classes of dynamical systems, these numbers are finite for all n, which we assume from now on. Thus, a and c are sequences of non-negative integers.

Clearly, a and c are related by Möbius inversion,

$$\mathrm{fix}(c)_n := a_n = \sum_{d|n} d\, c_d \quad \text{and} \quad \mathrm{orb}(a)_n := c_n = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) a_d. \tag{2}$$

Details on these transformations, considered as linear operators on the space of arithmetic functions, can be found in [6] and references given there. In what follows, we sometimes make use of the operator notation $\mathrm{orb}(a)$ and $\mathrm{fix}(c)$, referring to the transformations of the sequences a and c according to (2). Since fix and orb are inverses of each other, in the sense that $\mathrm{fix} \circ \mathrm{orb} = \mathrm{orb} \circ \mathrm{fix} = \mathrm{id}$, it is possible to implicitly define a sequence f by setting $\mathrm{orb}(f) = g$ or $\mathrm{fix}(f) = g$ for a given sequence g. Note that fix and orb are well-defined for arbitrary sequences of complex numbers. If, however, f is a sequence of integers, then so is $\mathrm{fix}(f)$, whereas the converse need not be true.

According to Puri's terminology [12], a *system* (X, T) comprises an arbitrary set X and a map $T : X \to X$. An integer sequence $(f_n)_{n\geq 1}$ is called *exactly realizable* if there is a system (X, T) whose fixed point counts are given by $(f_n)_{n\geq 1}$. Membership in the set $\mathcal{ER}$ of exactly realizable sequences is characterized by the Basic Lemma [12, Thm. 2.2]:

Theorem 1. *A sequence of non-negative integers $(f_n)_{n\geq 1}$ is exactly realizable if and only if, for all $n \geq 1$, the sum $\sum_{d|n} \mu\left(\frac{n}{d}\right) f_d$ yields a non-negative integer divisible by n, that is, if and only if* $\mathrm{orb}(f)$ *is a sequence of non-negative integers.*

In other words, the only restriction for an integer sequence to be exactly realizable is that it gives rise to an orbit counts sequence of non-negative integers. By compactification of $\mathbb{N}$ with respect to the discrete topology and the definition of an appropriate permutation of the resulting set, Puri shows that the realizing system can be chosen to be a homeomorphism on a compact space. Windsor [15] even gives a construction of a smooth system realizing an arbitrary sequence from $\mathcal{ER}$. Moss [10] systematically investigates the realization of integer sequences by algebraic dynamical systems. A collection of many exactly realizable sequences from the Online Encyclopedia of Integer Sequences (OEIS) [16] is listed by Puri and Ward [13].

A *(topological) factor* of a dynamical system (X, T) is a dynamical system (Y, S) for which there is a continuous surjection $\phi : X \to Y$ that makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y & \xrightarrow{S} & Y \end{array} \tag{3}$$

commutative, i.e., $\phi(T(x)) = S(\phi(x))$ for all $x \in X$ and, by induction, $\phi(T^n(x)) = S^n(\phi(x))$.

Being a factor of the dynamical system (X, T) is a much weaker condition than topological conjugacy, for which ϕ is required to be a homeomorphism. While the orbit statistics of both systems coincide in the latter case, a factor can have a completely different orbit structure. Going down to the factor, an arbitrary number of periodic points can be 'lost' since, for some arbitrary map U, the map $U \times S$ always gives rise to the factor S. An example of the number of periodic points in the factor system exceeding the number of periodic points in the original system is given by the maps dual to $x \mapsto 2x$ on $\mathbb{Q}$ and $\mathbb{Z}$, respectively, see example 11.

Since for many dynamical systems it is known that they are a factor of some well-understood dynamical system or, vice versa, that some well-studied system is a factor (cf. [5]), the question is raised, in what way their orbit statistics can be related. A natural approach to this question is the classification of integer sequences as *relative* fixed point counts, that is, as the difference of the fixed point counts sequences associated with (X, T) and (Y, S). It is clear, however, that this is only a very coarse way of relating the dynamics of the two systems. Example 9 later on gives an example of a dynamical system and a factor whose fixed point counts coincide, the factor map being strictly $2:1$, however.

