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BOUNDS FOR VANDERMONDE TYPE DETERMINANTS OF ORTHOGONAL POLYNOMIALS*

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Abstract. Let $(P_n)_{n\in\mathbb{N}_0}$ be a system of monic orthogonal polynomials. We establish upper and lower estimates for determinants of the form

$$V_n(z_1,\dots,z_k) \;:=\; \det\begin{pmatrix} P_n(z_1) & \dots & P_{n+k-1}(z_1) \\ \vdots & & \vdots \\ P_n(z_k) & \dots & P_{n+k-1}(z_k)\end{pmatrix}.$$

For the proofs, we have to study the monic orthogonal system $(P_n^{[w]})_{n\in\mathbb{N}_0}$ obtained by inserting the polynomial $w(x) := \prod_{\nu=1}^{k}(x-z_\nu)$ as a weight into the inner product defining $(P_n)_{n\in\mathbb{N}_0}$. We also express the recurrence formula for $(P_n^{[w]})_{n\in\mathbb{N}_0}$ in terms of Vandermonde type determinants.

Key words. Vandermonde type determinants, orthogonal systems, polynomial weights, inequalities.

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1. Introduction and statement of results.

First we want to introduce some terminology for orthogonal polynomials, referring to [1, 2, 7] for standard results.

We denote by σ an m-distribution, that is, a non-decreasing bounded function $\sigma : \mathbb{R} \to \mathbb{R}$ which attains infinitely many distinct values and is such that the moments

$$\mu_n \;:=\; \int_{-\infty}^{\infty} x^n \,\mathrm{d}\sigma(x) \qquad (n\in\mathbb{N}_0)$$

exist. Then there exists a uniquely determined sequence of polynomials

$$P_0(z),\; P_1(z),\; \dots,\; P_n(z),\; \dots\,,$$

called the sequence of *monic orthogonal polynomials with respect to* $\mathrm{d}\sigma(x)$, with the following properties:

(i) each P_n is a monic polynomial of degree n;

(ii) $\langle P_n, P_m\rangle \;:=\; \int_{-\infty}^{\infty} P_n(x)P_m(x)\,\mathrm{d}\sigma(x) = 0 \quad \text{for} \quad m\neq n.$

For any polynomial f, we define the norm

$$\|f\| \;:=\; \left(\int_{-\infty}^{\infty} |f(x)|^2 \;\mathrm{d}\sigma(x)\right)^{1/2} \tag{1.1}$$

and introduce the numbers

$$\gamma_n \;:=\; \|P_n\|^2 \qquad (n\in\mathbb{N}_0). \tag{1.2}$$

The system $(P_n)_{n\in\mathbb{N}_0}$ satisfies a recurrence formula

$$P_n(x) \;=\; (x-\alpha_n)P_{n-1}(x) \;-\; \beta_{n-1}P_{n-2}(x) \qquad (n\in\mathbb{N}), \tag{1.3}$$

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where $P_{-1}(x) \equiv 0$, $P_0(x) \equiv 1$, $\beta_0 = 1$, and

$$\alpha_n = \frac{1}{\gamma_{n-1}} \int_{-\infty}^{\infty} x P_{n-1}^2(x)\, \mathrm{d}\sigma(x) \quad \text{and} \quad \beta_n = \frac{\gamma_n}{\gamma_{n-1}} \quad (n \in \mathbb{N}); \tag{1.4}$$

see [7, § 3.2].

It is known that the polynomials P_n $(n \in \mathbb{N})$ have only real zeros. Denoting by J_n the smallest interval containing the zeros of P_n, we introduce the n-th *distance function*

$$d_n(z) := \min\left\{ |z - \zeta| \,:\, \zeta \in J_n \right\} \qquad (z \in \mathbb{C}). \tag{1.5}$$

Since the zeros of consecutive orthogonal polynomials interlace, we have

$$d_1(z) \geq d_2(z) \geq \cdots \geq d_n(z) \geq \cdots \geq |\Im z|\,.$$

In this paper, we want to estimate the following generalized Vandermonde type determinants:

$$V_n(z_1,\ldots,z_k) := \det \begin{pmatrix} P_n(z_1) & P_{n+1}(z_1) & \ldots & P_{n+k-1}(z_1) \\ \vdots & \vdots & & \vdots \\ P_n(z_k) & P_{n+1}(z_k) & \ldots & P_{n+k-1}(z_k) \end{pmatrix}. \tag{1.6}$$

It is easily seen that $V_0(z_1,\ldots,z_k)$ is equal to the classical Vandermonde determinant of $z_1,\ldots,z_k$. In fact, each polynomial P_m may be written as

$$P_m(z) = z^m + \sum_{\mu=0}^{m-1} c_{m\mu} P_\mu(z),$$

with certain constants $c_{m\mu}$. Hence, if we add to each column in (1.6) an appropriate linear combination of its predecessors, and do it first for the last column, then for the last but one and so on, we find that

$$V_0(z_1,\ldots,z_k) = \det \begin{pmatrix} 1 & z_1 & \ldots & z_1^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_k & \ldots & z_k^{k-1} \end{pmatrix} = \prod_{1 \leq \ell < j \leq k} \left(z_j - z_\ell\right). \tag{1.7}$$

There is no simple explicit formula for $V_n(z_1,\ldots,z_k)$ when $n \geq 1$, and therefore we are interested in bounds for these determinants.

It is usually not a big problem to find some upper bound for a determinant. For sake of completeness, we present the following result.

