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DIAMETER BOUNDS FOR EQUAL AREA PARTITIONS OF THE UNIT SPHERE*

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Abstract. The recursive zonal equal area (EQ) sphere partitioning algorithm is a practical algorithm for partitioning higher dimensional spheres into regions of equal area and small diameter. Another such construction is due to Feige and Schechtman. This paper gives a proof for the bounds on the diameter of regions for each of these partitions.

Key words. sphere, partition, area, diameter, zone

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1. Introduction. Stolarsky [12, p. 581] asserts the existence for any natural number N of a partition of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ into N regions of equal area and small diameter. The recursive zonal equal area (EQ) sphere partitioning algorithm [8, Section 3] is a practical means to achieve such a partition. Feige and Schechtman [5] give a construction which can easily be modified to give another such partition.

In this paper we prove that the both EQ partition and the modified Feige-Schechtman partition satisfy Stolarsky's assertion. This paper is the companion to [8] and is meant to be read in conjunction with that paper. Any definitions and notation not found here are to be found in [8]. The proofs given here are based on those in the Ph.D. thesis [7] and much of the technical detail which has been omitted here can be found there.

This paper is organized as follows. Section 2 repeats enough of the definitions and theorems of [8] to orient the reader. Section 3 contains the continuous model of the EQ partition which is used in the proof of the properties of this partition. Section 4 proves that the EQ partition satisfies Stolarsky's assertion. Section 5 contains estimates which will be used in the remainder of the paper. Section 6 provides a proof that the modified Feige-Schechtman construction satisfies Stolarsky's assertion. An appendix provides proofs for some of the lemmas. Further proofs and more details can be found in [7].

2. Preliminaries. For convenience, this section repeats some of the definitions and restates some of the theorems given in [8].

For any two points $\mathbf{a}, \mathbf{b} \in \mathbb{S}^d$, the Euclidean and spherical distances are related by

$$\|\mathbf{a}, \mathbf{b}\| = \Upsilon\left(s(\mathbf{a}, \mathbf{b})\right),$$

where

$$\Upsilon(\theta) := \sqrt{2 - 2\cos\theta} = 2\sin\frac{\theta}{2}. \tag{2.1}$$

For $d \geqslant 0$, the area of $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ is given by [9, p. 1]

$$\sigma(\mathbb{S}^d) = \frac{2\,\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}.$$

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For all that follows, we will use the following abbreviations. For $d \geqslant 1$, we define

$$\omega := \sigma(\mathbb{S}^{d-1}) \quad \text{and} \quad \Omega := \sigma(\mathbb{S}^d).$$

The area of a spherical cap $S(\mathbf{a}, \theta)$ of spherical radius θ and center $\mathbf{a}$ is [6, Lemma 4.1 p. 255]

$$\mathcal{V}(\theta) := \sigma\left(S(\mathbf{a}, \theta)\right) = \omega \int_0^\theta (\sin \xi)^{d-1} d\xi. \tag{2.2}$$

The function Θ is the inverse of $\mathcal{V}$.

This paper considers the Euclidean diameter of regions, defined as follows.

DEFINITION 2.1. *The* diameter *of a region $R \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$ is*

$$\operatorname{diam} R := \sup\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{x}, \mathbf{y} \in R\}.$$

The following definitions are specific to the main theorems stated here.

DEFINITION 2.2. *A set Z of partitions of $\mathbb{S}^d$ is said to be* diameter-bounded *with* diameter bound $K \in \mathbb{R}_+$ *if for all $P \in Z$, for each $R \in P$,*

$$\operatorname{diam} R \leqslant K \left|P\right|^{-\frac{1}{d}}.$$

DEFINITION 2.3. *The set of recursive zonal equal area partitions of $\mathbb{S}^d$ is defined as*

$$\mathrm{EQ}(d) := \{\mathrm{EQ}(d, N) \mid N \in \mathbb{N}_+\}.$$

where $\mathrm{EQ}(d, N)$ denotes the recursive zonal equal area partition of the unit sphere $\mathbb{S}^d$ into N regions, which is defined via the algorithm given in [8, Section 3].

The partition $\mathrm{EQ}(d, N)$ has the following properties.

THEOREM 2.4. *For $d \geqslant 1$ and $N \geqslant 1$, the partition $\mathrm{EQ}(d, N)$ is an equal area partition of $\mathbb{S}^d$.*

The proof of Theorem 2.4 is straightforward, following immediately from the construction of the EQ partition [8, Section 3].

THEOREM 2.5. *For $d \geqslant 1$, $\mathrm{EQ}(d)$ is diameter-bounded in the sense of Definition 2.2.*

Theorem 2.5 is a special case of Stolarsky's assertion:

THEOREM 2.6. [12, p. 581] *For each $d > 0$, there is a constant c_d such that for all $N > 0$, there is a partition of the unit sphere $\mathbb{S}^d$ into N regions, with each region having area Ω/N and diameter at most $c_d N^{-\frac{1}{d}}$.*

We will also often refer to the following quantities, defined in steps 1 to 3 of the EQ partition algorithm for $\mathrm{EQ}(d, N)$ [8, Section 3.2].

$$\mathcal{V}_R := \frac{\Omega}{N}, \quad \theta_c := \Theta(\mathcal{V}_R), \quad \delta_I := \mathcal{V}_R^{\frac{1}{d}}, \quad n_I := \frac{\pi - 2\theta_c}{\delta_I}. \tag{2.3}$$

3. A continuous model of the partition algorithm. Step 4 of the EQ partition algorithm [8, 3.2] is the first rounding step, which produces n from n_I. We define

$$\rho := \frac{n_I}{n}$$

so that

$$\delta_F = \rho \delta_I. \tag{3.1}$$

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For $N > 2$, if $n_I \geqslant \frac{1}{2}$ then Step 4 yields

$$n \in \left(n_I - \frac{1}{2}, n_I + \frac{1}{2}\right],$$

and therefore [7, Lemma 3.5.1, p. 87]

$$\rho \in \left[1 - \frac{1}{2n_I + 1}, 1 + \frac{1}{2n_I - 1}\right).$$

We see that bounds for ρ are given by lower bounds for n_I. The crudest such bound is given by $n_I > \frac{1}{2}$ which merely implies that $\rho > 1/2$.