The purpose of this short note is to derive a realizability criterion for integer sequences as such difference sequences, analogous to the one given in Theorem 1, and to consider a few consequences.

2 Relative realizability

An integer sequence h is called *relatively realizable* if there is a dynamical system (X, T) and a factor (Y, S) with fixed point counts sequences f and g, respectively, such that $h = f - g$. We denote the set of relatively realizable sequences by $\mathcal{ER}_{\mathrm{rel}}$.

By linearity of the mapping orb, the orbit counts sequences corresponding to f and g satisfy the same relation: $\mathrm{orb}(h) = \mathrm{orb}(f) - \mathrm{orb}(g)$. Since the realization is constructed orbit-wise, it is thus more convenient to consider orbit counts.

If the periodic orbits of (X, T) and (Y, S) partition the respective space, each surjective map on the set of equivalence classes defined by the periodic orbits can be extended to a surjection $\phi : X \to Y$. More precisely, if a T-orbit $\mathcal{O} = \{x, Tx, \ldots, T^{n-1}x\}$ is assigned an S-orbit $\{y, Sy, \ldots, S^{d-1}y\}$, the corresponding map $\phi : X \to Y$ is defined via

$$\phi(T^k(x)) = S^{r_k}(y), \quad r_k \leq d-1, \quad r_k \equiv k \mod d \tag{4}$$

for the elements of $\mathcal{O}$.

Proposition 2. *A sequence of integers h is relatively realizable if and only if the formal power series $H(x) := \sum_{n=1}^{\infty} h_n x^n \in \mathbb{Z}[[x]]$ associated with $\mathrm{orb}(h)$ admits a decomposition*

$$H(x) = \sum_{n=1}^{\infty} b_n x^n + \sum_{n>d|n} a_{d,n}(-x^d + x^n),$$

with $a_{d,n}, b_n \in \mathbb{N}_0$.

Proof. The grouping of the terms in $H(x)$ encodes the definition of an appropriate surjection ϕ. We first note that the n-th term has coefficient

$$b_n + \sum_{n>d|n} a_{d,n} - \sum_{k\geq 1} a_{n,kn}. \tag{5}$$

Let (X, T, Y, S, ϕ) relatively realize $h = f - g$ and set $\mathrm{orb}(f) = (\nu_1, \nu_2, \ldots)$, $\mathrm{orb}(g) = (\gamma_1, \gamma_2, \ldots)$. For each $\nu_k \geq 0$, let $C_1, \ldots, C_{\nu_k}$ denote the T-orbits of length k. For $d|k$ with $d < k$ set

$$a_{d,k} = |\{D \in \{\phi(C_1), \ldots, \phi(C_{\nu_k})\} : \ |D| = d, D \text{ has no preimage orbit of length } < k\}| \, .$$

In particular, $a_{d,k} = 0$ for divisors d of k that do not show up as the length of an image cycle and coinciding image cycles $\phi(C_i) = \phi(C_j)$ for $i \neq j$ are counted only once. Then, the number of ℓ with $a_{d,\ell d} \neq 0$ is finite, and $\sum_{\ell \geq 1} a_{d,\ell d} = \gamma_d$. Define $b_k := \nu_k - \sum_{n>d|n} a_{d,k}$ if $\nu_k > 0$, giving $b_k = 0$ otherwise. Thus, the $a_{d,k}, b_k$ define a formal power series of the structure indicated above whose n-th coefficient is, according to (5),

$$b_n + \sum_{n>d|n} a_{d,n} - \sum_{k \geq 1} a_{n,kn} = \nu_n - \gamma_n = \mathrm{orb}(h)_n.$$

Consequently, this is an appropriate decomposition of the power series of $\mathrm{orb}(h)$.