PROPOSITION 1.1. *Let $z_1,\ldots,z_k \in \mathbb{C}$. Then, with the preceding notations,*

$$|V_n(z_1,\ldots,z_k)| \leq \left[\gamma_n \gamma_{n+1} \cdots \gamma_{n+k-1} \prod_{j=1}^{k} \Big(\Delta_{n+k}(z_j) - \Delta_n(z_j) \Big) \right]^{1/2}, \tag{1.8}$$

where

$$\Delta_m(z) := \begin{cases} \dfrac{1}{\gamma_{m-1}} \Big(P_m'(z) P_{m-1}(z) - P_m(z) P_{m-1}'(z) \Big) & \text{if } z \in \mathbb{R} \\[2ex] \dfrac{1}{\gamma_{m-1}} \cdot \dfrac{\Im\left\{ P_m(z) P_{m-1}(\overline{z}) \right\}}{\Im z} & \text{if } z \in \mathbb{C} \setminus \mathbb{R} \end{cases} \qquad (m \in \mathbb{N}).$$

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Moreover, for any $z \in \mathbb{C}$,

$$\frac{1}{\gamma_{m-1}} \min_{1\le\mu\le m} \left|\frac{P_m(z)}{z-x_\mu}\right|^2 \le \Delta_m(z) \le \frac{1}{\gamma_{m-1}} \max_{1\le\mu\le m} \left|\frac{P_m(z)}{z-x_\mu}\right|^2, \tag{1.9}$$

where $x_1, \dots, x_m$ *are the zeros of* P_m.

Lower estimates for determinants in general and for our Vandermonde type determinants in particular are much more delicate. We shall establish a lower bound for the modulus of

$$\frac{V_{n+1}(z_1,\dots,z_k)}{V_n(z_1,\dots,z_k)}. \tag{1.10}$$

Used repeatedly for n, $n-1$, $n-2$, ... and combined with (1.7), it allows us to estimate $|V_{n+1}(z_1,\dots,z_k)|$ for $n \in \mathbb{N}_0$ from below. Here we admit that some (or all) of the points $z_1,\dots,z_k$ may coalesce. In this case, we define the quotient (1.10) by continuous continuation. More precisely, if $z_{j_0} = z_{j_1} = \cdots = z_{j_\ell}$, then we replace the polynomials in the j_1-st, j_2-nd, ..., j_ℓ-th row in (1.6) by their first, second, ..., ℓ-th derivative.

As usual, we shall denote by $\lfloor x\rfloor$ the largest integer not exceeding x. We are now ready for presenting the main result.

THEOREM 1.2. *Let* $(P_n)_{n\in\mathbb{N}_0}$ *be a sequence of monic orthogonal polynomials with associated intervals* J_n, *constants* γ_n, *and distance functions* d_n $(n \in \mathbb{N})$ *as specified in* (1.2)–(1.5). *Let* $w(x) = \prod_{j=1}^k (x - z_j)$ *be a real polynomial which has no zero in* $J_{n+\lfloor k/2\rfloor+1}$. *Denote by* m_1 *and* m_2 *the number of zeros (counted according to their multiplicities) in the left and the right component of* $\mathbb{R}\setminus J_{n+\lfloor k/2\rfloor+1}$, *respectively, Define*

$$m := \begin{cases} 0 & \text{if } m_1 \text{ and } m_2 \text{ are both even,} \\ 1 & \text{if exactly one of the numbers } m_1 \text{ and } m_2 \text{ is odd,} \\ 2 & \text{if } m_1 \text{ and } m_2 \text{ are both odd,} \end{cases}$$

and $\ell := (k-m)/2$. *Suppose that*

$$d_{n+\lfloor k/2\rfloor+1}(z_j) \ge r \qquad (j = 1,\dots,k),$$

with $r > 0$. *Then, for the determinants* (1.6),

$$\left|\frac{V_{n+1}(z_1,\dots,z_k)}{V_n(z_1,\dots,z_k)}\right| \ge \frac{r^m}{\gamma_n} \sum_{j=0}^{\ell} \binom{\ell}{j} \gamma_{n+\ell-j}\, r^{2j}. \tag{1.11}$$

Remark 1. Note that $m = 0$ if $w(x)$ does not change sign on $\mathbb{R}$. Furthermore, when $m = 0$, then the right-hand side of (1.11) remains positive even if $r \to 0$. Therefore (1.11) holds with a positive lower bound even if $w(x)$ has zeros on $J_{n+\lfloor k/2\rfloor+1}$ provided that their multiplicities are even and $m = 0$. However, if $w(x)$ changes sign on $J_{n+\lfloor k/2\rfloor+1}$, then the left-hand side of (1.11) may vanish, and so we cannot have a non-trivial lower bound.

The proof of Theorem 1.2 will show that (1.11) can be refined by working with individual bounds r_j instead of r such that $d_{n+\lfloor k/2\rfloor+1}(z_j) \ge r_j$ for $j = 1,\dots,k$.

Remark 2. At the conference in Inzell (3rd Workshop 'Orthogonal Polynomials, Approximation, and Harmonic Analysis', April 2000), Michael Skrzipek gave a lecture on the inversion of Vandermonde type matrices of orthogonal polynomials. Theorem 1.2 includes a sufficient condition for invertibility.

