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THE SINC-GALERKIN METHOD FOR SOLVING SINGULARLY-PERTURBED REACTION-DIFFUSION PROBLEM*

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Abstract. One of the new techniques used in solving boundary-value problems involving partial differential equations is the Sinc-Galerkin method. In this paper we solve the singularly-perturbed reaction-diffusion problem using the Sinc-Galerkin method. The scheme is tested on four problems and a comparison with finite element methods and the method of reduction of order is made. It is show that the Sinc-Galerkin method yields better results.

Key words. Sinc function, Sinc-Galerkin, singularly perturbed, reaction-diffusion, numerical solutions.

AMS subject classifications. 65N30,65T60, 35K05, 42C40

1. Introduction. There is a vast literature on numerical solutions of boundary-value problems involving ordinary and partial differential equations. Some of the well-known techniques used in solving these problems are the finite differences, finite elements, and multigrid methods.

The Galerkin method is another type of numerical technique used to solve partial differential equations with boundary or initial conditions. In this method the solution is assumed to be in a Hilbert space H with inner product $\langle , \rangle$, and one seeks an approximate solution to the problem in the form $\phi(x) = \sum_{k=1}^{N} a_k \psi_k(x)$, where $\{\psi_k(x)\}_{k=1}^{N}$ is a basis for an N-dimensional subspace of functions, S. The functions $\psi_k(x), k = 1, 2, \cdots, N$, are called test functions and the space S is called the test space.

There are number of hybrids of the Galerkin method that use different types of test functions. In the last decade or so, Sinc functions have been used in many applications, including numerical solutions of ordinary and partial differential equations. In the Sinc-Galerkin method, the test functions are translates of the sinc function, $S(x) = \sin \pi x/\pi x$. The sinc method, which was introduced and developed by F. Stenger more than twenty years ago [23], is based on the Whittaker-Shannon-Kotel'nikov sampling theorem for entire functions. This method, which uses entire functions as bases, has many advantages over classical methods that use polynomials as bases. For example, in the presence of singularities, it gives a much better rate of convergence and accuracy than polynomial methods.

In recent years, a lot of attention has been devoted to the study of the Sinc-Galerkin method to investigate various scientific models. The efficiency of the method has been formally proved by many researchers [3, 5, 6, 7, 15, 18, 22, 24]. For more details of the Sinc-Galerkin method see[16, 23] and the references therein.

In this paper, we will consider the Sinc-Galerkin method for the singularly perturbed elliptic boundary value problem

$$-\epsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+a\,u=f(x,y) \qquad \text{in} \quad \Omega=(0,1)\times(0,1), \tag{1.1}$$

and

$$u = 0, \quad \text{on} \quad \partial\,\Omega, \tag{1.2}$$

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where $0 < \epsilon \ll 1$, a is a finite constant and $f(x, y)$ is analytic in Ω. We shall assume that the solution $u(x, y)$ is analytic in Ω. For more details about the formulation of this equation, see Roos [20].

The numerical solution of singularly perturbed boundary value problems has recently received much attention. In fact, singularly perturbed problems appear in many branches of engineering, such as fluid mechanics, heat transfer, and problems in structural mechanics posed over thin domains. For more details see Roos [20]. Theorems that list conditions for the existence and uniqueness of solutions of such problems are thoroughly discussed in [20]. This kind of problem has been investigated by many researchers [1, 2, 4, 8, 9, 10, 13, 17, 19, 21].

The organization of the paper is as follows. In section 2, we review some basic facts about the sinc approximation that are necessary for the formulation of the discrete system. In section 3, the Sinc-Galerkin method is developed for linear second-order singularly perturbed boundary value problems with homogeneous boundary condition. The Sinc-Galerkin method is developed for nonlinear second-order singularly perturbed boundary value problems in section 4. Section 5 addresses singularly perturbed reaction diffusion equations, which will be solved using the Sinc-Galerkin method. Finally, some numerical examples will be presented in section 6, followed by the conclusions.

