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Integer Sequences Avoiding Prime Pairwise Sums

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Abstract

The following result is proved: If $A \subseteq \{1, 2, \ldots, n\}$ is the subset of largest cardinality such that the sum of no two (distinct) elements of A is prime, then $|A| = \lfloor (n+1)/2 \rfloor$ and all the elements of A have the same parity. The following open question is posed: what is the largest cardinality of $A \subseteq \{1, 2, \ldots, n\}$ such that the sum of no two (distinct) elements of A is prime and A contains elements of both parities?

1 Introduction

Some combinatorial problems have the following structure: find subsets $A \subseteq \{1, 2, \ldots, n\}$ such that the sum of no two (distinct) elements of A belongs to T, where T is a given set. We say that such a A is a *T-sumset-free set.* In this note we deal with the case $T = P$, the set of all primes. There appear to be no previous papers on this topic.

We try to determine all prime-sumset-free subsets of $\{1, 2, \ldots, n\}$ with the largest cardinality. Let the largest cardinality be U_n. It is clear that the set of all even (odd) integers in $\{1, 2, \ldots, n\}$ is a prime-sumset-free set. So $U_n \geq \lfloor (n+1)/2 \rfloor$. If $n+1$ is prime, then by considering a and $n+1-a$ we have $U_n \leq \lfloor (n+1)/2 \rfloor$. Thus $U_n = \lfloor (n+1)/2 \rfloor$ if $n+1$ is prime. By employing results about the distribution of primes we prove

Theorem 1. *For all $n \geq 1$ we have $U_n = \lfloor \frac{1}{2}(n+1) \rfloor$. Furthermore, if $A \subseteq \{1, 2, \ldots, n\}$ is a prime-sumset-free set with $|A| = U_n$, then all elements of A have the same parity.*

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A prime-sumset-free subset A of $\{1, 2, \ldots, n\}$ is called *an extremal prime-sumset-free subset* of $\{1, 2, \ldots, n\}$ if $A\cup\{a\}$ is not a prime-sumset-free subset for any $a \in \{1, 2, \ldots, n\}\setminus A$. Let $PF_k(n)$ $(k = 1, 2, \ldots)$ be the sequence of cardinalities of all extremal prime-sumset-free subsets of $\{1, 2, \ldots, n\}$ with $PF_1(n) > PF_2(n) > \ldots$. By the theorem we have $PF_1(n) = U_n = \lfloor (n+1)/2 \rfloor$. We pose the following open question:

Question 2. What are the values of $PF_k(n)$? In particular, What is the value of $PF_2(n)$?

Question 3. Determine all extremal prime-sumset-free subsets A with $|A| = PF_2(n)$.

2 Proof of the Theorem

Although the proof of the second part implies the first part, we give a proof of the first part by induction and the application of Bertrand's postulate here. It is easy to see that the conclusion is true for $n = 1$. Now we assume that the conclusion is true for $n < k (k \geq 2)$. By the Bertrand's postulate (see [1]) there exists a prime p with $k < p < 2k$. Assume that $A \subseteq \{1, 2, \ldots, n\}$ is prime-sumset-free. For $p - k \leq a \leq k$ we have $|\{a, p-a\} \cap A| \leq 1$. So

$$|A \cap [p-k,\, k]| \leq \frac{1}{2}(2k - p + 1).$$

By the induction hypothesis we have

$$|A \cap [1,\, p-k-1]| \leq \frac{1}{2}(p-k).$$

Hence

$$|A \cap [1,\, k]| \leq \frac{1}{2}(2k-p+1) + \frac{1}{2}(p-k) = \frac{1}{2}(k+1).$$

This implies that $U_k \leq [(k+1)/2]$. By the remark before the theorem we have $U_k \geq \lfloor (k+1)/2 \rfloor$. So $U_k = \lfloor (k+1)/2 \rfloor$. This completes the proof of the first part.

To prove the second part of Theorem 1, we need a lemma.

Lemma 4. *For any real number $x \geq 8$ we have*

$$\pi(\sqrt{2}x) - \pi(x) \geq 1.$$

In particular, if m, n are positive integers with $m > \sqrt{2}n$ and $n \geq 8$, then there exists at least one prime p with $m > p > n$.

Proof. By direct calculation we know that Lemma 4 is true for $8 \leq x \leq 25$. If $x > 25$, by Nagura [2] (see also [3, Lemma 4]) we have

$$\pi(\sqrt{2}x) - \pi(x) \geq \pi\Big(\frac{6}{5}x\Big) - \pi(x) \geq 1.$$

This completes the proof of Lemma 4. $\square$

Now we return to prove the second part of Theorem 1.