Remark 3. The proof of Theorem 1.2 rests on repeated application of Lemmas 2.2 and 2.3 given in § 2 below. In the proofs of these lemmas, all considerations are based on *equations*. It is only at the end that a lower bound is deduced from a mean value. We can as well deduce an upper bound from that mean value and establish an inequality analogous to (1.11), but in the opposite direction. More precisely, we can proceed as follows. Analogously to (1.5), we define

$$D_n(z) := \max\left\{ |z-\zeta| \,:\, \zeta \in J_n \right\} \qquad (z \in \mathbb{C}).$$

Then

$$D_1(z) \le D_2(z) \le \cdots \le D_n(z) \le \cdots .$$

Now suppose that in the situation of Theorem 1.2, we have

$$D_{n+\lfloor k/2\rfloor+1}(z_j) \le R \qquad (j = 1, \ldots, k).$$

Then, for the determinants (1.6),

$$\left| \frac{V_{n+1}(z_1,\ldots,z_k)}{V_n(z_1,\ldots,z_k)} \right| \le \frac{R^m}{\gamma_n} \sum_{j=0}^{\ell} \binom{\ell}{j} \gamma_{n+\ell-j}\, R^{2j} . \tag{1.12}$$

Remark 4. Except for trivial cases, inequality (1.11) is not sharp. In view of Remark 3 and an analysis of the proofs given below, we find that the precision of (1.11) depends on the length of $J_{n+\lfloor k/2\rfloor+1}$. If this interval is relatively small, then the estimate (1.11) is quite accurate. If $J_{n+\lfloor k/2\rfloor+1}$ is unbounded as $n\to\infty$, then (1.11) will be less accurate when n is large, but it will be non-trivial nevertheless.

The proof of Theorem 1.2 will show that the points $z_1,\ldots,z_k$ can be involved successively as real singles and pairs of conjugates. Therefore the location of these points, relative to one another, is not crucial for the accuracy of (1.11). This may be surprising since, on the left-hand side of (1.11), the numerator and the denominator tend to zero as two of the points $z_1,\ldots,z_k$ approach each other.

If in Theorem 1.2 the hypothesis on $w(x)$ holds for some $n\in\mathbb{N}$, then it automatically holds for all smaller indices n, and m and ℓ do not change when n is reduced. This allows us to deduce the following lower estimate for $|V_{n+1}(z_1,\ldots,z_k)|$ by iterating (1.11) and employing (1.7).

COROLLARY 1.3. *Suppose that in the statement of Theorem 1.2 the hypothesis on $w(x)$ holds for some $n\in\mathbb{N}$. Then, introducing the polynomials*

$$\phi_\nu(x) := \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{\gamma_{\nu+\ell-j}}{\gamma_\nu}\, x^{2j} \qquad (\nu = 0,\ldots,n),$$

we have

$$\begin{aligned}|V_{n+1}(z_1,\ldots,z_k)| &\ge r^{m(n+1)} \prod_{\nu=0}^{n} \phi_\nu(r) \prod_{1\le i<j\le k} |z_i - z_j| \\ &\ge r^{m(n+1)} \prod_{\nu=0}^{n} \frac{\gamma_{\nu+\ell}}{\gamma_\nu} \prod_{1\le i<j\le k} |z_i - z_j| .\end{aligned}$$

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For the proof of Theorem 1.2, we shall employ another orthogonal system which reveals why the determinants $V_n(z_1,\dots,z_k)$ are of interest.

Let $w(x)=\prod_{j=1}^{k}(x-z_\nu)$ be a real polynomial which is non-negative on the real line. Then there exists a uniquely determined sequence of monic orthogonal polynomials

$$P_0^{[w]}(z),\ P_1^{[w]}(z),\ \dots,\ P_n^{[w]}(z),\ \dots \tag{1.13}$$

with respect to $w(x)\mathrm{d}\sigma(x)$. We want to distinguish all the quantities associated with this system from those associated with $(P_n)_{n\in\mathbb{N}_0}$ by attaching a superscript $[w]$. Thus

$$\gamma_n^{[w]} := \int_{-\infty}^{\infty}\left(P_n^{[w]}(x)\right)^2 w(x)\,\mathrm{d}\sigma(x),$$

$J_n^{[w]}$ is the smallest interval containing the zeros of $P_n^{[w]}$, and $d_n^{[w]}(z)$ is the distance of z from $J_n^{[w]}$.

In 1858 already, Christoffel had observed (in the case where $\sigma(x)=x$) that

$$w(x)P_n^{[w]}(x) = (-1)^k\,\frac{V_n(x,z_1,\dots,z_k)}{V_n(z_1,\dots,z_k)} \qquad (n\in\mathbb{N}_0); \tag{1.14}$$

see [7, § 2.5], where the result is given for general σ.

When $w(x)$ changes sign on the real line, then $w(x)\mathrm{d}\sigma(x)$ may not be an admissible differential for defining an inner product in the space of polynomials. But if $w(x)$ is non-negative on $J_{n+\lfloor k/2\rfloor+1}$, then $w(x)\mathrm{d}\sigma(x)$ is admissible for the subspace $\mathcal{P}_n$ consisting of all polynomials of degree at most n. In fact, for any $f,\ g\in\mathcal{P}_n$, the integral

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}w(x)\,\mathrm{d}\sigma(x) \tag{1.15}$$

can be calculated by means of the Gaussian quadrature formula [7, § 3.4] whose nodes are the zeros of $P_{n+\lfloor k/2\rfloor+1}$, and so we need only the restriction of w to $J_{n+\lfloor k/2\rfloor+1}$. Thus we find that (1.15) defines an inner product on $\mathcal{P}_n$ and that the polynomials

$$P_0^{[w]}(z),\ P_1^{[w]}(z),\ \dots,\ P_n^{[w]}(z), \tag{1.16}$$

as given by (1.14), form an orthogonal basis for $\mathcal{P}_n$. Moreover, if the support of $\mathrm{d}\sigma(x)$ is contained in an interval J (such an interval is called an *interval of orthogonality*) and $w(x)$ is non-negative on J, then (1.14) defines an infinite sequence of orthogonal polynomials.

If, in the previous paragraph, $w(x)$ is non-positive on $J_{n+\lfloor k/2\rfloor+1}$ (respectively, on J), then the polynomials (1.16) (respectively, those in (1.13) with unrestricted n), exactly as defined by (1.14), form a sequence of monic orthogonal polynomials with respect to $-w(x)\mathrm{d}\sigma(x)$.

