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Bayesian Economic Cost Plans III. The Lot Mean Relative to a Quality Characteristic

Abraham F. Jalbout [1*], Hadi Y. Alkahby [2], Found N. Jalbout [3], Abdulla Darwish [3]

[1]Department of Chemistry, University of New Orleans, New Orleans, LA 70148-2820 USA, E-mail: Ajalbout@ejmaps.org
[2]Department of Mathematics, Dillard University, New Orleans, LA 70112 USA
[3]Department of Physics and Engineering, Dillard University, New Orleans, LA 70112 USA
*Author to whom correspondence should be addressed. [§]Guest Speaker at the Dillard University Seminar Series October 29, 2001

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Abstract: A quality control manufacturing process is designed to produce certain types of components (i.e. mechanical, electrical or chemical). The process is defined to be under control if the fraction of the items manufactured that are defective is reasonably small. The fraction of items defective varies from lot to lot, which is the main assumption that we will use in the mathematical development of our reliability model. It is logical to assume in this case that the mean of the lot is a random variable and so is the fraction defective. A relationship between the two quantities is the subject of this paper.

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1. Introduction.

When contemplating the application of this work it is best to describe an actual industrial process such as is producing items for commercial use in quantity. The items are tested relative to a certain property. Based on the information provided by testing a sample a decision can be made to accept, reject, scrap or rework these defective items [1-5]. Due to the similarity of these items produced, the upper and lower limits of the quality characteristic (X) are supposed to be the same. Another valuable assumption that can be made here is that if the items fail to meet the lower specification limit of X they will be rejected. However, if they fail to meet the upper specifications of X then they can be reworked. In this report, we shall not discuss the possibility of reworking items. By knowing the observations $(x_1, x_2, \ldots\ldots, x_n)$ in the lot, the mean values of each lot can be easily estimated. Assuming that the distribution of the lot means (μ) is normal then the areas at the ends of the normal curve can allow us to start with an expression of $p(\mu)$. A novel mathematical model is presented to generalize the problem of identifying the value of the fraction defective of many samples based on the mean of the lots (μ). This is an extremely important concept in reliability engineering and the applications range from chemical mechanisms [6] all the way to the design of any mechanical or electrical components that are commercially distributed.

2. Mathematical Model.

Define z as the difference of the lot mean μ and the lower specification limit L of the quality characteristic X. The general approach in this work is to derive a distribution for the fraction defective p as a function of the lot mean μ. In this work σ^2 is the variance of the quality characteristic (X), and $\sigma^2{}_\mu$ is variance of the lot mean (μ).

The fraction of items defective is given by previously derived work (see Dudewicz and Mishra [1]) as:

$$p(\mu) = \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\left(\frac{x-\mu}{\sigma}\right)^2\right] dx. \qquad (2.1)$$

Translating parameters about the origin can be performed by defining the following terms for simplification of the derivation (see Mitra and Amitava [2]):

$$z = \mu - L \qquad v = p(\mu) - \frac{1}{2}. \qquad (2.2)$$

Expression (2.2) and (2.1), yields the following:

$$p(\mu) = p(z+L) = \Phi(-\frac{z}{\sigma}) = v + \frac{1}{2} \quad . \qquad (2.3)$$

Where Φ is the cumulative probability distribution function and allowing $\Phi(-z/6) = h(z)$:

$$h^{(k)}(z) = \frac{d^{(k)}h(z)}{dz^k} \quad , \qquad (2.4)$$

$$for\ k = 0, \quad h^0(z) = h(z). \qquad (2.5)$$

For $k \geq 1$ equation (4) yields:

$$h^{(1)}(z) = -\frac{e^{-z^2/\sigma^2}}{\sqrt{2\pi\sigma}} \quad where \quad h^{(1)}(0) = -\frac{1}{\sqrt{2\pi\sigma}}. \qquad (2.6)$$

The expression for $h^{(2)}(z)$ is then:

$$h^{(2)}(z) = -\frac{1}{\sqrt{2\pi\sigma}}\left(-\frac{z}{\sigma^2} e^{-z^2/2\sigma^2}\right) \quad where \quad h^{(2)}(0) = 0 \ . \qquad (2.7)$$

Similarly the expressions of $h^{(3)}(z)$, $h^{(4)}(z)$, $h^{(5)}(z)$, $h^{(6)}(z)$ are:

$$h^{(3)}(z) = \frac{e^{-z^2/2\sigma^2}}{(2\pi)/2\sigma^3} - \frac{z^2 e^{-z^2/2\sigma^2}}{(2\pi)/2\sigma^5}, \qquad (2.8)$$

$$h^{(4)}(z)=\left(\frac{z^2}{\sigma^4}-\frac{1}{\sigma^2}\right)h^{(2)}(z)+\frac{2z}{\sigma^4}h^{(1)}(z)\ , \tag{2.9}$$

$$h^{(5)}(z)=\left(\frac{z^2}{\sigma^4}-\frac{1}{\sigma^2}\right)h^{(3)}(z)+\frac{4z}{\sigma^4}h^{(2)}(z)+2\frac{h^{(1)}(z)}{\sigma^4}\ , \tag{2.10}$$

$$h^{(6)}(z)=\left(\frac{z^2}{\sigma^4}-\frac{1}{\sigma^2}\right)h^{(4)}(z)+\frac{6}{\sigma^4}h^{(2)}(z)+6\frac{zh^{(3)}(z)}{\sigma^4}\ . \tag{2.11}$$

