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UNIFORM APPROXIMATION BY MINIMUM NORM INTERPOLATION*

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Abstract. Harmonic Hilbert spaces were introduced as an extension of periodic Hilbert spaces introduced by Babuška [1] to the non-periodic case [6]. In this paper we will investigate approximation by minimum norm interpolation in harmonic Hilbert spaces.

Key words. minimum norm interpolation, harmonic Hilbert spaces, remainders.

AMS subject classifications. 41A05, 41A25, 41A65, 65D05, 65T99.

1. Introduction. Periodic Hilbert spaces were introduced by Babuška [1] and minimum norm interpolation was investigated in these spaces [9, 3, 4]. We extended the construction of periodic Hilbert spaces to the non-periodic case by introducing harmonic Hilbert spaces and investigated minimum norm interpolation in these spaces [5]. In [6, 7] we studied approximation properties of classical cardinal interpolation in harmonic Hilbert spaces. We will investigate uniform approximation by minimum norm interpolation in the associated harmonic Hilbert space.

2. Minimum norm interpolation. The Wiener algebra $A(\mathbb{R})$ consists of those bounded continuous functions f which can represented as Fourier integrals of functions $F \in L_1(\mathbb{R})$:

$$f(x) = \int_{\mathbb{R}} F(t) e^{ixt} dt.$$

Let $b > 0$. Then the series

$$\sum_{k\in\mathbb{Z}} F(t + 2bk)$$

converges absolutely almost everywhere and defines a measurable function

$$F_{2b} \in L_1([-b, b])$$

which is called the periodization of $F \in L_1(\mathbb{R})$ (see [2]). The periodization $F_{2b} \in L_1([-b, b])$ and the function f are related by

$$f(r\frac{\pi}{b}) = \int_{[-b,b]} F_{2b}(t) e^{ir\frac{\pi}{b}t} dt,\ r \in \mathbb{Z}. \tag{2.1}$$

(See [2], p.33.)

The function of exponential-type

$$T_b(f)(x) = \int_{\mathbb{R}} \chi_{[-b,b]}(t) F_{2b}(t) e^{ixt} dt \tag{2.2}$$

is called the exponential-type interpolant of f in view of

$$T_b(f)(r\frac{\pi}{b}) = f(r\frac{\pi}{b}),\ r \in \mathbb{Z}. \tag{2.3}$$

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For $F_{2b} \in L_2([-b,b])$ it is just the cardinal function $C(f, \frac{\pi}{b})$ (see [11]). It possesses the cardinal series representation

$$T_b(f)(x) = \sum_{k \in \mathbb{Z}} f(kh) S(k,h)(x) = C(f,h)(x),\ h = \frac{\pi}{b}\ ,$$

where

$$S(k,h)(x) = \mathrm{sinc}(\frac{\pi}{h}(x - kh))$$

is the shifted Sinc function with step size $h = \frac{\pi}{b}$.

The harmonic Hilbert space $H_D(\mathbb{R})$ is the subspace of functions $f \in A(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \frac{|F(t)|^2}{D(t)} dt < \infty,$$

where $D \in L_1(\mathbb{R})$ is assumed to be non negative and bounded. D is called the defining function (see also [10] where $H_D(\mathbb{R})$ is called a weighted L_2-space). Note that if $D(t)$ vanishes at a set M of posivite measure the Fourier transform F has to vanish almost everywhere on the same set.

It turns out to be useful to add the additional assumptions:

$$D(-t) = D(t),\ D(t) \geq D(t+h) \geq 0, \tag{2.4}$$

and

$$0 < D(b) \leq D_{2b}(t) \leq \sum_{k \geq 0} D(kb) < \infty, \tag{2.5}$$

for almost all $t, h \geq 0$.

It was shown in [6] that for any $f \in H_D(\mathbb{R})$ we have

$$\sum_{k \in \mathbb{Z}} |f(kh)|^2 < \infty,\ h = \frac{\pi}{b}\ ,$$

which implies $F_{2b} \in L_2([-b,b])$ and

$$T_b(f) = C(f,h),\ f \in H_D(\mathbb{R})\ . \tag{2.6}$$

The Fourier integral of D

$$d(x) = \int_{\mathbb{R}} D(t) e^{ixt} dt$$

is called the generating function of the harmonic Hilbert space $H_D(\mathbb{R})$. With respect to the inner product

$$(f,g)_D = \int_{\mathbb{R}} F(t)\overline{G(t)}/D(t) dt$$

we have

$$f(x) = (f, d(\cdot - x))_D$$

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which shows that $H_D(\mathbb{R})$ is a reproducing kernel Hilbert space (see also [10]).