In order to establish the recurrence formula for the system $(P_n^{[w]})_{n\in\mathbb{N}_0}$, we need a modification of the determinants $V_n(z_1,\dots,z_k)$. We denote by $V_n^*(z_1,\dots,z_k)$ the determinant obtained from $V_n(z_1,\dots,z_k)$ by replacing the index n of the polynomials in the first column by $n-1$, that is,

$$V_n^*(z_1,\dots,z_k) = \det\begin{pmatrix} P_{n-1}(z_1) & P_{n+1}(z_1) & \dots & P_{n+k-1}(z_1)\\ \vdots & \vdots & & \vdots\\ P_{n-1}(z_k) & P_{n+1}(z_k) & \dots & P_{n+k-1}(z_k)\end{pmatrix}. \tag{1.17}$$

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THEOREM 1.4. *Let $(P_n)_{n\in\mathbb{N}_0}$ be a monic orthogonal system satisfying a recurrence formula*

$$P_n(x) \;=\; (x-\alpha_n)P_{n-1}(x) - \beta_{n-1}P_{n-2}(x) \qquad (n\in\mathbb{N})$$

with $P_{-1}(x)\equiv 0$ and $P_0(x)\equiv 1$. Let J be an interval of orthogonality, and suppose that $w(x)=\prod_{j=1}^{k}(x-z_j)$ is a real polynomial that does not change sign on J. Then

$$P_n^{[w]}(x) \;=\; (x-\alpha_n^{[w]})P_{n-1}^{[w]}(x) - \beta_{n-1}^{[w]}P_{n-2}^{[w]}(x) \qquad (n\in\mathbb{N}) \tag{1.18}$$

with $P_{-1}^{[w]}(x)\equiv 0,\; P_0^{[w]}(x)\equiv 1$, and

$$\alpha_n^{[w]} = \alpha_n \;+\; \beta_{n-1}\frac{V_{n-1}^*(z_1,\dots,z_k)}{V_{n-1}(z_1,\dots,z_k)} \;-\; \beta_n\frac{V_n^*(z_1,\dots,z_k)}{V_n(z_1,\dots,z_k)}\,, \tag{1.19}$$

$$\beta_n^{[w]} = \beta_n\frac{V_{n+1}(z_1,\dots,z_k)\,V_{n-1}(z_1,\dots,z_k)}{\big(V_n(z_1,\dots,z_k)\big)^2} \qquad (n\in\mathbb{N}). \tag{1.20}$$

Note that $\beta_0^{[w]}$ is not needed and may therefore be arbitrarily defined.

While (1.19) and (1.20) give explicit representations for $\alpha_n^{[w]}$ and $\beta_n^{[w]}$, Gautschi [3] proposed an algorithm for a recursive computation of these quantities. However, as far as the computation of the polynomials $P_n^{[w]}(x)$ is concerned, Skrzipek [6] pointed out that the use of the recurrence formula (1.18) may have disadvantages. He proposed an alternative approach.

In [5], we have proved several inequalities for $\gamma_n^{[w]}$ and $\|P_n^{[w]}\|$; see [5, Lemmas 3–5]. They imply further inequalities for $V_n(z_1,\dots,z_k)$ and its modifications. Some of these inequalities are sharp.

2. Lemmas. Continuing in using the notations of § 1, we shall prove the following auxiliary results.

LEMMA 2.1. *Let P_n and P_{n+1} be consecutive monic orthogonal polynomials, and denote by $x_1,\dots,x_{n+1}$ the zeros of P_{n+1}. Then*

$$\frac{P_n(z)}{P_{n+1}(z)} \;=\; \sum_{\nu=1}^{n+1}\frac{\lambda_\nu}{z-x_\nu} \qquad \textit{where} \quad \lambda_\nu>0 \;\; (\nu=1,\dots,n+1) \tag{2.1}$$

and

$$\sum_{\nu=1}^{n+1}\lambda_\nu \;=\; 1\,. \tag{2.2}$$

Proof. For (2.1), see [7, p. 47, Theorem 3.3.5]). Multiplying both sides of the equation in (2.1) by z and letting $z\to\infty$, we readily conclude that (2.2) holds. ∎

LEMMA 2.2. *Let $N\in\mathbb{N}$ and $w(x)=x-\xi$ with $\xi\in\mathbb{R}\setminus J_{N+1}$. Then*

$$J_n^{[w]} \;\subset\; J_{n+1} \tag{2.3}$$

and

$$\gamma_n^{[w]} \geq \gamma_n d_{n+1}(\xi) \tag{2.4}$$

for $n = 0, \dots, N$.

Proof. By (1.14),

$$w(x) P_n^{[w]}(x) = P_{n+1}(x) - \frac{P_{n+1}(\xi)}{P_n(\xi)} P_n(x) \quad (n = 0, \dots, N). \tag{2.5}$$

Now let $x_1, \dots, x_{n+1}$ be the zeros of P_{n+1} in increasing order. Then

$$w(x_\nu) P_n^{[w]}(x_\nu) = -\frac{P_{n+1}(\xi)}{P_n(\xi)} P_n(x_\nu) \qquad (\nu = 1, \dots, n+1).$$

Since the polynomials P_n and P_{n+1} are monic and their zeros interlace, we have

$$\operatorname{sgn} P_n(x_\nu) = (-1)^{n+1-\nu} \qquad (\nu = 1, \dots, n+1). \tag{2.6}$$

Taking into account that $w(x)$ does not change sign on J_{N+1}, we find that

$$\operatorname{sgn} P_n^{[w]}(x_\nu) = (-1)^{n-\nu} \operatorname{sgn} \frac{P_{n+1}(\xi)}{w(x_1) P_n(\xi)} \qquad (\nu = 1, \dots, n+1).$$

Hence the zeros of $P_n^{[w]}$ and P_{n+1} interlace, and this implies (2.3).