We can re-express the bound $n_I > \frac{1}{2}$ in terms of a lower bound on N by means of the function ν, where

$$\nu(x) := \left(\frac{x}{\Omega}\right)^{\frac{1}{d}} \left(\pi - 2\Theta\left(\frac{\Omega}{x}\right)\right). \tag{3.2}$$

The function ν defined by (3.2) satisfies $\nu(2) = 0$, $\nu(N) = n_I$, and $\nu(x)$ is monotonically increasing in x for $x \geqslant 2$ [7, Lemma 3.5.2, p. 87]. As a consequence, it is possible to define the inverse function $\mathcal{N}_0$ where

$$\mathcal{N}_0(y) := \nu^{-1}(y) \tag{3.3}$$

for $y \geqslant 0$. We then have $\mathcal{N}_0\big(\nu(x)\big) = x$ and $\nu\big(\mathcal{N}_0(y)\big) = y$ for $x \geqslant 2$ and $y \geqslant 0$, and by the inverse function theorem, $\mathcal{N}_0(y)$ is monotonic increasing in y for $y \geqslant 0$.

For $N > x$ such that $x > \mathcal{N}_0(1/2)$, we then have

$$n_I > \nu(x) > \frac{1}{2} \tag{3.4}$$

and

$$\rho \in [\rho_L(x), \rho_H(x)], \tag{3.5}$$

where

$$\rho_L(x) := 1 - \frac{1}{2\nu(x) + 1} \quad \text{and} \quad \rho_H(x) := 1 + \frac{1}{2\nu(x) - 1}. \tag{3.6}$$

We can make $\rho_L(x)$ and $\rho_H(x)$ arbitrarily close to 1 by making x large enough. More precisely,

$$\rho_L(x) \nearrow 1, \text{ and } \rho_H(x) \searrow 1 \text{ as } x \to \infty. \tag{3.7}$$

Step 6 of the EQ partition algorithm is the second rounding step, which produces m_i from y_i. By examining steps 5 to 7 of the EQ partition algorithm, it is straightforward to verify that for $d > 1$, $N > 1$ and $i \in \{1, \ldots, n\}$ the following relationships hold [7, Lemmas 3.5.3, 3.5.4, pp. 88–89]:

$$a_i \in \left[-\frac{1}{2}, \frac{1}{2}\right], \quad a_n = 0, \quad \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} m_i = N - 2,$$
$$m_i = y_i + a_{i-1} - a_i = \frac{\mathcal{V}(\vartheta_{i+1}) - \mathcal{V}(\vartheta_i)}{\mathcal{V}_R} \in \mathbb{N}_0,$$
$$\mathcal{V}(\vartheta_i) = \mathcal{V}(\vartheta_{F,i}) + a_{i-1}\mathcal{V}_R.$$

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To make it easier to find bounds for functions which vary from zone to zone, such as y and m, we define and use continuous analogs of these functions. This way, instead of having to find a bound for a function value over $n+2$ points, where n varies with N, we need only find a bound for a function over a fixed number of points and continuous intervals. We therefore define the functions

$$\begin{aligned}
\mathcal{Y}(\vartheta) &:= \frac{\mathcal{V}(\vartheta+\delta_F)-\mathcal{V}(\vartheta)}{\mathcal{V}_R}, & \mathcal{M}(\tau,\beta,\vartheta) &:= \mathcal{Y}(\vartheta)+\tau+\beta,\\
\mathcal{T}(\tau,\vartheta) &:= \Theta\big(\mathcal{V}(\vartheta)-\tau\mathcal{V}_R\big), & \mathcal{B}(\beta,\vartheta) &:= \Theta\big(\mathcal{V}(\vartheta+\delta_F)+\beta\mathcal{V}_R\big),\\
\Delta(\tau,\beta,\vartheta) &:= \mathcal{B}(\beta,\vartheta)-\mathcal{T}(\tau,\vartheta), & \mathcal{W}(\tau,\beta,\vartheta) &:= \max_{\xi\in[\mathcal{T}(\tau,\vartheta),\mathcal{B}(\beta,\vartheta)]}\sin\xi,\\
\mathcal{P}(\tau,\beta,\vartheta) &:= \mathcal{W}(\tau,\beta,\vartheta)\,\mathcal{M}(\tau,\beta,\vartheta)^{\frac{1}{1-d}}, & &
\end{aligned} \tag{3.8}$$

so that for $i\in\{1,\ldots,n\}$, we have

$$\begin{aligned}
\mathcal{Y}(\vartheta_{F,i}) &= y_i, & \mathcal{M}(-a_{i-1},a_i,\vartheta_{F,i}) &= m_i,\\
\mathcal{T}(-a_{i-1},\vartheta_{F,i}) &= \vartheta_i, & \mathcal{B}(a_i,\vartheta_{F,i}) &= \vartheta_{i+1},\\
\Delta(-a_{i-1},a_i,\vartheta_{F,i}) &= \delta_i, & \mathcal{W}(-a_{i-1},a_i,\vartheta_{F,i}) &= w_i,\\
\mathcal{P}(-a_{i-1},a_i,\vartheta_{F,i}) &= p_i. & &
\end{aligned}$$

These functions have symmetries which follow from the symmetries of the trigonometric functions. The function $\mathcal{Y}$ satisfies

$$\mathcal{Y}(\pi-\vartheta)=\mathcal{Y}(\vartheta-\delta_F).$$

The functions $\mathcal{T}$ and $\mathcal{B}$ satisfy the identities

$$\begin{aligned}
\mathcal{T}(\tau,\pi-\vartheta) &= \pi-\mathcal{B}(\tau,\vartheta-\delta_F) \quad \text{and}\\
\mathcal{B}(\beta,\pi-\vartheta) &= \pi-\mathcal{T}(\beta,\vartheta-\delta_F).
\end{aligned}$$

For each $f\in\{\mathcal{M},\Delta,\mathcal{W},\mathcal{P}\}$, the function f satisfies

$$f(\tau,\beta,\pi-\vartheta)=f(\beta,\tau,\vartheta-\delta_F).$$

For our feasible domain we therefore use the set $\mathbb{D}$, defined as follows.