2. Sinc Interpolation. The goal of this section is to recall notation and definitions of the Sinc function, state some known results, and derive useful formulas that are important for this paper. First we denote the set of all integers, the set of all real numbers, and the set of all complex numbers by R, Z and C, respectively.

The sinc function is defined on R by

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \quad -\infty < x < \infty.$$

For $h > 0$, the translated sinc functions with evenly spaced nodes are given as

$$S(k, h)(x) = \text{sinc}\left(\frac{x - kh}{h}\right), \quad \text{k} = 0, \pm 1, \pm 2, \ldots.$$

If f is defined on R, then for $h > 0$ the series

$$C(f, h) = \sum_{k=-\infty}^{\infty} f(hk)\,\text{sinc}\left(\frac{x - hk}{h}\right)$$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of Whittaker cardinal expansions have been studied and are thoroughly surveyed in [23]. These properties are derived in the infinite strip D_d of the complex plane where for $d > 0$

$$D_d = \left\{\zeta = \xi + i\eta : |\eta| < d \leq \frac{\pi}{2}\right\}.$$

Approximations can be constructed for infinite, semi-finite, and finite intervals. To construct approximations on the interval $(0, 1)$ which are used in this paper, consider the conformal map

$$\phi(z) = \ln\left(\frac{z}{1 - z}\right),$$

which maps the eye-shaped region,

$$D = \left\{z = x + iy : \left|\arg\left(\frac{z}{1 - z}\right)\right| < d \leq \frac{\pi}{2}\right\},$$

onto the infinite strip D_d.

The sinc-Galerkin method requires that the derivatives of composite sinc functions be evaluated at the nodes. We need the following lemma.

LEMMA 2.1. [23] *Let ϕ be a conformal one-to one map of the arbitrary simply connected domain D onto D_d. Then*

$$\delta_{jk}^{(0)} = [S(j,h)\circ\phi(x)]\,|_{x=x_k} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k, \end{cases} \tag{2.1}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)\circ\phi(x)]\,|_{x=x_k} = \begin{cases} 0, & \text{if } j = k, \\ \frac{(-1)^{k-j}}{k-j}, & \text{if } j \neq k, \end{cases} \tag{2.2}$$

and

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)\circ\phi(x)]\,|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & \text{if } j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & \text{if } j \neq k. \end{cases} \tag{2.3}$$

In equations (2.1)-(2.3) h is step size and x_k is a sinc grid point given by

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1+e^{kh}}.$$

3. Linear Second Order Singularly Perturbed Boundary Value Problem. In this section, we shall study the Sinc-Galerkin scheme for the singularly perturbed boundary value problem

$$\epsilon u'' + a(x)u' + b(x)u = f(x) \qquad \text{for } 0 < x < 1 \tag{3.1}$$

subject to boundary conditions

$$u(0) = 0, \quad u(1) = 0. \tag{3.2}$$

We assume an approximate solution of the form

$$u_m(x) = \sum_{j=-M}^{N} c_j\, S_j(x), \quad m = M + N + 1. \tag{3.3}$$

where $S_j(x)$ is the function $S(j,h)\circ\phi(x)$ for some fixed step size h. The unknown coefficients $\{c_j\}_{-M}^{N}$ in (3.3) are determined by orthogonalizing the residual with respect to the basis functions$\{S_k\}_{k=-M}^{N}$, i.e.,

$$\langle \epsilon u'', S_k\rangle + \langle a(x)u', S_k\rangle + \langle b(x)u, S_k\rangle = \langle f, S_k\rangle, \qquad k = -M, \ldots, N. \tag{3.4}$$

The weighted inner product $\langle , \rangle$ is taken to be

$$\langle g(x), f(x)\rangle = \int_0^1 g(x)f(x)w(x)dx.$$

Here $w(x)$ plays the role of a weight function which is chosen depending on the boundary conditions, the domain, and the differential equation. For the case of second order boundary value problems, it is convenient to take

$$w(x) = \frac{1}{\phi'(x)}.$$

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A complete discussion of the choice of the weight function can be found in [16, 23].