The polynomials $P_0^{[w]}, \dots, P_N^{[w]}$ are orthogonal with respect to $\pm w(x) \mathrm{d}\sigma(x)$, the sign depending on the sign of $w(x)$ on J_{N+1}. In any case,

$$\gamma_n^{[w]} = \left| \int_{-\infty}^{\infty} \left(P_n^{[w]}(x) \right)^2 w(x) \, \mathrm{d}\sigma(x) \right| = \left| \int_{-\infty}^{\infty} w(x) P_n^{[w]}(x) P_n(x) \, \mathrm{d}\sigma(x) \right|.$$

Substituting (2.5) into the right-hand side, we readily find that

$$\gamma_n^{[w]} = \gamma_n \left| \frac{P_{n+1}(\xi)}{P_n(\xi)} \right|. \tag{2.7}$$

Now, by Lemma 2.1,

$$\left| \frac{P_n(\xi)}{P_{n+1}(\xi)} \right| \leq \frac{1}{\min_{1 \leq \nu \leq n+1} |\xi - x_\nu|} \leq \frac{1}{d_{n+1}(\xi)},$$

and so (2.4) follows from (2.7). □

While in Lemma 2.2 $w(x)$ was linear, we now establish a corresponding result for a quadratic $w(x)$.

LEMMA 2.3. *Let* $N \in \mathbb{N}$ *and* $w(x) = (x - \zeta_1)(x - \zeta_2)$, *where either* $\zeta_2 = \overline{\zeta_1}$ *or* $\zeta_1, \zeta_2 \in \mathbb{R} \setminus J_{N+2}$. *Then*

$$J_n^{[w]} \subset J_{n+1} \qquad (n = 0, \dots, N). \tag{2.8}$$

If, in addition, ζ_1 *and* ζ_2 *do not lie in different components of* $\mathbb{R} \setminus J_{N+2}$, *then*

$$\gamma_n^{[w]} \geq \gamma_{n+1} + \gamma_n \, d_{n+1}(\zeta_1) \, d_{n+1}(\zeta_2) \qquad (n = 0, \dots, N). \tag{2.9}$$

Proof. We shall prove the lemma under the additional hypothesis that $\zeta_1 \neq \zeta_2$. An extension to $\zeta_1 = \zeta_2$ will be achieved by continuous continuation, as we have explained in the paragraph following (1.10).

Using the notation (1.17), we deduce from (1.14) by Laplace expansion (with respect to the first row) of the determinant in the numerator that

$$w(x)P_n^{[w]}(x) \;=\; \frac{V_{n+1}(\zeta_1,\zeta_2)}{V_n(\zeta_1,\zeta_2)}\,P_n(x) - \frac{V_{n+1}^*(\zeta_1,\zeta_2)}{V_n(\zeta_1,\zeta_2)}\,P_{n+1}(x) + P_{n+2}(x). \tag{2.10}$$

Now let $x_1,\dots,x_{n+1}$ be again the zeros of P_{n+1} in increasing order. It follows from the recurrence formula (1.3) and from (1.4) that

$$P_{n+2}(x_\nu) \;=\; -\frac{\gamma_{n+1}}{\gamma_n}\,P_n(x_\nu) \qquad (\nu = 1,\dots,n+1).$$

Hence (2.10) gives

$$w(x_\nu)P_n^{[w]}(x_\nu) \;=\; P_n(x_\nu)\left(\frac{V_{n+1}(\zeta_1,\zeta_2)}{V_n(\zeta_1,\zeta_2)} - \frac{\gamma_{n+1}}{\gamma_n}\right) \qquad (\nu = 1,\dots,n+1).$$

The term in parentheses must be different from zero since $P_n^{[w]}$ would have $n+1$ zeros otherwise. Recalling (2.6), we easily conclude that the zeros of $P_n^{[w]}$ and P_{n+1} interlace. This shows that (2.8) holds.

Now we want to estimate $\gamma_n^{[w]}$ from below. Clearly

$$\gamma_n^{[w]} \;=\; \int_{-\infty}^{\infty}\left(P_n^{[w]}(x)\right)^2 w(x)\,\mathrm{d}\sigma(x) \;=\; \int_{-\infty}^{\infty} w(x)P_n^{[w]}(x)P_n(x)\,\mathrm{d}\sigma(x).$$

Substituting (2.10) into the right-hand side, we readily find that

$$\gamma_n^{[w]} \;=\; \gamma_n\,\frac{V_{n+1}(\zeta_1,\zeta_2)}{V_n(\zeta_1,\zeta_2)}\,. \tag{2.11}$$

Employing the recurrence formula (1.3), we obtain that

$$\begin{aligned} V_{n+1}(\zeta_1,\zeta_2) &= \det\begin{pmatrix} P_{n+1}(\zeta_1) & P_{n+2}(\zeta_1) \\ P_{n+1}(\zeta_2) & P_{n+2}(\zeta_2)\end{pmatrix} \\ &= \det\begin{pmatrix} P_{n+1}(\zeta_1) & (\zeta_1-\alpha_{n+2})P_{n+1}(\zeta_1) - \beta_{n+1}P_n(\zeta_1) \\ P_{n+1}(\zeta_2) & (\zeta_2-\alpha_{n+2})P_{n+1}(\zeta_2) - \beta_{n+1}P_n(\zeta_2)\end{pmatrix} \\ &= (\zeta_2-\zeta_1)P_{n+1}(\zeta_1)P_{n+1}(\zeta_2) \;+\; \frac{\gamma_{n+1}}{\gamma_n}\,V_n(\zeta_1,\zeta_2)\,. \end{aligned}$$