DEFINITION 3.1. *The* feasible domain $\mathbb{D}$ *is defined as*

$$\mathbb{D}:=\mathbb{D}_t\cup\mathbb{D}_m\cup\mathbb{D}_b,$$

where

$$\begin{aligned}
\mathbb{D}_t &:= \{(\tau,\beta,\vartheta)\mid \tau=0,\beta\in\left[-\frac{1}{2},\frac{1}{2}\right],\vartheta=\vartheta_c\},\\
\mathbb{D}_m &:= \{(\tau,\beta,\vartheta)\mid \tau\in\left[-\frac{1}{2},\frac{1}{2}\right],\beta\in\left[-\frac{1}{2},\frac{1}{2}\right],\vartheta\in[\vartheta_{F,2},\pi-\vartheta_c-2\delta_F]\},\\
\mathbb{D}_b &:= \{(\tau,\beta,\vartheta)\mid \tau\in\left[-\frac{1}{2},\frac{1}{2}\right],\beta=0,\vartheta=\pi-\vartheta_c-\delta_F\}.
\end{aligned} \tag{3.9}$$

We can now use the feasible domain $\mathbb{D}$ and the analogue functions Δ and $\mathcal{P}$ to bound the maximum diameter of regions of the EQ partition.

LEMMA 3.2. [7, Lemma 3.5.11] *Assume that $d > 1$ and that* $\mathrm{EQ}(d-1)$ *has diameter bound κ. Then for $N > 2$, if we define*

$$\operatorname{maxdiam}(d, N) := \max_{R \in \mathrm{EQ}(d,N)} \operatorname{diam} R,$$

then

$$\operatorname{maxdiam}(d, N) \leqslant \sqrt{(\max_{\mathbb{D}} \Delta)^2 + \kappa^2 (\max_{\mathbb{D}} \mathcal{P})^2}.$$

We need only consider the northern hemisphere to obtain a valid bound for the diameter of a region of the recursive zonal equal area partition of $\mathbb{S}^d$. First define the following subdomains of the feasible domain $\mathbb{D}$,

$$\mathbb{D}_+ := \left\{ (\tau, \beta, \vartheta) \in \mathbb{D} \;\middle|\; \vartheta \leqslant \frac{\pi}{2} - \frac{\delta_F}{2} \right\},$$

$$\mathbb{D}_- := \left\{ (\tau, \beta, \vartheta) \in \mathbb{D} \;\middle|\; \vartheta > \frac{\pi}{2} - \frac{\delta_F}{2} \right\},$$

$$\mathbb{D}_{m+} := \mathbb{D}_m \cap \mathbb{D}_+. \tag{3.10}$$

The following result then holds.

LEMMA 3.3. [7, Lemma 3.5.12] *For $f \in \{\mathcal{M}, \Delta, \mathcal{W}, \mathcal{P}\}$ and $(\tau, \beta, \vartheta) \in \mathbb{D}_-$, we can find $(\tau', \beta', \vartheta') \in \mathbb{D}_+$ such that $f(\tau', \beta', \vartheta') = f(\tau, \beta, \vartheta)$. In particular, if $(\tau, \beta, \vartheta) \in \mathbb{D}_b$, then $(\tau', \beta', \vartheta') \in \mathbb{D}_t$, and if $(\tau, \beta, \vartheta) \in \mathbb{D}_{m-}$, then $(\tau', \beta', \vartheta') \in \mathbb{D}_{m+}$.*

COROLLARY 3.4. *For $f \in \{\mathcal{M}, \Delta, \mathcal{W}, \mathcal{P}\}$,*

$$\max_{\mathbb{D}} f = \max_{\mathbb{D}_+} f.$$

An analysis of the diameter of the polar caps is not needed for the proof of Theorem 2.5. It is included for completeness, and for comparison to the Feige–Schechtman bound to be examined below. This is a consequence of the isodiametric inequality for $\mathbb{S}^d$.

THEOREM 3.5. (Isodiametric inequality for $\mathbb{S}^d$) *Any region $R \subset \mathbb{S}^d$ of spherical diameter $\delta < \pi$ has area bounded by*

$$\sigma(R) \leqslant \mathcal{V}\left(\frac{\delta}{2}\right).$$

Equality holds only for spherical caps of spherical radius $\frac{\delta}{2}$.

This result is well known; see [2] for a proof of a generalized version of this inequality, based on the proof of [1].

We have the following upper bound for the diameter of a polar cap of $\mathrm{EQ}(d, N)$.

LEMMA 3.6. *For $d > 1$ and $N \geqslant 2$, the diameter of each polar cap of* $\mathrm{EQ}(d, N)$ *is bounded above by $K_c N^{-\frac{1}{d}}$, where*

$$K_c := 2 \left(\frac{\Omega\, d}{\omega} \right)^{\frac{1}{d}}.$$

The following two bounds are used in the proof of Theorem 2.5.

LEMMA 3.7. *For $d > 1$, there is a positive constant $N_\Delta \in \mathbb{N}$ and a monotonic decreasing positive real function K_Δ such that for each partition* $\mathrm{EQ}(d, N)$ *with*

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$N > x \geqslant N_\Delta$,

$$\max_{\mathbb{D}} \Delta \leqslant K_\Delta(x) N^{-\frac{1}{d}}.$$

LEMMA 3.8. *For $d > 1$, there is a positive constant $N_P \in \mathbb{N}$ and a monotonic decreasing positive real function C_P such that for each partition* $\mathrm{EQ}(d, N)$ *with*
$N > x \geqslant N_P$,

$$\max_{\mathbb{D}} \mathcal{P} \leqslant C_P(x) N^{-\frac{1}{d}}.$$

4. Proofs of main theorems.

Proof of Theorem 2.5. The theorem is true for $d = 1$, with $\mathrm{EQ}(1)$ having diameter bound $K_1 = 2\pi$, since the recursive zonal equal area partition algorithm partitions the circle $\mathbb{S}^1$ into N equal segments, each of arc length $2\pi/N$, and therefore each segment has diameter less than $2\pi/N$.