For the inverse direction, we define

$$\nu_n := b_n + 1 + \sum_{n>d|n} a_{d,n} \text{ and } \gamma_n := \sum_{k \geq 1} a_{n,kn} + 1$$

and realize the sequences $\mathrm{orb}(f) := (\nu_1, \nu_2, \ldots)$ and $\mathrm{orb}(g) := (\gamma_1, \gamma_2, \ldots)$ on the one-point compactification $(\mathbb{N}_*, \tau_*)$ of $\mathbb{N}$. If 'the point at infinity' ∞, added for the purpose of compactification, is a fixed point of the permutation σ on $\mathbb{N}_* = \mathbb{N} \cup \{\infty\}$, σ is continuous in the obtained topology, see [12] and the reference given therein. In each of the resulting systems $(\mathbb{N}_*, T)$ and $(\mathbb{N}_*, S)$, there is at least one cycle of length n for all $n \in \mathbb{N}$. By construction, $(\mathbb{N}_*, S)$ and $(\mathbb{N}_*, T)$ have $a_{n,kn}$ corresponding cycles of length n and kn, respectively. ϕ can be defined on these as described by (4). For the b_n further T-cycles of length n, one of the n-S-cycles can be chosen as the image cycle. If the 1-orbit $\{\infty\}$ is mapped to its analogue in the factor system, it is straightforward to check that the map ϕ is continuous with respect to the considered topologies. It is surjective by construction, thus turning $(\mathbb{N}_*, S)$ into a factor of $(\mathbb{N}_*, T)$ and therefore yielding a relative realization of the sequence h. □

As a consequence, we obtain that each integer sequence is the difference orbit counts sequence of a dynamical system and a factor.

Theorem 3. *A sequence h of integers is relatively realizable if and only if $\mathrm{orb}(h)$ is a sequence of integers.*

Proof. Since it is clear that the condition is necessary, it suffices to give a decomposition of an arbitrary element $H(x) = \sum_{k=1}^{\infty} \eta_k x^k$ from $\mathbb{Z}[[x]]$ as in Proposition 2. A possible approach for doing so is to select a divisor $d|n$ and a multiple ℓn of n and to decompose η_n into a sum of the shape $\eta_n = b_n + a_{d,n} - a_{n,\ell n}$ with non-negative integers $b_n, a_{d,n}$ and $a_{n,\ell n}$.

Consider the case n odd first. If $\eta_n \geq 0$, define $b_n = \eta_n$ and $a_{n,\ell n} = 0$ for all $\ell \geq 1$. If $\eta_n < 0$, set $a_{n,2n} = -\eta_n$, $a_{n,\ell n} = 0$ for all $\ell \neq 2$ and $b_n = 0$. For n even and $\eta_n - a_{n/2,n} \geq 0$, set $b_n = \eta_n - a_{n/2,n}$, $a_{d,n} = 0$ for all $d|n$; for $\eta_n - a_{n/2,n} < 0$ set $a_{n,2n} = -(\eta_n - a_{n/2,n})$, $a_{n,\ell n} = 0$ for all $\ell \neq 2$ and $b_n = 0$. Thus (5) yields $b_n + a_{n/2,n} - a_{n,2n}$ as the n-th coefficient, which, in each of the cases considered above, coincides with η_n. Hence, $H(x)$ can be written as in Proposition 2. □

Remark 4. Analogous to the proof of the Basic Lemma [12], Proposition 2 and Theorem 3 together give a construction of a realizing permutation system for a given integer sequence.

Remark 5. A combinatorially easier construction can be obtained by defining an infinite preimage-cycle for each cycle of the factor system (Y, S) and, on the other hand, providing (Y, S) with some 1-orbit to which all orbits of (X, T) can be sent. The drawback of such a construction is that, due to the infinite preimages of the periodic orbits in the factor system, there is no direct way of obtaining a result about the existence of a realization by compact dynamical systems as given in the Basic Lemma. In this context, it makes sense to introduce a notion like 'factor surjective on the set of periodic points' referring to the restriction of the factor map to the sets of periodic points being surjective. In other words, this defines a subclass of dynamical systems with factors in which every periodic orbit of the factor has a periodic orbit in its preimage. Since the permutation systems in Proposition 2 are of that type, it follows that requiring dynamical systems to be factor surjective on the set of periodic points is not restrictive with regard to relative realizability.

Corollary 6. *Every exactly realizable sequence is relatively realizable:* $\mathcal{ER} \subset \mathcal{ER}_{\mathrm{rel}}$.