The method of approximating the integrals in (3.4) begins by integrating by parts to transfer all derivatives from u to S_k. We need the following theorem.

THEOREM 3.1. *The following relations hold*

$$\langle \epsilon u'', S_k \rangle \approx h \sum_{j=-M}^{N} \sum_{i=0}^{2} \frac{u(x_j)}{\phi'(x_j)h^i} \delta_{kj}^{(i)} g_{2,i}(x_j), \tag{3.5}$$

for some functions $g_{2,i}$ to be determined.

Proof. The inner product with sinc basis element is given by

$$\langle \epsilon\, u'', S_k \rangle = \int_0^1 \epsilon\, u''(x) S_k(x) w(x) dx. \tag{3.6}$$

This expression contains the second derivative of u, but the desired result is the variable u with no derivatives. Integrating by parts to remove second derivatives from the dependent variable u leads to the equality,

$$\langle \epsilon\, u''(x), S_k(x) \rangle = B_x + \int_0^1 u(x)(\epsilon\, S_k(x) w(x))'' dx, \tag{3.7}$$

where the boundary term

$$B_x = \left[\sum_{i=0}^{1} (-1)^i u^{(1-i)} (\epsilon\, S_k w)^{(i)} \right]_{x=0}^{1} = 0.$$

Setting

$$\frac{d^n}{d\phi^n}[S_k(x)] = S_k^{(n)}(x), \qquad 0 \le n \le 2,$$

and noting that

$$\frac{d}{dx}[S_k(x)] = S_k^{(1)}(x)\phi'(x),$$

by expanding the derivatives under the integral in (3.7), we obtain

$$\langle \epsilon\, u'', S_k(x) \rangle = \int_0^1 \left(\sum_{i=0}^{2} u(x) S_k^{(i)}(x)\; g_{2,i} \right) d\,x, \tag{3.8}$$

where

$$g_{2,2}(x) = \epsilon\, w(\phi')^2, \qquad g_{2,0}(x) = \epsilon\, w'',$$

and

$$g_{2,1}(x) = \epsilon\, w(\phi)'' + 2\epsilon\, w'\phi'.$$

Applying the Sinc quadrature rule to the right-hand side of (3.8) and deleting the error terms yields (3.5). □

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The approximation of the last three inner products on the right-hand side of (3.4) has been thoroughly treated in [16]. We will list them for convenience:

$$\langle a(x)u', S_k\rangle \approx -h\sum_{j=-M}^{N}\frac{u(x_j)}{\phi'(x_j)}\left[\frac{1}{h}\delta_{kj}^{(1)}\,(a\,w)\phi' + \delta_{kj}^{(0)}(a\,w)'\right] \tag{3.9}$$

and

$$\langle G, S_k\rangle \approx h\frac{G(x_k)w(x_k)}{\phi'(x_k)}, \tag{3.10}$$

where G is $b(x)u$ or f, respectively.

Replacing each term of (3.4) with the approximation defined in (3.5), (3.9), and (3.10), respectively, and replacing $u(x_j)$ by c_j and dividing by h, we obtain the following theorem.

THEOREM 3.2. *If the assumed approximate solution of the boundary-value problem (3.1)-(3.2) is (3.3), then the discrete Sinc-Galerkin system for the determination of the unknown coefficients $\{c_j\}_{j=-M}^{N}$ is given by*

$$\begin{aligned}\sum_{j=-M}^{N}\sum_{i=0}^{2}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2,i}(x_j)}{\phi'(x_j)}c_j - \sum_{j=-M}^{N}\left[\frac{1}{h}\delta_{kj}^{(1)}\,(a(x_j)\,w) + \delta_{kj}^{(0)}\frac{(a(x_j)\,w)'}{\phi'(x_j)}\right]c_j \\ +\frac{b(x_k)w(x_k)}{\phi'(x_k)}c_k = \frac{f(x_k)w(x_k)}{\phi'(x_k)}\end{aligned} \tag{3.11}$$