Now assume that $d > 1$ and $N > 2$. We know from Lemma 3.2 that

$$\operatorname{maxdiam}(d, N) \leqslant \sqrt{\left(\max_{\mathbb{D}} \Delta\right)^2 + \kappa^2 \left(\max_{\mathbb{D}} \mathcal{P}\right)^2}.$$

From Lemma 3.7, we know that there is a positive constant $N_\Delta \in \mathbb{N}$ and a monotonic decreasing positive real function K_Δ such that for each partition $\mathrm{EQ}(d, N)$ with $N > x \geqslant N_\Delta$,

$$\max_{\mathbb{D}} \Delta \leqslant K_\Delta(x) N^{-\frac{1}{d}}.$$

From Lemma 3.8, we know that there is a positive constant $N_P \in \mathbb{N}$ and a monotonic decreasing positive real function C_P such that for each partition $\mathrm{EQ}(d, N)$ with $N > x \geqslant N_P$,

$$\max_{\mathbb{D}} \mathcal{P} \leqslant C_P(x) N^{-\frac{1}{d}}.$$

Define

$$N_H := \max(N_\Delta, N_P).$$

Assuming that $\mathrm{EQ}(d-1)$ is diameter bounded, with diameter bound κ, then for $N > N_H$, we have $\operatorname{maxdiam}(d, N) \leqslant K_H N^{-\frac{1}{d}}$, where

$$K_H := \sqrt{K_\Delta(N_H)^2 + \kappa^2 C_P(N_H)^2}.$$

For $d > 1$ and $N \leqslant N_H$, we note that the diameter of $\mathbb{S}^d$ is 2, and so the diameter of any region is bounded by 2. Therefore for $N \leqslant N_H$, $\operatorname{maxdiam}(d, N) \leqslant K_L N^{-\frac{1}{d}}$, where

$$K_L := 2 N_H^{\frac{1}{d}}.$$

Finally, we see by induction that for $d > 1$, $\operatorname{maxdiam}(d, N) \leqslant K_d N^{-\frac{1}{d}}$, where

$$K_d := \max(K_L, K_H). \quad \square$$

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5. Estimates for caps. Later we will need to compare $\sin\theta$ with $\sin(\theta+\phi)$, for various θ and ϕ. The following estimate is useful for this task. For all $\theta, \phi \in \mathbb{R}$, we have

$$\sin(\theta+\phi) - \sin\theta = 2\ \sin\frac{\phi}{2}\ \cos\left(\theta + \frac{\phi}{2}\right).$$

Therefore for $\phi \in (0, \pi]$, $\theta \in (0, \pi/2 - \phi/2]$, we have $\sin(\theta+\phi) > \sin\theta > 0$.

In the estimate below, we assume that $\theta \in (0, \xi]$, $\xi \in (0, \pi/2]$, and use the well-known sine ratio function

$$\operatorname{sinc}\theta := \frac{\sin\theta}{\theta}.$$

We have the well-known estimate

$$\sin\theta \in [\operatorname{sinc}\xi, 1]\,\theta. \tag{5.1}$$

In the estimates below we assume that $\theta \in (0, \xi]$, $\xi \in (0, \pi/2]$. From (2.2) we have $D\mathcal{V}(\theta) = \omega \sin^{d-1}\theta$. Using the estimate (5.1) therefore gives us

$$D\mathcal{V}(\theta) \in [(\operatorname{sinc}\xi)^{d-1}, 1]\ \omega\theta^{d-1}.$$

Thus,

$$\mathcal{V}(\theta) \in [(\operatorname{sinc}\xi)^{d-1}, 1]\ \frac{\omega}{d}\theta^{d}. \tag{5.2}$$

If we then substitute $\Theta(v)$ for θ, we obtain for $v \in [0, \mathcal{V}(\xi)]$ that

$$\Theta(v) \in [1, (\operatorname{sinc}\xi)^{\frac{1-d}{d}}]\ \left(\frac{d}{\omega}\right)^{\frac{1}{d}}\ v^{\frac{1}{d}}. \tag{5.3}$$

The estimates (5.2) and (5.3) are crude. There are instances where we need a sharper upper bound than that given by (5.2). The estimate below is more accurate for large d for θ away from $\pi/2$.

LEMMA 5.1. [7, Lemma 2.3.18] *For $d \geqslant 2$ and $\theta \in [0, \pi/2)$ we have*

$$\mathcal{V}(\theta) \leqslant \frac{\omega}{d}\frac{\sin^d\theta}{\cos\theta}, \tag{5.4}$$

with equality only when $\theta = 0$.

If we combine (5.2) with (5.4), we obtain the following result.

COROLLARY 5.2. *For $d \geqslant 2$ and $\theta \in [0, \pi/2)$ we have*

$$\mathcal{V}(\theta) \in \left[\frac{1}{\operatorname{sinc}\theta}, \frac{1}{\cos\theta}\right]\ \frac{\omega}{d}\ \sin^d\theta. \tag{5.5}$$

Recall from (2.3) that $\vartheta_c = \Theta\left(\frac{\Omega}{N}\right)$ and define

$$J_c(x) := \operatorname{sinc}\Theta\left(\frac{\Omega}{x}\right). \tag{5.6}$$

As a result of (5.3), for $N \geqslant x \geqslant 2$, we have

$$\vartheta_c \in [1, J_c(x)^{\frac{1-d}{d}}]\ \left(\frac{d}{\omega}\right)^{\frac{1}{d}}\ \delta_I. \tag{5.7}$$

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Using Lemma 5.1, we obtain the following upper bound for $\sin\vartheta_c$.

LEMMA 5.3. [7, Lemma 3.5.14] *For $x \geqslant 2$,*

$$x^{\frac{1}{d}} \sin\Theta\left(\frac{\Omega}{x}\right) \leqslant \left(\frac{\Omega\, d}{\omega}\right)^{\frac{1}{d}}. \tag{5.8}$$

Therefore, for $N \geqslant 2$,

$$\sin\vartheta_c \leqslant \left(\frac{d}{\omega}\right)^{\frac{1}{d}} \delta_I. \tag{5.9}$$

Combining (5.6), (5.7), and (5.9), we have the estimate

$$\sin\vartheta_c \in [J_c(x), 1] \left(\frac{d}{\omega}\right)^{\frac{1}{d}} \delta_I \tag{5.10}$$

for $N \geqslant x \geqslant 2$.

6. The modified Feige and Schechtman construction. Feige and Schechtman [5] give a constructive proof of the following lemma, which can be used to prove Stolarsky's assertion.

LEMMA 6.1. [5, Lemma 21, pp. 430–431] *For each $0 < \gamma < \pi/2$ the sphere $\mathbb{S}^{d-1}$ can be partitioned into $N = \big(O(1)/\gamma\big)^d$ regions of equal area, each of diameter at most γ.*

Lemma 6.1 corresponds to a diameter bound of order $\mathrm{O}(N^{\frac{1}{d+1}})$ rather than $\mathrm{O}(N^{\frac{1}{d}})$, but the construction given in the proof [5, pp. 430–431] is easily modified to yield the following upper bound on the smallest maximum diameter of an equal area partition of $\mathbb{S}^d$.