Minimum norm interpolation in $H_D(\mathbb{R})$ with respect to the uniform mesh $\frac{k\pi}{b},\ k \in \mathbb{Z}$, is solved theoretically by the projection theorem. Consider the closed linear subspace

$$N_b^D = \left\{ f \in H_D(\mathbb{R}) : f(\frac{k\pi}{b}) = 0,\ k \in \mathbb{Z} \right\},$$

and its orthogonal complement $(N_b^D)^\perp$ which is the closure of the linear subspace generated by shifted generating functions $d(\cdot - k\frac{k\pi}{b}),\ k \in \mathbb{Z}$:

$$(N_b^D)^\perp = \operatorname{lin} \left\{ d(\cdot - k\frac{k\pi}{b}) :\ k \in \mathbb{Z} \right\}^- .$$

There is a unique orthogonal projector T_b^D with range $(N_b^D)^\perp$ and kernel N_b^D. The projection theorem states that $T_b^D(f)$ is the unique function of minimum norm in the linear manifold

$$f + N_b^D.$$

PROPOSITION 2.1 (Minimum norm interpolation).
Given $f \in H_D(\mathbb{R})$, its projection $T_b^D(f)$ on $(N_b^D)^\perp$ is the unique minimum norm interpolant of $f \in H_D(\mathbb{R})$, i. e., it minimizes the norm $\|g\|_D$ among all functions $g \in H_D(\mathbb{R})$ satisfying $g(\frac{k\pi}{b}) = f(\frac{k\pi}{b}),\ k \in \mathbb{Z}$.

The minimum norm interpolant of the Sinc function $S(0,h)$ is denoted by

$$S^D(0,h) = T_b^D(S(0,h)) =: g_b^D.$$

The generalized Sinc function $S^D(0,h)$ is given by the Fourier integral

$$S^D(0,h)(x) = \frac{1}{2b} \int_{\mathbb{R}} \frac{D(t)}{D_{2b}(t)} e^{ixt} dt = g_b^D(x)\ ,\ h = \frac{\pi}{b}.$$

It was shown in [6] that for any $f \in H_D(\mathbb{R})$ the associated minimum norm interpolant $T_b^D(f)$ possesses a generalized cardinal series $C^D(f,h)$:

$$T_b^D(f)(x) = \sum_{k \in \mathbf{Z}} f(kh) g_b^D(x - kh) = C^D(f,h)(x),\ h = \frac{\pi}{b}.$$

Note that for $D(t) = \chi_{[-b,b]}(t)$ we obtain $C^D(f,h) = C(f,h)$. For deriving error estimates an integral representation of the minimum norm interpolant $T_b^D(f)$ is of importance.

PROPOSITION 2.2. *The generalized cardinal series $C^D(f,h)$ possesses the integral representation*

$$T_b^D(f)(x) = \int_{\mathbb{R}} \frac{F_{2b}(t)}{D_{2b}(t)} D(t) e^{ixt} dt = C^D(f,h)(x), \qquad h = \frac{\pi}{b}. \tag{2.7}$$

Proof. Put

$$C_n^D(f,h)(x) = \sum_{k=-n}^{n} f(kh) g_b^D(x - kh),$$

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$$F_{2b}^n(t) = \frac{1}{2b}\sum_{k=-n}^{n} f(kh)e^{-ikht},\ h = \frac{\pi}{b}.$$

Since

$$C_n^D(f,h)(x) = \int_{\mathbb{R}} \frac{F_{2b}^n(t)}{D_{2b}(t)} D(t)e^{ixt}dt,$$

we can conclude

$$\begin{aligned}
\left|T_b^D(f)(x) - C_n^D(f,h)(x)\right| &\le \int_{\mathbb{R}} |F_{2b}(t) - F_{2b}^n(t)| \frac{D(t)}{D_{2b}(t)}dt \\
&\le \left(\int_{\mathbb{R}} |F_{2b}(t) - F_{2b}^n(t)|^2 \frac{D(t)}{D_{2b}(t)^2}dt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} D(t)dt\right)^{\frac{1}{2}} \\
&= \left(\int_{[-b,b]} |F_{2b}(t) - F_{2b}^n(t)|^2 \frac{1}{D_{2b}(t)}dt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} D(t)dt\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{D(b)}\int_{[-b,b]} |F_{2b}(t) - F_{2b}^n(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} D(t)dt\right)^{\frac{1}{2}}.
\end{aligned}$$