$k=-M,\cdots,N.$

The following notation will be necessary for writing down the system. Let $D(g)$ be the $m\times m$ diagonal matrix

$$\mathbf{D}(g) = \begin{pmatrix} g(x_{-M}) & & & \\ & g(x_{-M+1}) & & \\ & & \ddots & \\ & & & g(x_N)\end{pmatrix}.$$

Define the $m\times m$ matrices $\mathbf{I}^{(p)}$ (see [11]) for $0\le p\le 2$ by

$$I^{(p)} = \left[\delta_{jk}^{(p)}\right], \quad \mathrm{j}, k=-M,\ldots,N.$$

Let $\mathbf{c}$ be the m-vector with j-th component given by c_j, and $\mathbf{1}$ be an m-vector each of whose components is 1. In this notation the system in (3.11) takes the matrix form

$$\mathbf{A}\begin{pmatrix} c_{-M}\\ c_{-M+1}\\ \vdots \\ c_N\end{pmatrix} = \Theta, \tag{3.12}$$

where

$$\mathbf{A} = \sum_{j=0}^{2}\frac{1}{h^j}\mathbf{I}^{(j)}\mathbf{D}\left(\frac{g_{2,j}}{\phi'}\right) - \left[\frac{1}{h}\mathbf{I}^{(1)}\mathbf{D}\,(a\,w) + \mathbf{D}\left(\frac{(a\,w)'}{\phi'}\right)\right] + \mathbf{D}\left(\frac{b\,w}{\phi'}\right), \tag{3.13}$$

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and

$$\Theta = D\left(\frac{wf}{\phi'}\right)\mathbf{1}. \tag{3.14}$$

Now we have a linear system of m equations of the m unknown coefficients, namely, $\{c_j, j = -M, \ldots, N\}$. We can obtain the coefficients of the approximate solution by solving this linear system. This system (3.12) may be easily solved by a variety of methods. In this paper we use the QR method [12]. The solution $\mathbf{c} = (c_{-M}, c_{-M+1}, \ldots c_N)^T$ gives the coefficients in the Sinc-Galerkin approximation $u_m(x)$ of $u(x)$.

4. Non-Linear Second Order Singularly Perturbed Boundary Value Problem. Consider a nonlinear, second order BVP of the form

$$\epsilon\, u'' + a(x)u' + b(x)u + \sigma(x)\, Q(u) = f(x) \qquad \text{for } 0 < x < 1, \tag{4.1}$$

subject to boundary conditions

$$u(0) = 0, \quad u(1) = 0, \tag{4.2}$$

where $Q(u)$ may be a polynomial, a rational function, or an exponential. Due to the large number of different possibilities, our work will be focused mainly on the following forms $Q(u)$:

1. $Q(u) = u^n, \quad n > 1,$
2. $Q(u) = \exp(\pm\, u),\ \cos(u),\ \sin(u),\ \sinh(u),\ \cosh(u), \ldots,$
3. $Q(u) = \dfrac{1}{(1 \pm u)^n}, \dfrac{1}{(1 \pm u^2)^n}, \dfrac{1}{(u^2 \pm 1)^n}, \quad n \neq 0,$

or any analytic function of u which has a power series expansion.