Remark 7. Obviously, the criterion for exact realizability is subsumed by the one for relative realizability, but it is also easy to construct a relative realization for a sequence from $\mathcal{ER}$: for f realized by (X, T), the sequence $2f$ is realized by the induced mapping on the topological sum $X + X$ whose factor is the original map, resulting in a relative realization of the sequence f. A further construction is given by the exact realization of a sequence $f + u$, where $u_n = 1$ for all n and the trivial system $(\{0\}, id)$, being a factor of any dynamical system.

The following result shows that $\mathcal{ER}_{\mathrm{rel}}$ shares many properties with $\mathcal{ER}$ (cf. [12, Section 2]). The proofs are either based on the integrality condition of the orbit counts sequences or on the construction of realizing dynamical systems. They are very similar to those for $\mathcal{ER}$ [12] and can be found in [11].

Theorem 8. *The set* $\mathcal{ER}_{\mathrm{rel}}$ *of relatively realizable sequences satisfies the following properties:*

1. *There are no zero divisors in* $\mathcal{ER}_{\mathrm{rel}}$*:* $fg = 0$ *for* $f, g \in \mathcal{ER}_{\mathrm{rel}}$ *implies* $f = 0$ *or* $g = 0$.

2. $\mathcal{ER}_{\mathrm{rel}}$ *contains the constant sequences over* $\mathbb{Z}$.

3. $\mathcal{ER}_{\mathrm{rel}}$ *is closed under addition, multiplication und multiplication with elements* $z \in \mathbb{Z}$.

4. *The constant sequence* $u = (1)_{n \geq 1}$ *is the only completely multiplicative sequence in* $\mathcal{ER}_{\mathrm{rel}}$. □

Example 9. The *Thue-Morse*- and *Period Doubling*-chains are examples of inflation dynamical systems that can be treated symbolically. For background information, consider the articles by Queffelec [1], Allouche and Mendès France [14] and the references given there. Define, over the finite alphabets $\{a, b\}$ and $\{A, B\}$, respectively, the substitution rules

$$\mathrm{TM} : a \to ab,\ b \to ba \text{ and } \mathrm{PD} : A \to AB,\ B \to AA.$$

The squares TM^2 and PD^2 of each of the two mappings, iterated on the respective one-letter seeds, produce the fixed points $\{..a|a..,..b|b..\}$, $\{..a|b..,..b|a..\}$ and $\{..A|A..,..B|B..\}$, $\{..B..,..A..\}$, respectively, i.e., 2-cycles of the original maps. The inflations TM and PD define maps on the corresponding LI classes which coincides with the hulls that are obtained as the orbit closures under the continuous translation action of $\mathbb{R}$.

The factor map ϕ is given by the block map

$$aa \mapsto B,\ bb \mapsto B,\ ab \mapsto A,\ ba \mapsto A.$$

Since the sequences $..A|A..$ and $..B|A..$ correspond to the two sequences $\{..a|b..,..b|a..\}$ and $\{..a|a..,..b|b..\}$, respectively, the map ϕ is strictly $2:1$. (The vertical line indicates the reference point of the bi-infinite chain.) Following the method described by Anderson and Putnam [2], the dynamical zeta function of both systems can be calculated as [9]

$$\zeta(z) = \frac{1-z}{(1+z)(1-2z)},$$

giving rise to the fixed point counts sequence $a_m = 2^m + (-1)^m - 1$ (A099430 in the OEIS [16]). Thus, these two dynamical systems share the same fixed point counts and relatively realize the sequence $(0, 0, 0, \ldots)$ (A000004). In fact, this and the property of being $2:1$ completely determine the combinatorics of the map ϕ. Let

$$\begin{aligned} \alpha(n) &:= \#\ n\text{-orbits of PD whose preimages are two } n\text{-orbits of TM and} \\ \beta(n) &:= \#\ n\text{-orbits of PD whose preimages are } 2n\text{-orbits of TM.} \end{aligned}$$

Clearly, $c_n = \alpha(n) + \beta(n)$. Furthermore, $\beta(n) = \frac{c_n}{2}$ if n is odd and $\beta(n) = (c_n + \beta(n/2))/2$ if n is even. Since a_n and c_n are related via (2), the calculation of $\beta(n)$ yields

$$\beta(n) = \frac{1}{2n} \sum_{d|n,\ d \text{ odd}} \mu(d) \cdot 2^{n/d} - \delta_{n,1},$$

where $\delta_{n,1} = 1$ for $n = 1$ and 0 otherwise. Except for $\beta(1) = 0$, this is A000048.