LEMMA 6.2. *For $d > 1$, $N > 2$, there is a partition $FS(d, N)$ of the unit sphere $\mathbb{S}^d$ into N regions, with each region $R \in FS(d, N)$ having area Ω/N and Euclidean diameter bounded above by*

$$\operatorname{diam} R \leqslant \Upsilon\left(\min(\pi, 8\vartheta_c)\right),$$

with Υ defined by (2.1) and ϑ_c defined by (2.3).

We now use the modified Feige–Schechtman construction to prove Stolarsky's assertion, Theorem 2.6.

Proof of Theorem 2.6. For $d = 1$, we partition the circle into equal segments and the proof is as per the proof of Theorem 2.5. For $d > 1$ and $N = 1$, there is one region of diameter $2 = 2N^{-\frac{1}{d}}$. For $d > 1$ and $N = 2$, there are two regions, each of diameter $2 = 2^{\frac{d+1}{d}} N^{-\frac{1}{d}}$. Otherwise, we use Lemma 6.2 and the estimates (5.7) and (5.9). Define

$$N_{FS} = \frac{\Omega}{\mathcal{V}\left(\frac{\pi}{8}\right)}.$$

Then for $N \geqslant N_{FS}$,

$$\vartheta_c = \Theta\left(\frac{\Omega}{N}\right) \leqslant \frac{\pi}{8},$$

with equality only when $N = N_{FS}$. Therefore, for $N \geqslant N_{FS}$, Lemmas 5.3 and 6.2 give us

$$\max_{R \in FS(d,N)} \operatorname{diam} R \leqslant 2\, \sin 4\vartheta_c < 8\, \sin\vartheta_c < K_{FS} N^{-\frac{1}{d}},$$

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where

$$K_{FS} := 8 \left(\frac{\Omega\, d}{\omega} \right)^{\frac{1}{d}}.$$

For $2 < N < N_{FS}$, we have

$$\operatorname{maxdiam} FS(d,N) \leqslant 2 = 2\, N^{\frac{1}{d}} N^{-\frac{1}{d}} < 2\, N_{FS}^{\frac{1}{d}} N^{-\frac{1}{d}}.$$

Let $K_{FSL} := 2\, N_{FS}^{\frac{1}{d}}$. Using (5.5), we have

$$\mathcal{V}\left(\frac{\pi}{8}\right) \geqslant \frac{1}{\operatorname{sinc} \frac{\pi}{8}} \frac{\omega}{d} \sin^d \frac{\pi}{8} > \frac{\omega}{d} \sin^d \frac{\pi}{8}.$$

We also have $\sin \frac{\pi}{8} > \frac{1}{4}$, so that

$$\mathcal{V}\left(\frac{\pi}{8}\right) > \frac{\omega}{4^d\, d}.$$

Therefore

$$N_{FS} = \frac{\Omega}{\mathcal{V}\left(\frac{\pi}{8}\right)} < 4^d\, \frac{\Omega\, d}{\omega}.$$

In other words,

$$K_{FSL}^d = 2^d\, N_{FS} < 8^d\, \frac{\Omega\, d}{\omega} = K_{FS}^d.$$

We therefore have $K_{FSL} < K_{FS}$. For $d \geqslant 2$ we have [7, Lemma 2.3.20]

$$\frac{\Omega}{\omega} > \sqrt{\frac{2\pi}{d}}. \tag{6.1}$$

For the case $N = 2$, from (6.1) we obtain

$$2^{d+1} < 8^d\, \sqrt{2\pi d} < 8^d\, \frac{\Omega\, d}{\omega} = K_{FS}^d.$$

Therefore Theorem 2.6 is satisfied by $c_d = K_{FS}$. □

REMARK 6.3. The Feige–Schechtman constant K_{FS} thus provides an upper bound for the minimum constant for the diameter bound of an equal area partition of $\mathbb{S}^d$. Theorems 2.4 and 2.5 yield an alternate proof of Theorem 2.6, with $c_d = K_d$.

Appendix A. Proofs of Lemmas. The definitions of the functions Δ and $\mathcal{P}$ and the definition of the feasible domain $\mathbb{D}$ depend on the fitting collar angle δ_F. Thus the proofs of Lemmas 3.7 and 3.8 need an estimate for δ_F. Recall from (3.1) that $\delta_F = \rho \delta_I$. Therefore, from (3.5), for $N > x > \mathcal{N}_0(1/2)$, where $\mathcal{N}_0$ is defined by (3.3), we have

$$\delta_F \in [\rho_L(x), \rho_H(x)]\delta_I. \tag{A.1}$$

We also need estimates for $\vartheta_{F,i}$, as defined by Step 5 of the EQ partition algorithm [8, Section 3.2], and for $\sin \vartheta_{F,i}$ and $\mathcal{V}(\vartheta_{F,i})$. Here and below, we generalize the definition of $\vartheta_{F,i}$, by defining

$$\vartheta_{F,\iota} := \vartheta_c + (\iota - 1)\delta_F,$$

for $\iota \in [1, n+1]$. For $N > x > \mathcal{N}_0(1/2)$, where $\mathcal{N}_0$ is defined by (3.3), the estimates (5.7) and (A.1) now yield

$$\vartheta_{F,\iota} \in \left[\left(\frac{d}{\omega}\right)^{\frac{1}{d}} + (\iota-1)\rho_L(x), \left(\frac{d}{\omega}\right)^{\frac{1}{d}} J_c(x)^{\frac{1-d}{d}} + (\iota-1)\rho_H(x) \right] \delta_I. \tag{A.2}$$

The estimates for $\sin\vartheta_{F,\iota}$ and $\mathcal{V}(\vartheta_{F,\iota})$ below assume that $N > x > \mathcal{N}_0(1/2)$, where $\mathcal{N}_0$ is defined by (3.3), and the lower bounds for these estimates also assume that

$$\Theta\left(\frac{\Omega}{x}\right) + (\iota-1)\rho_H(x)\left(\frac{\Omega}{x}\right)^{\frac{1}{d}} \leqslant \frac{\pi}{2}. \tag{A.3}$$