For $n \le 8$ we can verify Theorem 1 directly. Now we assume that $k > 8$ and Theorem 1 is true for all $n < k$. Let $A \subseteq \{1, 2, \ldots, k\}$ be a prime-sumset-free set with $|A| = U_k$. Let q_k be the largest prime q with $q \le 2k$. By Lemma 4 we have $q_k > \sqrt{2}k$. If $8 < k \le 20$, by direct verification we have $q_k - k \ge 8$. If $k \ge 21$, then $q_k - k > (\sqrt{2} - 1)k \ge 8$. For any $q_k - k \le a \le k$ we have $|A \cap \{a,\ q_k - a\}| \le 1$. Hence

$$|A \cap [q_k - k,\ k]| \le \frac{1}{2}(2k - q_k + 1).$$

Since $A \cap [1,\ q_k - k - 1]$ is a prime-sumset-free set, we have

$$|A \cap [1,\ q_k - k - 1]| \le U_{q_k - k - 1} = \left\lfloor \frac{1}{2}(q_k - k) \right\rfloor.$$

By the assumption $|A| = U_k = \lfloor (k+1)/2 \rfloor$ we have

$$\begin{aligned} \left\lfloor \frac{1}{2}(k+1) \right\rfloor = |A| &= |A \cap [1,\ q_k - k - 1]| + |A \cap [q_k - k,\ k]| \\ &\le \left\lfloor \frac{1}{2}(q_k - k) \right\rfloor + \frac{1}{2}(2k - q_k + 1) \\ &= \left\lfloor \frac{1}{2}(k+1) \right\rfloor. \end{aligned}$$

So

$$|A \cap [1,\ q_k - k - 1]| = \left\lfloor \frac{1}{2}(q_k - k) \right\rfloor = U_{q_k - k - 1}.$$

If $2|k$, then by the induction hypothesis we have

$$A \cap [1,\ q_k - k - 1] = \{1, 3, \ldots, q_k - k - 2\} \text{ or } \{2, 4, \ldots, q_k - k - 1\}.$$

If $2 \not| k$, then by the induction hypothesis we have

$$A \cap [1,\ q_k - k - 1] = \{1, 3, \ldots, q_k - k - 1\}.$$

Case 1: $2|k$ and $A \cap [1,\ q_k - k - 1] = \{1, 3, \ldots, q_k - k - 2\}$.

Let $2m \in [q_k - k,\ k]$. Then

$$\frac{2m + q_k - k}{2m} = 1 + \frac{q_k - k}{2m} > 1 + \frac{\sqrt{2}k - k}{k} = \sqrt{2}.$$

By $q_k - k \ge 8$ and Lemma 4 there exists at least one prime p with $2m < p < 2m + q_k - k$. So $1 \le p - 2m \le q_k - k - 2$. Thus $p - 2m \in A \cap [1,\ q_k - k - 1]$. Hence $2m \notin A$. So

$$A \subseteq \{1, 3, 5, \ldots, k - 1\}.$$

Since $|A| = U_k = \frac{1}{2}k$, we have $A = \{1, 3, 5, \ldots, k - 1\}$.

Case 2: $2|k$ and $A \cap [1,\ q_k - k - 1] = \{2, 4, \ldots, q_k - k - 1\}$.

Let $2m+1 \in [q_k - k,\ \ k]$. Then

$$\frac{2m+1+q_k-k}{2m+1} = 1 + \frac{q_k-k}{2m+1} > 1 + \frac{\sqrt{2}k-k}{k} = \sqrt{2}.$$

By $q_k - k \geq 8$ and Lemma 4 there exists at least one prime p with $2m+1 < p < 2m+1+q_k-k$. So $1 \leq p-2m-1 \leq q_k-k-1$. Thus $p-2m-1 \in A \cap [1,\ \ q_k-k-1]$. Hence $2m+1 \notin A$. So

$$A \subseteq \{2,\, 4,\, \ldots,\, k\}.$$

Since $|A| = U_k = \frac{1}{2}k$, we have $A = \{2,\, 4,\, \ldots,\, k\}$.

Case 3: $2 \not| k$. Then

$$A \cap [1,\ \ q_k-k-1] = \{1,\, 3,\, \ldots,\, q_k-k-1\}.$$

Let $2m \in [q_k - k,\ \ k]$. Then

$$\frac{2m+q_k-k}{2m} = 1 + \frac{q_k-k}{2m} > 1 + \frac{\sqrt{2}k-k}{k} = \sqrt{2}.$$

By $q_k - k \geq 8$ and Lemma 4 there exists at least one prime p with $2m < p < 2m+q_k-k$. So $1 \leq p-2m \leq q_k-k-1$. Thus $p-2m \in A \cap [1,\ \ q_k-k-1]$. Hence $2m \notin A$. So

$$A \subseteq \{1,\, 3,\, 5,\, \ldots,\, k-1\}.$$

Since $|A| = U_k = \frac{1}{2}(k-1)$, we have $A = \{1,\, 3,\, 5,\, \ldots,\, k-1\}$.

This completes the proof.

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