Example 10. The *torus parametrization* of substitution tilings [5] yields a large class of dynamical systems with a torus automorphism as a topological factor. The one-dimensional Fibonacci chain, obtained by the standard projection method, gives rise to the relative fixed point counts sequence $h_n = (-1)^n$ (arising from the fixed point counts A001610 and A001350), which corresponds to an orbit counts sequence of $\mathrm{orb}(h) = (-1, 1, 0, 0, \ldots)$. More complicated examples are provided by higher dimensional projections. The Penrose tiling and its torus parametrization, for instance, lead to the relative fixed point and orbit counts sequences

$$(-1, 9, -16, 29, -51, 84, -141, \ldots) \text{ and } (-1, 5, -5, 5, -10, 15, -20, \ldots)$$

which can be calculated from the corresponding dynamical zeta functions stated explicitly by Baake and Grimm [4]. The more complicated difference sequences in the last case reflect the phenomenon of a large (though always finite) number of singular tilings being sent to the same torus parameter occurring in higher dimensional systems, whereas the first case illustrates that the one-dimensional Fibonacci torus parametrization is 'nearly one-to-one'.

Another large group of dynamical systems with non-trivial factors is provided by S-integer dynamical systems, cf. [7], [8] and references given therein. Let Q, P be subsets of the set of all primes with $Q \subset P$. Then the map S, dual to $x \mapsto 2x$ on the ring of Q-integers, is a factor of T, the map dual to $x \mapsto 2x$ on the ring of P-integers via the dual of the inclusion map from the ring of P-integers to the ring of Q-integers.

Example 11. Choose P to be the set of all rational primes, $Q = \varnothing$ and $\alpha : \mathbb{Q} \to \mathbb{Q}$, $x \mapsto 2x$. The resulting commutative diagramme is obtained by setting $X = \hat{\mathbb{Q}}$ and $Y = \hat{\mathbb{Z}}$ in (3). The dual map $\hat{\alpha} : \hat{\mathbb{Q}} \to \hat{\mathbb{Q}}$ is characterized by $\chi \mapsto \chi \circ \alpha$ for all $\chi \in \hat{\mathbb{Q}}$. Thus, the fixed point equation $\hat{\alpha}^k(\chi) = \chi$ is equivalent with $\chi(2^k x) = \chi(x)$ for all $x \in \mathbb{Q}$ or, by the properties of characters, $\chi((2^k - 1)x) = 1$ for all $x \in \mathbb{Q}$. An appropriate choice of x shows that only the trivial character satisfies this condition, yielding a fixed point counts sequence of $(1, 1, 1, 1, \ldots)$ (A000012). On Y, each $k \in \mathbb{Z}$ gives rise to the element $e^{2\pi i k} \in \hat{\mathbb{Z}}$, such that the $(2^n - 1)$-th roots of unity constitute the $2^n - 1$ fixed points of S^n (A000225).

For P chosen as above and $Q = P \setminus \{3\}$ we obtain $X = \hat{\mathbb{Q}}$ and $Y = \widehat{\mathbb{Z}_{(3)}}$, where $\mathbb{Z}_{(3)} = \mathbb{Z}[\frac{1}{p} : p \text{ prime}, \ p \neq 3]$. According to [8, Example 4.1], the n-periodic points of S are given by $|2^n - 1|_3^{-1}$, i.e., $(1, 3, 1, 3, 1, 9, 1, 3, 1, 3, 1, 9, 1, 3, 1, \ldots)$, yielding a relative fixed point sequence of $(0, -2, 0, -2, 0, -8, 0, -2, 0, -2, 0, -8, 0, -2, 0, \ldots)$.

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