If we define

$$J_{F,\iota}(x) := \operatorname{sinc}\left(\Theta\left(\frac{\Omega}{x}\right) + (\iota-1)\rho_H(x)\left(\frac{\Omega}{x}\right)^{\frac{1}{d}}\right),$$

then from (5.1) and (A.2) we have the estimate

$$\sin\vartheta_{F,\iota} \in \left[J_{F,\iota}(x)\left(\left(\frac{d}{\omega}\right)^{\frac{1}{d}} + (\iota-1)\rho_L(x)\right), \left(\frac{d}{\omega}\right)^{\frac{1}{d}} J_c(x)^{\frac{1-d}{d}} + (\iota-1)\rho_H(x) \right] \delta_I$$

and from (5.2) we have the estimate

$$\mathcal{V}(\vartheta_{F,\iota}) \in [s_{L,\iota}(x), s_{H,\iota}(x)]\mathcal{V}_R,$$

where

$$s_{L,\iota}(x) := J_{F,\iota}(x)^{d-1}\left(1 + (\iota-1)\rho_L(x)\left(\frac{\omega}{d}\right)^{\frac{1}{d}}\right)^d,$$
$$s_{H,\iota}(x) := \left(J_c(x)^{\frac{1-d}{d}} + (\iota-1)\rho_H(x)\left(\frac{\omega}{d}\right)^{\frac{1}{d}}\right)^d.$$

If we define

$$s_\iota := \left(1 + (\iota-1)\left(\frac{\omega}{d}\right)^{\frac{1}{d}}\right)^d,$$

then, since $J_{F,\iota}(x) \nearrow 1$, $J_c(x) \nearrow 1$, $\rho_L(x) \nearrow 1$ and $\rho_H(x) \searrow 1$, as $x \to \infty$ we see that

$$s_{L,\iota}(x) \nearrow s_\iota \text{ and } s_{H,\iota}(x) \searrow s_\iota \text{ as } x \to \infty.$$

By making x large enough and ι small enough, we can ensure that (A.3) holds.

LEMMA A.1. [7, Lemma 3.5.16] *If $x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.3), then (A.3) holds for*

$$\iota \in \left[1, \frac{13}{4}\right].$$

For the remainder of this paper we use the abbreviation

$$\eta := \frac{1}{\sqrt{8\pi d}}.$$

The proofs of Lemmas 3.7 and 3.8 require the following results, which are proved in [7, Chapter 3].

LEMMA A.2. [7, Lemma 3.5.17] *There is an* $x \geqslant \mathcal{N}_0(5)$*, such that*

$$J_{F,(1+\eta)}(x)^{d-1}\left(1+\eta\,\rho_L(x)\,\left(\frac{\omega}{d}\right)^{\frac{1}{d}}\right)^d > \frac{3}{2}. \tag{A.4}$$

LEMMA A.3. [7, Lemma 3.5.19] *If* $x \geqslant \mathcal{N}_0(5)$*, and* x *satisfies (A.4), then for* $N > x$ *we have*

$$\mathcal{V}(\vartheta_c + \eta\delta_F) > \frac{3}{2}\mathcal{V}_R. \tag{A.5}$$

As a result of (A.5), we have

$$\mathcal{V}(\vartheta_c + \eta\delta_F) - \mathcal{V}(\vartheta_c) > \frac{\mathcal{V}_R}{2}.$$

From (2.2) and the symmetries of the sine function, for $\vartheta \in (0, \pi/2 - \eta\delta_F/2]$, we have

$$\begin{aligned}\frac{\partial}{\partial\vartheta}\left(\mathcal{V}(\vartheta+\eta\delta_F) - \mathcal{V}(\vartheta)\right) &= D\mathcal{V}(\vartheta+\eta\delta_F) - D\mathcal{V}(\vartheta)\\ &= \omega\left(\sin^{d-1}(\vartheta+\eta\delta_F) - \sin^{d-1}\vartheta\right) \geqslant 0,\end{aligned} \tag{A.6}$$

with equality only when $\vartheta = \frac{\pi}{2} - \eta\frac{\delta_F}{2}$. This results in the following corollary.

COROLLARY A.4. [7, Corollary 3.5.20] *If* $x \geqslant \mathcal{N}_0(5)$*, and* x *satisfies (A.4), then for* $N > x$ *and* $\vartheta \in [\vartheta_c, \pi - \vartheta_c - \eta\delta_F]$ *we have*

$$\mathcal{V}(\vartheta + \eta\delta_F) - \mathcal{V}(\vartheta) > \frac{\mathcal{V}_R}{2}. \tag{A.7}$$

If $x \geqslant \mathcal{N}_0(5)$, and $N > x$ then $n \geqslant 5$, so $\vartheta_{F,2} < \frac{\pi}{2}$. Since $8\pi d \geqslant 16\pi > 49$, we therefore have

$$\eta\delta_F < \frac{\delta_F}{7}. \tag{A.8}$$

Proof of Lemma 3.6. Assume that $d > 1$ and $N > 1$. From (2.3) we know that the diameter of each of the polar caps of the partition $\mathrm{EQ}(d,N)$ is $2\sin\vartheta_c$. From (5.9) we have the estimate

$$2\sin\vartheta_c \leqslant 2\left(\frac{\Omega\, d}{\omega}\right)^{\frac{1}{d}} N^{-\frac{1}{d}}.$$

for $N \geqslant x \geqslant 2$. ☐

Proof of Lemma 3.7. Throughout this proof, we assume that $N > x$ where $x \geqslant \mathcal{N}_0(5)$, with $\mathcal{N}_0$ defined by (3.3), so that $n \geqslant 5$. Using Corollary 3.4, we also assume that $(\tau,\beta,\vartheta) \in \mathbb{D}_+$. For the top collar, $(\tau,\beta,\vartheta) \in \mathbb{D}_t$, (3.9) gives $\tau = 0$, $\beta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\vartheta = \vartheta_c$. From (2.2) we have

$$\mathcal{V}\left(\mathcal{B}(\beta,\vartheta_c)\right) = \mathcal{V}(\vartheta_c + \delta_F) + \beta\mathcal{V}_R \leqslant \mathcal{V}(\vartheta_c + \delta_F) + \frac{\mathcal{V}_R}{2}.$$

Since $n \geqslant 5$, we have $\vartheta_c + \delta_F \in [\vartheta_c, \pi - \vartheta_c - \eta\delta_F]$, and we can use (A.7) to obtain

$$\mathcal{V}\left(\mathcal{B}(\beta, \vartheta_c)\right) \leqslant \mathcal{V}(\vartheta_c + \delta_F) + \frac{\mathcal{V}_R}{2} < \mathcal{V}\big(\vartheta_c + (1+\eta)\delta_F\big),$$

and therefore

$$\mathcal{B}(\beta, \vartheta_c) < \vartheta_c + (1+\eta)\delta_F.$$

Therefore (3.8) yields

$$\Delta(\tau, \beta, \vartheta) = \Delta(0, \beta, \vartheta_c) = \mathcal{B}(\beta, \vartheta_c) - \mathcal{T}(0, \vartheta_c) = \mathcal{B}(\beta, \vartheta_c) - \vartheta_c < (1+\eta)\delta_F.$$