We start with the case $Q(u) = u^n$, where n is a non-negative integer, or a fraction. The approximate solution for $u(x)$ is represented by the formula

$$u_m(x) = \sum_{j=-M}^{N} c_j\, S_j(x), \quad m = M + N + 1. \tag{4.3}$$

The unknown coefficients c_j in equation (4.3) are determined by orthogonalizing the residual with respect to the basis functions, i.e.,

$$\langle \epsilon u'', S_k\rangle + \langle a(x)u', S_k\rangle + \langle b(x)u, S_k\rangle + \langle \sigma u^n, S_k\rangle = \langle f, S_k\rangle\,. \tag{4.4}$$

We need the following lemma

LEMMA 4.1. [7] *The following relations hold*

$$\langle \sigma(x)u^n, S_k\rangle \approx h\frac{w(x_k)u^n(x_k)\sigma(x_k)}{\phi'(x_k)}. \tag{4.5}$$

Replacing each term of (4.4) with the approximations defined in (3.5),(3.9),(3.10), (4.5), and replacing $u(x_j)$ by c_j and dividing by h, we obtain the following theorem.

THEOREM 4.2. *If the assumed approximate solution of the boundary-value problem (4.1)and (4.2) is (4.3), then the discrete Sinc-Galerkin system for the determination of the*

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unknown coefficients $\{c_j\}_{j=-M}^{N}$ *is given by*

$$\sum_{j=-M}^{N}\sum_{i=0}^{2}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2,i}(x_j)}{\phi'(x_j)}c_j-\sum_{j=-M}^{N}\left[\frac{1}{h}\delta_{kj}^{(1)}\,(a(x_j)\,w)+\delta_{kj}^{(0)}\frac{(a(x_j)\,w)'}{\phi'(x_j)}\right]c_j$$
$$+\frac{b(x_k)w(x_k)}{\phi'(x_k)}c_k+\frac{w(x_k)\sigma(x_k)}{\phi'(x_k)}c_k^n=\frac{f(x_k)w(x_k)}{\phi'(x_k)} \tag{4.6}$$

$k=-M,\cdots,N.$

Proof. Combine Lemma 4.1 and Theorem 3.2 . □

Using the notation in the previous section, let $\mathbf{c}^n$ be the m-vector with j-th component given by c_j^n . The system in (4.6) takes the matrix form

$$\mathbf{A}\,\mathbf{c}+\mathbf{E}\,\mathbf{c}^n=\Theta, \tag{4.7}$$

where

$$\mathbf{E}=\mathbf{D}\left(\frac{\sigma w}{\phi'}\right) \tag{4.8}$$

with $\mathbf{A}$ and Θ defined by equations (3.13) and (3.14), respectively.

Now we have a nonlinear system of $m = M + N + 1$ equations of the m unknown coefficients, namely, $\{c_j\}_{j=-M}^{N}$. We can obtain the coefficients of the approximate solution by solving this nonlinear system by *Newton's method* [7]. The solution $\mathbf{c}=(c_{-M},\ldots,c_N)^T$ gives the coefficients in the Sinc-Galerkin approximation $u_m(x)$ of $u(x)$.

5. Singularly Perturbed Reaction-Diffusion Equation. In this section the Sinc-Galerkin method is given for the approximate solution of the equations (1.1)-(1.2). The assumed approximate solution takes the form

$$u_{m_x,m_y}(x,y)=\sum_{j=-M_y}^{N_y}\sum_{i=-M_x}^{N_x}u_{ij}\,S_{ij}(x,y), \tag{5.1}$$

where $m_x = M_x + N_x + 1$, $m_y = M_y + N_y + 1$. The basis functions $\{S_{ij}(x,y)\}$ for $-M_x \le i \le N_x, -M_y \le j \le N_y$ are given as the product of basis functions. In this paper we take

$$S_{ij}(x,y)=S_i(x)S_j(y)=[S(i,h_x)\circ\phi(x)][S(j,h_y)\circ\gamma(y)].$$

The unknown coefficients $\{u_{ij}\}$ in equation (5.1) are determined by orthogonalizing the residual with respect to the functions $\{S_{kl}(x,y)\}$, $-M_x \le k \le N_x$, $-M_y \le l \le N_y$. This yields the discrete Galerkin system

$$-\langle\epsilon u_{xx},S_kS_l\rangle-\langle\epsilon u_{yy},S_kS_l\rangle+\langle a\,u,S_kS_l\rangle=\langle f,S_kS_l\rangle\,. \tag{5.2}$$

We need the following lemma.