This shows

$$C^D(f,h)(x) = \lim_{n\to\infty} C_n^D(f,h)(x) = T_b^D(f)(x).\ \square$$

3. Approximation by minimum norm interpolation. We start with approximation by functions of exponential type which are defined as Fourier integrals

$$S_b(f)(x) = \int_{\mathbb{R}} \chi_{[-b,b]}(t)F(t)e^{ixt}dt.$$

PROPOSITION 3.1. *For any $f \in H_D(\mathbb{R})$ the error estimate*

$$|f(x) - S_b(f)(x)| \le \left(2b\sum_{k\ge 1} D(kb)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right) \frac{|F(t)|^2}{D(t)}dt\right)^{\frac{1}{2}}$$

holds.

Proof . Since

$$\begin{aligned}
|f(x) - S_b(f)(x)| &\le \int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right)|F(t)|\,dt \\
&= \int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right) \frac{|F(t)|}{D(t)^{\frac{1}{2}}} D(t)^{\frac{1}{2}}dt \\
&\le \left(\int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right) D(t)dt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right) \frac{|F(t)|^2}{D(t)}dt\right)^{\frac{1}{2}},
\end{aligned}$$

and

$$\int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right) D(t)dt\ = 2\int_b^\infty D(t)dt\ \le 2b\sum_{k=1}^{\infty} D(bk),$$

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the proof is complete. ∎

PROPOSITION 3.2. *Let $f \in H_D(\mathbb{R})$ and define $h(x) = S_b(f)(x)$. Then the error estimate*

$$\left|h(x) - T_b^D(h)(x)\right| \leq 2\left(2b\sum_{k\geq 1} D(kb)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \chi_{[-b,b]}(t)\frac{|F(t)|^2}{D(t)}dt\right)^{\frac{1}{2}}$$

holds.

Proof. Note that

$$h(x) = \int_{\mathbb{R}} H(t)e^{ixt}dt,\ H(t) = \chi_{[-b,b]}(t)F(t).$$

Moreover, we have

$$H_{2b}(t)\chi_{[-b,b]}(t) = \chi_{[-b,b]}(t)F(t)$$

for almost all t. Then we can conclude

$$\begin{aligned}
\left|h(x) - T_b^D(h)(x)\right| &\leq \int_{\mathbb{R}} \left|\chi_{[-b,b]}(t)F(t) - \frac{H_{2b}(t)}{D_{2b}(t)}D(t)\right| dt \\
&\leq \int_{\mathbb{R}} \chi_{[-b,b]}(t)\left|F(t) - \frac{H_{2b}(t)}{D_{2b}(t)}D(t)\right| dt \\
&+ \int_{\mathbb{R}} \left(1-\chi_{[-b,b]}(t)\right)\left|\frac{H_{2b}(t)}{D_{2b}(t)}D(t)\right| dt \\
&= \int_{\mathbb{R}} \chi_{[-b,b]}(t)\frac{D_{2b}(t)-D(t)}{D_{2b}(t)}\left|F(t)\right| dt \\
&+ \int_{\mathbb{R}} \left|\frac{H_{2b}(t)}{D_{2b}(t)}\right| D(t)dt\ - \int_{\mathbb{R}} \chi_{[-b,b]}(t)\left|\frac{H_{2b}(t)}{D_{2b}(t)}\right| D(t)dt \\
&= \int_{\mathbb{R}} \chi_{[-b,b]}(t)\frac{D_{2b}(t)-D(t)}{D_{2b}(t)}\left|F(t)\right| dt \\
&+ \int_{\mathbb{R}} \chi_{[-b,b]}(t)\left|H_{2b}(t)\right| dt\ - \int_{\mathbb{R}} \chi_{[-b,b]}(t)\frac{D(t)}{D_{2b}(t)}\left|H_{2b}(t)\right| dt \\
&= \int_{\mathbb{R}} \chi_{[-b,b]}(t)\frac{D_{2b}(t)-D(t)}{D_{2b}(t)}\left|F(t)\right| dt \\
&+ \int_{\mathbb{R}} \chi_{[-b,b]}(t)\left|F(t)\right| dt\ - \int_{\mathbb{R}} \chi_{[-b,b]}(t)\frac{D(t)}{D_{2b}(t)}\left|F(t)\right| dt \\
&= 2\int_{\mathbb{R}} \chi_{[-b,b]}(t)\left(D_{2b}(t)-D(t)\right)\frac{|F(t)|}{D_{2b}(t)}dt.
\end{aligned}$$