For $(\tau, \beta, \vartheta) \in \mathbb{D}_{m+}$ (3.10) gives $\tau \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\beta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\vartheta \in \left[\vartheta_{F,2}, \frac{\pi}{2} - \frac{\delta_F}{2}\right]$. Since $n \geqslant 5$, we have $\vartheta + \delta_F \in [\vartheta_c, \pi - \vartheta_c - \eta\delta_F]$, since

$$\vartheta_c + \frac{3}{2}\delta_F < \vartheta_c + 2\delta_F < \frac{\pi}{2},$$

yielding

$$\vartheta + \delta_F \leqslant \frac{\pi}{2} + \frac{\delta_F}{2} < \pi - \vartheta_c - \delta_F.$$

From (2.2), (3.8), and (A.7), we now have

$$\mathcal{V}\big(\mathcal{B}(\beta, \vartheta)\big) = \mathcal{V}(\vartheta + \delta_F) + \beta\mathcal{V}_R \leqslant \mathcal{V}(\vartheta + \delta_F) + \frac{\mathcal{V}_R}{2} < \mathcal{V}\big(\vartheta + (1+\eta)\delta_F\big).$$

We therefore have

$$\mathcal{B}(\beta, \vartheta) < \vartheta + (1+\eta)\delta_F. \tag{A.9}$$

Since $\vartheta - \eta\delta_F > \vartheta_c$, using (2.2), (3.8), and (A.7), we also have

$$\mathcal{V}\big(\mathcal{T}(\tau, \vartheta)\big) = \mathcal{V}(\vartheta) + \tau\mathcal{V}_R \geqslant \mathcal{V}(\vartheta) - \frac{\mathcal{V}_R}{2} > \mathcal{V}(\vartheta - \eta\delta_F),$$

so that

$$\vartheta - \eta\delta_F < \mathcal{T}(\tau, \vartheta). \tag{A.10}$$

Combining (A.9) and (A.10) and using (3.8), we therefore have

$$\Delta(\tau, \beta, \vartheta) = \mathcal{B}(\beta, \vartheta) - \mathcal{T}(\tau, \vartheta) < (1+2\eta)\delta_F.$$

The estimate (A.1) now yields

$$\Delta(\tau, \beta, \vartheta) < K_\Delta(x) N^{-\frac{1}{d}},$$

where

$$K_\Delta(x) := (1+2\eta)\, \rho_H(x)\, \Omega^{\frac{1}{d}},$$

with $\rho_H(x)$ defined by (3.6). We also have

$$K_\Delta(x) \searrow K_\Delta(\infty) := (1+2\eta)\, \Omega^{\frac{1}{d}} \text{ as } x \to \infty,$$

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since $\rho_H(x) \searrow 1$ as $x \to \infty$, by(3.7). $\square$

Proof of Lemma 3.8. Throughout this proof, we assume that $N > x$ where $x \geqslant \mathcal{N}_0(5)$, with $\mathcal{N}_0$ defined by (3.3), so that $n \geqslant 5$. Using Corollary 3.4, we also assume that $(\tau, \beta, \vartheta) \in \mathbb{D}_+$. We will show that

$$\begin{aligned} \mathcal{W}(\tau, \beta, \vartheta) &\leqslant C_1(x) \sin \vartheta, \\ \mathcal{M}(\tau, \beta, \vartheta) &\geqslant C_2(x) \sin^{d-1} N^{\frac{d-1}{d}}, \end{aligned}$$

with C_1 monotonic non-increasing and C_2 monotonic non-decreasing. We first examine $\mathcal{W}$. Using (A.9) for $\vartheta \leqslant \pi/2 - (1+\eta)\delta_F$, we have

$$\mathcal{W}(\tau, \beta, \vartheta) \leqslant \sin(\vartheta + (1+\eta)\delta_F) < \sin \vartheta + (1+\eta)\delta_F.$$

For $\vartheta \in [\pi/2(1+\eta)\delta_F, \pi/2 - \delta_F/2]$, we have

$$\sin \vartheta + (1+\eta)\delta_F \geqslant \frac{2}{\pi}\left(\frac{\pi}{2} - (1+\eta)\delta_F\right) + (1+\eta)\delta_F \geqslant 1.$$

Hence, $\mathcal{W}(\tau, \beta, \vartheta) < \sin \vartheta + (1+\eta)\delta_F$. Since $\vartheta \in [\vartheta_c, \pi - \vartheta_c]$ we have $\sin \vartheta \geqslant \sin \vartheta_c$ and therefore

$$\mathcal{W}(\tau, \beta, \vartheta) < \left(1 + (1+\eta)\frac{\delta_F}{\sin \vartheta_c}\right) \sin \vartheta.$$

From (A.1) we have $\delta_F \leqslant \rho_H(x) \Omega^{\frac{1}{d}} N^{\frac{-1}{d}}$. From (5.10) we have

$$\sin \vartheta_c \geqslant J_c(x) \left(\frac{\Omega d}{\omega}\right)^{\frac{1}{d}} N^{\frac{-1}{d}},$$

so that $\mathcal{W}(\tau, \beta, \vartheta) \leqslant C_1(x) \sin \vartheta$, with

$$C_1(x) := 1 + (1+\eta)\frac{\rho_H(x)}{J_c(x)} \left(\frac{\omega}{d}\right)^{\frac{1}{d}},$$

with $\frac{\rho_H(x)}{J_c(x)} \searrow 1$ as $x \to \infty$, since $J_c(x) \nearrow 1$ and $\rho_H(x) \searrow 1$ as $x \to \infty$. Thus $C_1(x)$ is monotonic nonincreasing as $x \to \infty$. Now for $\mathcal{M}$. From (3.8) we have

$$\mathcal{M}(\tau, \beta, \vartheta) \geqslant \frac{\mathcal{V}(\vartheta + \delta_F) - \mathcal{V}(\vartheta)}{\mathcal{V}_R} - 1.$$