LEMMA 5.1. *The following relations hold*

$$\langle\epsilon u_{xx},S_kS_l\rangle=h_x\,h_y\frac{\upsilon(y_l)}{\phi_2'(y_l)}\sum_{i=-M_x}^{N_x}\sum_{j=0}^{2}\frac{u(x_i,y_l)}{\phi_1'(x_i)}\left[\frac{1}{h_x^j}\,\delta_{ki}^{(j)}\,\rho_j\right], \tag{5.3}$$

$$\langle \epsilon u_{yy}, S_k S_l\rangle = h_x\, h_y\, \frac{w(x_k)}{\phi_1'(x_k)} \sum_{i=-M_y}^{N_y} \sum_{j=0}^{2} \frac{u(x_k,y_i)}{\phi_2'(y_i)} \left[\frac{1}{h_y^j}\, \delta_{ki}^{(j)}\, \eta_j\right], \tag{5.4}$$

$$\langle a\, u, S_k S_l\rangle = a\, h_x\, h_y\, \frac{w(x_k)\, u(x_k,y_l)\, \upsilon(y_l)}{\phi_1(x_k)\, \phi_2(y_l)}, \tag{5.5}$$

and

$$\langle f, S_k\, S_l\rangle = h_x\, h_y\, \frac{w(x_k)\, f(x_k,y_l)\, \upsilon(y_l)}{\phi_1(x_k)\, \phi_2(y_l)}, \tag{5.6}$$

where

$$\rho_2 = \epsilon\, (\phi_1')^2\, w, \quad \rho_1 = \epsilon\, \phi_1'\, w + 2\, \epsilon\, \phi_1'\, w', \quad \rho_0 = \epsilon\, w'',$$

and

$$\eta_2 = \epsilon\, (\phi_2')^2\, \upsilon, \quad \eta_1 = \epsilon\, \phi_2'\, \upsilon + 2\, \epsilon\, \phi_2'\, \upsilon', \quad \eta_0 = \epsilon\, \upsilon''.$$

Replacing each term of (5.2) with the approximation defined in (5.3)–(5.6) and replacing $u(x_k,y_l)$ by u_{kl} and dividing by $h_x h_y$ we obtain the following theorem.

THEOREM 5.2. *If the assumed approximate solution of the problem (1.1)-(1.2) is (5.1), then the discrete Sinc-Galerkin system for the determination of the unknown coefficients* $\{u_{kj}, -M_x < k < N_x,\ -M_t < j < N_t\}$ *is given by*

$$\begin{aligned} & -\frac{\upsilon(y_l)}{\phi_2'(y_l)} \sum_{i=-M_x}^{N_x} \sum_{j=0}^{2} \frac{u(x_i,y_l)}{\phi_1'(x_i)} \left[\frac{1}{h_x^j}\, \delta_{ki}^{(j)}\, \rho_j\right] \\ & -\frac{w(x_k)}{\phi_1'(x_k)} \sum_{i=-M_y}^{N_y} \sum_{j=0}^{2} \frac{u(x_k,y_i)}{\phi_2'(y_i)} \left[\frac{1}{h_y^j}\, \delta_{ki}^{(j)}\, \eta_j\right] + \\ & a\, \frac{w(x_k)\, u(x_k,y_l)\, \upsilon(y_l)}{\phi_1(x_k)\, \phi_2(y_l)} = \frac{w(x_k)\, f(x_k,y_l)\, \upsilon(y_l)}{\phi_1(x_k)\, \phi_2(y_l)}. \end{aligned} \tag{5.7}$$