This shows

$$\left|h(x) - T_b^D(h)(x)\right| \leq 2\int_{\mathbb{R}} \chi_{[-b,b]}(s)\left|D_{2b}(t) - D(t)\right| \frac{|F(t)|}{D_{2b}(t)} dt$$
$$\leq 2\left(\int_{\mathbb{R}} \chi_{[-b,b]}(s)\left|D_{2b}(t) - D(t)\right|^2 \frac{1}{D_{2b}(t)} dt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \chi_{[-b,b]}(s)\frac{|F(t)|^2}{D_{2b}(t)} dt\right)^{\frac{1}{2}}$$
$$\leq 2\left(\int_{\mathbb{R}} \chi_{[-b,b]}(s)\left|D_{2b}(t) - D(t)\right| dt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \chi_{[-b,b]}(s)\frac{|F(t)|^2}{D(t)} dt\right)^{\frac{1}{2}}$$
$$\leq 2\left(2b\sum_{k\geq 1} D(kb)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \chi_{[-b,b]}(t)\frac{|F(t)|^2}{D(t)} dt\right)^{\frac{1}{2}}$$

which completes the proof. $\square$

Our main result is the error estimate for minimum norm interpolation in the associated harmonic Hilbert space.

THEOREM 3.3. *Let $f \in H_D(\mathbb{R})$. Then the estimate*

$$\left|f(x) - T_b^D(f)(x)\right| \leq 4\left(b\sum_{k\geq 1} D(kb)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{|F(t)|^2}{D(t)} dt\right)^{\frac{1}{2}}$$

holds.

Proof . We have

$$|f(x) - S_b(f)(x)| \leq \left(2b\sum_{k\geq 1} D(kb)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right) \frac{|F(t)|^2}{D(t)} dt\right)^{\frac{1}{2}},$$

$$\left|S_b(f)(x) - T_b^D(S_b(f))(x)\right| \leq 2\left(2b\sum_{k\geq 1} D(kb)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \chi_{[-b,b]}(t)\frac{|F(t)|^2}{D(t)} dt\right)^{\frac{1}{2}}.$$

Moreover,

$$\left|T_b^D(g)(x)\right| \leq \int_{\mathbb{R}} \frac{|G_{2b}(t)|}{D_{2b}(t)} D(t) dt = \int_{\mathbb{R}} \chi_{[-b,b]}(t)\left|G_{2b}(t)\right| dt \leq \int_{\mathbb{R}} |G(t)|\, dt.$$

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Then we obtain

$$
\begin{aligned}
&\left|f(x) - T_b^D(f)\right| \\
&\le |f(x) - S_b(f)(x)| + \left|S_b(f)(x) - T_b^D(S_b(f))(x)\right| + \left|T_b^D(S_b(f))(x) - T_b^D(f)(x)\right| \\
&\le 2\int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right) |F(t)|\, dt \; + \left|S_b(f)(x) - T_b^D(S_b(f))(x)\right| \\
&\le 2\left(2b\sum_{k\ge 1} D(kb)\right)^{\frac{1}{2}} \left[\left(\int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right) \frac{|F(t)|^2}{D(t)} dt\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}} \chi_{[-b,b]}(t) \frac{|F(t)|^2}{D(t)} dt\right)^{\frac{1}{2}}\right] \\
&\le 4\left(b\sum_{k\ge 1} D(kb)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{|F(t)|^2}{D(t)} dt\right)^{\frac{1}{2}}. \qquad \square
\end{aligned}
$$

We consider two examples.

COROLLARY 3.4. *Let $f \in H_D(\mathbb{R})$ and $D(t) = \left(1 + t^{2r}\right)^{-1},\ r \ge 1$. Then the estimate*

$$
\left|f(x) - T_b^D(f)(x)\right| \le 4b^{-r+1/2} \left(\frac{2r}{2r-1}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |F(t)|^2 \left(1 + t^{2r}\right) dt\right)^{\frac{1}{2}}
$$

holds.