But

$$\frac{\mathcal{V}(\vartheta + \delta_F) - \mathcal{V}(\vartheta)}{\mathcal{V}_R} = \omega \int_{\vartheta}^{\vartheta + \delta_F} \sin^{d-1} \xi \, d\xi \geqslant \omega \delta_F \sin^{d-1} \vartheta$$

for $\vartheta \in [0, \pi/2 - \delta_F/2]$. Therefore,

$$\begin{aligned} \mathcal{M}(\tau, \beta, \vartheta) &\geqslant \omega \sin^{d-1} \vartheta \frac{\delta_F}{\mathcal{V}_R} - 1 \\ &\geqslant \left(\rho \omega \Omega^{\frac{1-d}{d}} - \frac{1}{\sin^{d-1} \vartheta_c N^{\frac{d-1}{d}}}\right) \sin^{d-1} \vartheta N^{\frac{d-1}{d}} \end{aligned}$$

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since $\vartheta \geqslant \vartheta_c$. Using (3.5) and (5.10) we therefore have

$$\mathcal{M}(\tau, \beta, \vartheta) \geqslant C_2(x) \sin^{d-1} \vartheta N^{\frac{d-1}{d}},$$

where

$$C_2(x) := \rho\omega\Omega^{\frac{1-d}{d}} - J_c(x)^{1-d} \left(\frac{\omega}{\Omega d}\right)^{\frac{d-1}{d}}.$$

If $J_c(x)^{d-1}\rho_L(x)\omega^{\frac{1}{d}} d^{\frac{d-1}{d}} > 1$ then we have $C_2(x) > 0$. This is true for x sufficiently large since $\omega d^{d-1} > 1$ and since both $J_c(x) \nearrow 1$ and $\rho_L(x) \nearrow 1$ as $x \to \infty$. We also see that C_2 is monotonically nondecreasing. □

Proof of Lemma 6.2. This proof uses a modified version of the construction given the proof of [5, Lemma 21] in [5, p. 430–431].

1. Given $d > 1$, $N > 2$, use (2.3) to determine ϑ_c. Then we have $\mathcal{V}(\vartheta_c) = \mathcal{V}_R = \Omega/N$, with $\mathcal{V}_R$ being the area we need for each region of the partition.
2. A *saturated packing* of packing radius ρ is a packing of spherical caps of packing radius ρ such that another cap cannot be added without moving the existing caps. Create a saturated packing of $\mathbb{S}^d$ by caps of spherical radius ϑ_c, constructed via a greedy algorithm so that each cap kisses at least one other cap. Let m be the number of caps in the packing. We see that no point of $\mathbb{S}^d$ is more than $2\vartheta_c$ from the centre of a cap, otherwise we could have added another cap. Therefore the m centre points of the packing are also the centres of a covering of $\mathbb{S}^d$ by spherical caps of spherical radius $2\vartheta_c$ [13, p. 1091] [14, Lemma 1, p. 2112].
3. Now partition $\mathbb{S}^d$ into Voronoi cells V_i, $i \in \{1, \ldots, m\}$ based on these m centre points. The Voronoi cell V_i corresponding to centre point i consists of those points of $\mathbb{S}^d$ which are at least as close to the centre point i as they are to of any of the other centre points. We see that the Voronoi cells must contain the packing caps and be contained in the covering caps. Thus each V_i has area at least $\mathcal{V}_R$ and spherical diameter at most $\min(\pi, 2\vartheta_c)$.
4. Now create a graph Γ with a node for each centre point and an edge for each pair of kissing packing caps.
5. Take any spanning tree S of Γ (also known as a *maximal tree* [10, Section 6.2 pp. 101–103]). The tree S has leaves, which are nodes having only one edge, and either a single centre node, or a bicentre, which is a pair of nodes joined by an edge. The centre or bicentre nodes are the nodes for which the shortest path to any leaf has the maximum number of edges [3] [4, Volume 9, p. 430] [11, Chapter 6, Section 9, p. 135]. If there is a single centre, mark it as the root node. If there is a bicentre, arbitrarily mark one of the two nodes as the root node. Now create the directed tree T from S by directing the edges from the leaves towards the root [11, Chapter 6, Section 7, p. 129].
6. For each leaf j, of T define $n_j := \lfloor \sigma(V_j)/\mathcal{V}_R \rfloor$, (with $\lfloor x \rfloor$ denoting the least integer function of x).
7. Partition V_j into the super-region U_j with $\sigma(U_j) = n_j \mathcal{V}_R$ and the remainder $W_j := V_j \setminus U_j$.
8. For each nonleaf node k other than the root, define $X_k = V_k \cup \bigcup_{(j,k)\in T} W_j$, that is, we add all the remainders of the daughters of k to V_k to obtain X_k.
9. Now define $n_k := \lfloor \sigma(X_k)/\mathcal{V}_R \rfloor$ and partition X_k into the super-region U_k with $\sigma(U_k) = n_k \mathcal{V}_R$ and the remainder $W_k := X_k \setminus U_k$.
10. Continue until only the root node is left.

11. For the root node ℓ, if we define $U_\ell := V_\ell \cup \bigcup_{(k,\ell)\in T} W_k$, we see that we must have $\sigma(U_\ell) = n_\ell \mathcal{V}_R$, where
$$n_\ell := N - \sum_{i \neq \ell} n_i.$$
that is, the area of the super-region corresponding to the root node must be an integer multiple of $\mathcal{V}_R$. Since at each step we have assembled U_i only from the Voronoi cells corresponding to kissing packing caps, each U_i is contained in a spherical cap with centre the same as the centre of the corresponding packing cap, and spherical radius $\min(\pi, 4\vartheta_c)$, and so the spherical diameter of each U_i is at most $\min(\pi, 8\vartheta_c)$.
12. Now partition each U_i into n_i regions of area $\mathcal{V}_R$, and let $FS(d, N)$ be the resulting partition of $\mathbb{S}^d$. Then $FS(d, N)$ is a partition of $\mathbb{S}^d$ into N regions, with each region $R \in FS(d, N)$ having area Ω/N and Euclidean diameter bounded above by
$$\operatorname{diam} R \leqslant \Upsilon\left(\min(\pi, 8\vartheta_c)\right) = 2\sin\left(\min\left(\frac{\pi}{2}, 4\vartheta_c\right)\right). \quad \square$$

REMARK A.5. Feige and Schechtman's proof uses a maximal packing instead of a saturated packing, but maximality is harder to achieve and the proof of Lemma 6.2 only needs a saturated packing.

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