Introducing the notation of Toeplitz matrices in equation (5.7) leads to the matrix form

$$\begin{aligned} & -\left[\sum_{j=0}^{2} \frac{1}{h_x^j} I^{(j)} \mathbf{D}\left(\frac{\rho_j}{\phi_1'\, w}\right)\right] \mathbf{D}(\phi_1')\mathbf{D}(w)\mathbf{U}\,\mathbf{D}\left(\frac{\upsilon}{\phi_2'}\right) \\ & -\mathbf{D}\left(\frac{w}{\phi_1'}\right) \mathbf{U}\mathbf{D}(\upsilon)\,\mathbf{D}(\phi_2') \left[\sum_{j=0}^{1} \frac{1}{h_y^j} I^{(j)} \mathbf{D}\left(\frac{\eta_j}{\phi_2' \upsilon}\right)\right]^t \\ & +a\,\mathbf{D}\left(\frac{w}{\phi_1'}\right) \mathbf{U}\mathbf{D}\left(\frac{\upsilon}{\phi_2'}\right) = \mathbf{D}\left(\frac{w}{\phi_1'}\right) \mathbf{F}\mathbf{D}\left(\frac{\upsilon}{\phi_2'}\right). \end{aligned}$$

Note that $[\ \ ]^t$, denotes the transpose of the matrix $[\ \ ]$. Premultiplying by $\mathbf{D}(\phi_1'(x_k))$ and postmultiplying by $\mathbf{D}(\phi_2'(y_l))$ yields the equivalent system

$$\mathbf{AX} + \mathbf{XB} = \mathbf{G}, \tag{5.8}$$

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where

$$\mathbf{A} = \mathbf{A1} + a\,\mathbf{I},$$

$$\mathbf{A1} = -\mathbf{D}(\phi_1')\left[\sum_{j=0}^{2} \frac{1}{h_x^j}\mathbf{I}^{(j)}\mathbf{D}\left(\frac{\rho_j}{\phi_1'\,w}\right)\right]\mathbf{D}(\phi_1'),$$

$$\mathbf{B} = -\mathbf{D}(\phi_2')\left[\sum_{j=0}^{2} \frac{1}{h_y^j}\mathbf{I}^{(j)}\mathbf{D}\left(\frac{\eta_j}{\phi_2'\upsilon}\right)\right]^t\mathbf{D}(\phi_2'),$$

$$\mathbf{G} = \mathbf{D}(w)\,\mathbf{F}\,\mathbf{D}(\upsilon),$$

and

$$\mathbf{X} = \mathbf{D}(w)\,\mathbf{U}\;\mathbf{D}(\upsilon).$$

The four matrices **A**, **B**, **X** and **G** have dimension $m_x \times m_x$, $m_y \times m_y$, $m_x \times m_y$ and $m_x \times m_y$, respectively. Lastly, the $m_x \times m_y$ matrices **U** and **F** have kl-th entries given by u_{kl} and $f(x_k, y_l) = f(\frac{e^{kh}}{1+e^{kh}}, \frac{e^{lh}}{1+e^{lh}})$, respectively.

To obtain the approximate solution of equation (5.1), we need to solve the system for **U** which requires solving (5.8) for **X**. To solve (5.8), see[5].

6. Numerical Examples. In this section, four examples will be tested by using the Sinc-Galerkin method discussed above. For purposes of comparison, contrast and performance, examples with known solutions were chosen. For the sake of comparison only, we will discuss the first and the third examples that were investigated by Reddy [19] and Navon [14].

In all examples, d is taken to be $\pi/2$ and we report absolute error which is defined as

$$\text{absolute error} = \left| u_{\text{exact solution}} - U_{\text{Sinc-Galerkin}} \right|$$

EXAMPLE 6.1. [19] Consider the boundary-value problem

$$\epsilon\,\frac{d^2 u}{d\,x^2} + \frac{d\,u}{d\,x} = 1 + 2\,x, \quad 0 \le x \le 1,$$

and

$$u(0) = 1, \quad u(1) = 1,$$

whose exact solution is

$$u(x) = x\,(x + 1 - 2\,\epsilon) + \frac{(2\,\epsilon - 1)\,(1 - e^{-x/\epsilon})}{1 - e^{-1/\epsilon}}.$$

The parameters $\alpha = \beta = \frac{1}{2}$ and $N = 100$ are used. Maximum absolute error are tabulated in Table 6.1 for Sinc-Galerkin together with the analogous results of Reddy [19], who use the method of reduction of order.