Proof: We have

$$
\sum_{k\ge 1} D(kb) \le D(b) + \int_1^\infty \left(1 + (bt)^{2r}\right)^{-1} dt \; \le b^{-2r} + \int_1^\infty (bt)^{-2r}\, dt = b^{-2r}\frac{2r}{2r-1}. \; \square
$$

COROLLARY 3.5. *Let $f \in H_D(\mathbb{R})$ and $D(t) = e^{-\alpha|t|}, \alpha > 0$. Then the estimate*

$$
\left|f(x) - T_b^D(f)(x)\right| \le 4e^{-\alpha b/2} \left(\frac{\alpha b + 1}{\alpha}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |F(t)|^2 e^{\alpha|t|} dt\right)^{\frac{1}{2}}
$$

holds.

Proof: We have

$$
\sum_{k\ge 1} D(kb) \le e^{-\alpha b} + \int_1^\infty e^{-\alpha bt} dt = e^{-\alpha b}\left(1 + \frac{1}{\alpha b}\right) = \frac{\alpha b + 1}{\alpha b} e^{-\alpha b}. \square
$$

4. Improved error estimates. We recall that the error estimate is based on investigating

$$
|f(x) - S_b(f)(x)| \le \int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right) |F(t)|\, dt,
$$

$$
\left|T_b^D(f)(x) - T_b^D(S_b(f))(x)\right| \le \int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right) |F(t)|\, dt,
$$

$$
\left|S_b(f)(x) - T_b^D(S_b(f))(x)\right| \le 2\int_{\mathbb{R}} \chi_{[-b,b]}(t) \left(D_{2b}(t) - D(t)\right) \frac{|F(t)|}{D_{2b}(t)} dt.
$$

Following the idea of Golomb [8, 3, 4] for the periodic case, we assume that the function

$$f(x) = \int_{\mathbb{R}} F(t) e^{ixt} dt$$

satisfies the smoothness condition

$$\int_{\mathbb{R}} |F(t)| / D(t) dt < \infty.$$

Then we can conclude

$$|f(x) - S_b(f)(x)| \leq D(b) \int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right) |F(t)| / D(t) dt \, ,$$

$$\left|T_b^D(f)(x) - T_b^D(S_b(f))(x)\right| \leq D(b) \int_{\mathbb{R}} \left(1 - \chi_{[-b,b]}(t)\right) |F(t)| / D(t) dt \, ,$$

$$\left|S_b(f)(x) - T_b^D(S_b(f))(x)\right| \leq 2 \sum_{k=1}^{\infty} D(kb) \int_{\mathbb{R}} \chi_{[-b,b]}(t) |F(t)| / D(t) dt.$$

This implies

PROPOSITION 4.1. *Assume that f satisfies the smoothness condition*

$$\int_{\mathbb{R}} |F(t)| / D(t) dt < \infty.$$

Then the error estimate

$$\left|f(x) - T_b^D(f)(x)\right| \leq 2 \sum_{k=1}^{\infty} D(kb) \int_{\mathbb{R}} |F(t)| / D(t) dt$$

holds.

We consider two examples.

COROLLARY 4.2. *Let $D(t) = \left(1 + t^{2r}\right)^{-1}, r \geq 1$ and assume that f satisfies*

$$\int_{\mathbb{R}} |F(t)| \left(1 + t^{2r}\right) dt < \infty.$$

Then the estimate

$$\left|f(x) - T_b^D(f)(x)\right| \leq b^{-2r} \frac{4r}{2r-1} \int_{\mathbb{R}} |F(t)| \left(1 + t^{2r}\right) dt$$

holds.

Proof. We have

$$\sum_{k \geq 1} D(kb) \leq b^{-2r} \frac{2r}{2r-1} \, . \; \square$$

COROLLARY 4.3. *Let $D(t) = e^{-\alpha|t|}$, $\alpha > 0$, and assume that f satisfies*

$$\int_{\mathbb{R}} |F(t)|\, e^{\alpha|t|} dt < \infty.$$

Then the estimate

$$\left|f(x) - T_b^D(f)(x)\right| \leq e^{-\alpha b} \left(2 + \frac{2}{\alpha b}\right) \int_{\mathbb{R}} |F(t)|\, e^{\alpha|t|} dt < \infty$$

holds.

Proof . We have

$$\sum_{k\geq 1} D(kb) \leq \frac{\alpha b + 1}{\alpha b} e^{-\alpha b}. \square$$

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