Hence (2.11) may be rewritten as

$$\gamma_n^{[w]} \;=\; \gamma_{n+1} + \gamma_n\,\frac{(\zeta_2-\zeta_1)P_{n+1}(\zeta_1)P_{n+1}(\zeta_2)}{V_n(\zeta_1,\zeta_2)}\,. \tag{2.12}$$

Using Lemma 2.1 for a partial fraction decomposition of P_n/P_{n+1}, we find that

$$\frac{V_n(\zeta_1,\zeta_2)}{P_{n+1}(\zeta_1)P_{n+1}(\zeta_2)} \;=\; \frac{P_n(\zeta_1)}{P_{n+1}(\zeta_1)} - \frac{P_n(\zeta_2)}{P_{n+1}(\zeta_2)} \;=\; (\zeta_2-\zeta_1)\sum_{\nu=1}^{n+1}\frac{\lambda_\nu}{(\zeta_1-x_\nu)(\zeta_2-x_\nu)}\,.$$

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Since ζ_1, and ζ_2 are either a pair of conjugate zeros or a pair of real zeros lying in the same component of $\mathbb{R} \setminus J_{N+2}$, we see that the last sum is a mean value of positive terms. Therefore

$$\frac{V_n(\zeta_1, \zeta_2)}{(\zeta_2 - \zeta_1) P_{n+1}(\zeta_1) P_{n+1}(\zeta_2)} \le \frac{1}{d_{n+1}(\zeta_1) d_{n+1}(\zeta_2)} .$$

Combining this estimate with (2.12), we obviously obtain (2.9). $\square$

3. Proofs of the results in § 1.

Proof of Proposition 1.1. Let $A = (a_{\mu\nu})$ be a matrix in $\mathbb{C}^{n\times n}$. Then, by an inequality of Hadamard [4, p. 418, Theorem 13.5.3] applied to the transpose of A,

$$|\det A| \le \prod_{\mu=1}^{n} \left(\sum_{\nu=1}^{n} |a_{\mu\nu}|^2 \right)^{1/2} .$$

This estimate can be generalized. If $D \in \mathbb{C}^{n\times n}$ is a non-singular diagonal matrix with diagonal entries $d_1, \ldots, d_n$, then $\det(D^{-1}AD) = \det A$, and therefore

$$|\det A| \le \frac{1}{|d_1 \cdots d_n|} \prod_{\mu=1}^{n} \left(\sum_{\nu=1}^{n} |a_{\mu\nu} d_\nu|^2 \right)^{1/2} .$$

This shows that $V_n(z_1, \ldots, z_k)$ may be estimated as

$$|V_n(z_1, \ldots, z_k)| \le \left[\gamma_n \gamma_{n+1} \cdots \gamma_{n+k-1} \prod_{j=1}^{k} \left(\sum_{\nu=n}^{n+k-1} \frac{1}{\gamma_\nu} |P_\nu(z_j)|^2 \right) \right]^{1/2} . \tag{3.1}$$

By the Christoffel–Darboux formula (see [7, p. 43, (3.2.3) and (3.2.4)])

$$\sum_{\nu=0}^{m-1} \frac{1}{\gamma_\nu} |P_\nu(z)|^2 = \begin{cases} \dfrac{1}{\gamma_{m-1}} \Big(P_m'(z) P_{m-1}(z) - P_m(z) P_{m-1}'(z) \Big) & \text{if} \quad z \in \mathbb{R}, \\ \dfrac{1}{\gamma_{m-1}} \cdot \dfrac{P_m(z) P_{m-1}(\overline{z}) - P_m(\overline{z}) P_{m-1}(z)}{z - \overline{z}} & \text{if} \quad z \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

But this is the quantity $\Delta_m(z)$, defined in Proposition 1.1. Thus (3.1) gives (1.8).

Employing Lemma 2.1, we can avoid the distinction between real and non-real z in the definition of $\Delta_m(z)$. In fact, let $x_1, \ldots, x_m$ be the zeros of P_m, and let $\lambda_1, \ldots, \lambda_m$ be the coefficients in the partial fraction decomposition of P_{m-1}/P_m according to Lemma 2.1.

If $z \in \mathbb{R}$, then

$$\begin{aligned} \gamma_{m-1} \Delta_m(z) &= P_m'(z) P_{m-1}(z) - P_m(z) P_{m-1}'(z) \\ &= -\, P_m^2(z) \frac{\mathrm{d}}{\mathrm{d}z} \frac{P_{m-1}(z)}{P_m(z)} \\ &= P_m^2(z) \sum_{\mu=1}^{m} \frac{\lambda_\mu}{(z - x_\mu)^2} \\ &= |P_m(z)|^2 \sum_{\mu=1}^{m} \frac{\lambda_\mu}{|z - x_\mu|^2} , \end{aligned}$$

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and if $z \in \mathbb{C} \setminus \mathbb{R}$, then

$$\begin{aligned}\gamma_{m-1}\Delta_m(z) &= \frac{P_m(z)P_{m-1}(\overline{z}) - P_m(\overline{z})P_{m-1}(z)}{z-\overline{z}} \\ &= \frac{|P_m(z)|^2}{z-\overline{z}}\left(\frac{P_{m-1}(\overline{z})}{P_m(\overline{z})} - \frac{P_{m-1}(z)}{P_m(z)}\right) \\ &= \frac{|P_m(z)|^2}{z-\overline{z}}\left(\sum_{\mu=1}^{m}\frac{\lambda_\mu}{\overline{z}-x_\mu} - \sum_{\mu=1}^{m}\frac{\lambda_\mu}{z-x_\mu}\right) \\ &= |P_m(z)|^2\sum_{\mu=1}^{m}\frac{\lambda_\mu}{|z-x_\mu|^2}\,.\end{aligned}$$