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TABLE 6.1
Maximum absolute error for Example 6.1

ϵ	The method of reduction of order [19]	Sinc-Galerkin method
10^{-3}	0.213E-02	0.358E-08
10^{-4}	0.112E-04	0.271E-08
10^{-6}		0.797E-10

EXAMPLE 6.2. Consider the boundary-value problem

$$\epsilon \frac{d^2 u}{d x^2} + 2 \frac{d u}{d x} + u^2 = \left(-\frac{1}{\epsilon} + e^{-x/\epsilon} \right) e^{-x/\epsilon}, \quad 0 \le x \le 1,$$

and

$$u(0) = 1, \quad u(1) = e^{-1/\epsilon}$$

whose exact solution is

$$u(x) = e^{-x/\epsilon}.$$

The parameters $\alpha = \beta = \frac{1}{2}$ and $N = 100$ are used. Maximum absolute error at different ϵ are tabulated in Table 6.2.

TABLE 6.2
Maximum absolute error for Example 6.2

ϵ	Maximum absolute error
10^{-2}	0.135E-07
10^{-4}	0.432E-08
10^{-6}	0.297E-10

EXAMPLE 6.3. [14] Consider the boundary-value problem

$$-\epsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + 2 u = f(x, y),$$

and

$$u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0,$$

where $f(x, y)$ is chosen such that the solution is

$$u(x, y) = \left(1 - \frac{e^{-x/\epsilon} + e^{-(1-x)/\epsilon}}{1 - e^{-1/\epsilon}} \right) \left(1 - \frac{e^{-y/\epsilon} + e^{-(1-y)/\epsilon}}{1 - e^{-1/\epsilon}} \right).$$

The parameters $M_x = N_x = M_y = N_y = 100$ and $\alpha = \beta = \frac{3}{2}$ are used for the Sinc-Galerkin solution. Table 6.3 exhibits a comparison between the errors obtained by using Sinc-Galerkin method and errors of Li and Navon [14], who use the Finite Element method.

EXAMPLE 6.4. Consider the boundary-value problem

$$-\epsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - u = f(x, y),$$

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TABLE 6.3
$||u_{exact} - u_{App}||_{L2}$ *for Example 6.3*

ϵ	The Finite Element Method [14] with $N = 5329$	Sinc-Galerkin Method
10^{-2}	1.978E-03	0.387E-05
10^{-3}	0.457E-03	0.321E-05
10^{-5}	0.704E-03	0.112E-06
10^{-6}	0.317E-04	0.137E-06
10^{-7}	0.136E-04	0.133E-06

and

$$u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0,$$

where $f(x, y)$ is chosen such that the solution is

$$u(x, y) = x\, y\, \ln\, x\, \ln\, y.$$

The parameters $M_x = N_x = M_y = N_y = 100$ and $\alpha = \beta = \frac{3}{2}$ are used for the Sinc-Galerkin solution. Table 6.4 exhibits the maximum absolute errors at different ϵ.

TABLE 6.4
Maximum absolute error for Example 6.4

ϵ	Maximum absolute error
10^{-4}	0.1846E-07
10^{-6}	0.4697E-09
10^{-8}	0.4032E-09
10^{-10}	0.2643E-09

The computations associated with the four examples discussed above were performed using MATLAB.

7. Conclusions. The Sinc-Galerkin method was tested on four problems. A comparison with finite element methods and the method of reduction of order is made and it was seen that the Sinc-Galerkin method yields good results. The results of Example 6.4 clearly indicate that our methods are accurate even when singularities occur at the boundaries.

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