Hence for any $z \in \mathbb{C}$,

$$\Delta_m(z) \;=\; \frac{1}{\gamma_{m-1}}\sum_{\mu=1}^{m}\lambda_\mu\left|\frac{P_m(z)}{z-x_\mu}\right|^2,$$

which gives (1.9) at once. ◻

Proof of Theorem 1.2. First we note that

$$\left|\frac{V_{n+1}(z_1,\dots,z_k)}{V_n(z_1,\dots,z_k)}\right| \;=\; \frac{\gamma_n^{[w]}}{\gamma_n}\,. \tag{3.2}$$

In fact, using (1.14), we see that

$$\begin{aligned}\gamma_n^{[w]} &= \left|\int_{-\infty}^{\infty}\left(P_n^{[w]}(x)\right)^2 w(x)\,\mathrm{d}\sigma(x)\right| \;=\; \left|\int_{-\infty}^{\infty} w(x)P_n^{[w]}(x)P_n(x)\,\mathrm{d}\sigma(x)\right| \\ &= \left|\frac{1}{V_n(z_1,\dots,z_k)}\int_{-\infty}^{\infty} V_n(x,z_1,\dots,z_k)P_n(x)\,\mathrm{d}\sigma(x)\right|.\end{aligned}$$

Now, expanding the determinant inside the integral with respect to the first row and paying attention to the orthogonality of the system $(P_n)_{n\in\mathbb{N}_0}$, we readily obtain (3.2).

In view of (3.2), we have to estimate $\gamma_n^{[w]}$ from below. For this we can use Lemmas 2.2 and 2.3 repeatedly, taking advantage of the obvious fact that the operation of attaching a superscript $[w]$ is multiplicative in the following sense. If $w = uv$, then $P_n^{[w]} = (P_n^{[u]})^{[v]}$ and, consequently, $\gamma_n^{[w]} = (\gamma_n^{[u]})^{[v]}$.

Obviously, we may factor the polynomial w as

$$w(x) \;=\; p(x)q_1(x)\cdots q_\ell(x),$$

where p is a monic real polynomial of degree m such that, if $m = 2$, then the zeros of p lie in different components of $\mathbb{R} \setminus J_{n+\lfloor k/2\rfloor+1}$, and where q_λ $(\lambda = 1,\dots,\ell)$ are monic real polynomials of degree two, each having either a pair of conjugate zeros or a pair of real zeros lying in the same component of $\mathbb{R} \setminus J_{n+\lfloor k/2\rfloor+1}$. In particular, each $q_\lambda(x)$ is positive for $x \in J_{n+\lfloor k/2\rfloor+1}$.

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Applying Lemma 2.2 m times, we readily see that

$$\gamma_\nu^{[p]} \geq \gamma_\nu \, r^m \qquad (\nu = 0, \ldots, n+\ell).$$

Now we define $w_0(x) := p(x)$ and

$$w_\lambda(x) := p(x) q_1(x) \cdots q_\lambda(x) \qquad (\lambda = 1, \ldots, \ell).$$

We claim that

$$\gamma_\nu^{[w_\lambda]} \geq r^m \sum_{j=0}^{\lambda} \binom{\lambda}{j} \gamma_{\nu+\lambda-j} r^{2j} \quad \text{and} \quad d_\nu^{[w_\lambda]}(z) \geq d_{\nu+m+\lambda}(z) \tag{3.3}$$
$$(\lambda = 0, \ldots, \ell;\ \nu = 0, \ldots, n+\ell-\lambda).$$

The inequality for $d_\nu^{[w_\lambda]}(z)$ is an easy consequence of Lemmas 2.2 and 2.3. The inequality for $\gamma_\nu^{[w_\lambda]}$ may be proved by induction on λ as follows.

Let ζ_1 and ζ_2 be the zeros of $q_{\lambda+1}$. Using again Lemma 2.3, we conclude that

$$\begin{aligned}
\gamma_\nu^{[w_{\lambda+1}]} = \left(\gamma_\nu^{[w_\lambda]}\right)^{[q_{\lambda+1}]} &\geq \gamma_{\nu+1}^{[w_\lambda]} + \gamma_\nu^{[w_\lambda]} d_{\nu+1}^{[w_\lambda]}(\zeta_1)\, d_{\nu+1}^{[w_\lambda]}(\zeta_2) \\
&\geq \gamma_{\nu+1}^{[w_\lambda]} + \gamma_\nu^{[w_\lambda]} d_{\nu+m+\lambda+1}(\zeta_1)\, d_{\nu+m+\lambda+1}(\zeta_2) \\
&\geq \gamma_{\nu+1}^{[w_\lambda]} + \gamma_\nu^{[w_\lambda]} r^2 .
\end{aligned}$$

Now the induction hypothesis applies and gives

$$\begin{aligned}
\gamma_{\nu+1}^{[w_\lambda]} + \gamma_\nu^{[w_\lambda]} r^2 &\geq r^m \left[\sum_{j=0}^{\lambda} \binom{\lambda}{j} \gamma_{\nu+1+\lambda-j} r^{2j} + \sum_{j=0}^{\lambda} \binom{\lambda}{j} \gamma_{\nu+\lambda-j} r^{2j+2}\right] \\
&= r^m \left[\gamma_{\nu+\lambda+1} + \sum_{j=1}^{\lambda} \left\{\binom{\lambda}{j} + \binom{\lambda}{j-1}\right\} \gamma_{\nu+\lambda+1-j} r^{2j} + \gamma_\nu r^{2\lambda+2}\right] \\
&= r^m \sum_{j=0}^{\lambda+1} \binom{\lambda+1}{j} \gamma_{\nu+\lambda+1-j} r^{2j} .
\end{aligned}$$

This completes the proof of (3.3).

Finally, noting that $w(x) = w_\ell(x)$, and combining (3.2) and (3.3), we readily obtain (1.11). $\square$

Proof of Theorem 1.4. Let $\operatorname{sgn} w(x) =: \varepsilon$ for $x \in J$. It is clear, from the general theory of orthogonal polynomials, that a recurrence formula of the form (1.18) holds, where, according to (1.4),

$$\alpha_n^{[w]} = \frac{\varepsilon}{\gamma_{n-1}^{[w]}} \int_{-\infty}^{\infty} x \left(P_{n-1}^{[w]}(x)\right)^2 w(x)\, \mathrm{d}\sigma(x) \tag{3.4}$$

and

$$\beta_n^{[w]} = \frac{\gamma_n^{[w]}}{\gamma_{n-1}^{[w]}} \tag{3.5}$$

for $n \in \mathbb{N}$.

¿From (3.2) and the discussion of the influence of the sign of w (see § 1), we know that

$$\gamma_n^{[w]} = \varepsilon(-1)^k \gamma_n \frac{V_{n+1}(z_1,\ldots,z_k)}{V_n(z_1,\ldots,z_k)} \qquad (n \in \mathbb{N}_0), \tag{3.6}$$

which gives (1.20) at once.

A verification of (1.19) is more sophisticated. In view of (1.14), we may write

$$x\left(P_{n-1}^{[w]}(x)\right)^2 w(x) = \det \begin{pmatrix} P_{n-1}(x) & P_n(x) & \ldots & P_{n+k-1}(x) \\ P_{n-1}(z_1) & P_n(z_1) & \ldots & P_{n+k-1}(z_1) \\ \vdots & \vdots & \ldots & \vdots \\ P_{n-1}(z_k) & P_n(z_k) & \ldots & P_{n+k-1}(z_k) \end{pmatrix} \frac{(-1)^k x P_{n-1}^{[w]}(x)}{V_{n-1}(z_1,\ldots,z_k)}.$$

By the orthogonality of the system $(P_n)_{n\in\mathbb{N}_0}$, we have

$$\int_{-\infty}^{\infty} x P_{n-1+j}(x) P_{n-1}^{[w]}(x)\,\mathrm{d}\sigma(x) = \begin{cases} 0 & \text{if} \quad j \geq 2, \\ \gamma_n & \text{if} \quad j = 1. \end{cases}$$

Hence, when we expand the determinant with respect to the first row and calculate $\alpha_n^{[w]}$ according to (3.4), we find that

$$\alpha_n^{[w]} = \frac{\varepsilon(-1)^k}{\gamma_{n-1}^{[w]}} \left[\frac{V_n(z_1,\ldots,z_k)}{V_{n-1}(z_1,\ldots,z_k)} \int_{-\infty}^{\infty} x P_{n-1}(x) P_{n-1}^{[w]}(x)\,\mathrm{d}\sigma(x) - \gamma_n \frac{V_n^*(z_1,\ldots,z_k)}{V_{n-1}(z_1,\ldots,z_k)} \right]. \tag{3.7}$$

Next, we have to calculate the integral on the right-hand side. By the recurrence formula for the system $(P_n)_{n\in\mathbb{N}_0}$, we have

$$x P_{n-1}(x) = P_n(x) + \alpha_n P_{n-1}(x) + \beta_{n-1} P_{n-2}(x).$$

This implies that

$$\int_{-\infty}^{\infty} x P_{n-1}(x) P_{n-1}^{[w]}(x)\,\mathrm{d}\sigma(x) = \alpha_n \gamma_{n-1} + \beta_{n-1} \int_{-\infty}^{\infty} P_{n-2}(x) P_{n-1}^{[w]}(x)\,\mathrm{d}\sigma(x)\,. \tag{3.8}$$

It remains to calculate the integral on the right-hand side. For this, we proceed as follows. By (1.14),

$$w(x) P_{n-2}^{[w]}(x) = (-1)^k \frac{V_{n-2}(x, z_1,\ldots,z_k)}{V_{n-2}(z_1,\ldots,z_k)}\,.$$

Multiplying both sides by $P_{n-1}^{[w]}(x)$, expanding the Vandermonde type determinant in the numerator, with respect to the first row, and integrating with respect to $\mathrm{d}\sigma(x)$, we obtain

$$0 = \int_{-\infty}^{\infty} w(x) P_{n-2}^{[w]}(x) P_{n-1}^{[w]}(x)\,\mathrm{d}\sigma(x) =$$

$$(-1)^k \left[\frac{V_{n-1}(z_1,\ldots,z_k)}{V_{n-2}(z_1,\ldots,z_k)} \int_{-\infty}^{\infty} P_{n-2}(x) P_{n-1}^{[w]}(x)\,\mathrm{d}\sigma(x) - \gamma_{n-1} \frac{V_{n-1}^*(z_1,\ldots,z_k)}{V_{n-2}(z_1,\ldots,z_k)} \right],$$

and so

$$\int_{-\infty}^{\infty} P_{n-2}(x) P_{n-1}^{[w]}(x)\,\mathrm{d}\sigma(x) = \gamma_{n-1} \frac{V_{n-1}^*(z_1,\ldots,z_k)}{V_{n-1}(z_1,\ldots,z_k)}\,. \tag{3.9}$$

Finally, combining (3.6)–(3.9), we arrive at (1.19). $\square$

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