Expressions for $h^{(k)}(z)$ that contain any positive integer value of k can be derived. This provides for a general expression to any moment. In table I all the values of $h^{(k)}(z)$ for $z=0$ all the way to $k=12$ are shown. Thus, from equation (2.2) we can write:

$$v=f(z)=h(z)-\frac{1}{2}\ , \tag{2.12}$$

expression (2.12) yields:

$$z=f^{-1}(z)\ . \tag{2.13}$$

Employing expressions (2.12) and (2.13) it is possible to write:

$$\frac{dz}{dv}=\frac{1}{h^{(1)}(z)}\ . \tag{2.14}$$

An equation for $z^{(k)}(v)$ similar to expression (2.4) can be defined as follows:

$$z^{(k)}(v)=\frac{d^{(k)}(z)}{dv^k}, \tag{2.15}$$

where:

$$z^0(v)=z(v)\ , \tag{2.16}$$

k	$z^{k}(v)\|_{v=0}$	$h^{(k)}(v)\|_{v=0}$
0	0	½
1	$-2\sqrt{\pi}\sigma$	$-(-2\sqrt{\pi}\sigma)^{-1}$
2	0	0
3	$-(2\pi)^{3/2}\sigma$	$(2\sqrt{\pi}\sigma^{3})^{-1}$
4	0	0
5	$-7(2\pi)^{5/2}\sigma$	$-3(2\sqrt{\pi}\sigma^{5})^{-1}$
6	0	0
7	$-127(2\pi)^{7/2}\sigma$	$15(2\sqrt{\pi}\sigma^{7})^{-1}$
8	0	0
9	$-4369(2\pi)^{9/2}\sigma$	$-105(2\sqrt{\pi}\sigma^{9})^{-1}$
10	0	0
11	$-318493(2\pi)^{11/2}\sigma$	$945(2\sqrt{\pi}\sigma^{11})^{-1}$
12	0	0

TABLE 1. Values of $z^{(k)}(0)$ and $h^{(k)}(0)$ for $k = 0,1,\ldots 12$.

Note: $z^{k}(v)\|_{v=0}$ and $h^{(k)}(v)\|_{v=0}$ are zero for all even values of k.

$$z^{0}(0) = z(0) = 0 \quad , \tag{2.17}$$

if:

$$v = 0 \quad \text{then} \quad p = \frac{1}{2} \quad \text{and} \quad z = 0 \ . \tag{2.18}$$

Employing expression (15), $z^{(2)}(v)$ can be written as:

$$z^{(2)}(v) = \frac{d^{(2)}z}{dv^{2}} = -\frac{h^{(2)}z}{[h^{(1)}(z)]^{3}} . \tag{2.19}$$

To simplify these equations we will define the following:

$$E^{k} = h^{(k)}(z) \quad , \tag{2.20}$$

$$E = \frac{1}{h^{(1)}(z)} \quad . \tag{2.21}$$

By using (2.18)-(2.20), $z^{(1)}(v)$ and $z^{(2)}(v)$ can be written as:

$$z^{(1)}(v) = E \ , \tag{2.22}$$

$$z^{(2)}(v) = -E_2E^3 = E' \ , \tag{2.23}$$

where E' is the derivative of E relative to v. Thus taking the derivative of E_k relative to v yields:

$$\frac{dE_k}{dv} = \frac{dh^{(k)}(z)}{dv} = \frac{dh^{(k)}(z)}{dz} \cdot \frac{dz}{dv} = E_{k+1} \cdot E \ . \tag{2.24}$$

Then it can be derived that:

$$E(0) = \frac{1}{h^{(1)}(0)} = -\sqrt{2\pi\sigma} \ . \tag{2.25}$$

By applying equation (2.23) we can derive an expression for $z^{(3)}(v)$ as follows:

$$z^{(3)}(v) = -E_3E^4 + 3E^2{}_2 \ . \tag{2.26}$$

The following are the equations of $z^{(4)}(v)$, $z^{(5)}(v)$, $z^{(6)}(v)$ derived in the same way as $z^{(3)}(v)$:

$$z^{(4)}(v) = -E^4 \frac{dE_3}{dv} - 4E^3E_3E' + 6E_2E_3EE^5 + 3E^2{}_2 \cdot 5E^4 \cdot (-E_2E^3). \tag{2.27}$$

Thus equation (2.27) can be simplified continuously to the following form:

$$= -E^5E_4 + 10E_3E_2E^6 - 15E^3{}_2E^7 \ . \tag{2.28}$$

We can solve for $z^{(5)}(v)$ by using the previously mentioned equations:

$$\begin{aligned} z^{(5)}(v) = &-5E^4 \cdot (-E_2E^3)E_4 - E^5E_5E + 10E_4E_2E^6 \cdot E \\ &+ 10E_3 \cdot E_3E^6E + 10E_2 \cdot E_3 \cdot 6E^5 \cdot (-E_2E^3) - 15E^3{}_2 \cdot 7E^6 \cdot (-E_2E^3) \ . \\ &- 15(z)(E^2{}_2) \cdot E_3 \cdot E \cdot E^7 \end{aligned} \tag{2.29}$$

The above equation can be further simplified:

$$z^{(5)}(v) = 5E^7E_2E_4 - E^6E_5 - 60E^8E^2{}_2E_3 + 10E_4E_2E^7 + 10E^2{}_3E^7 - 45E^2{}_2 + 105E^4{}_2E^6. \tag{2.30}$$

From these equations an expression for $z^{(6)}(v)$ is given by:

$$z^{(6)}(v) = -E^7E_6 + 21E^8E_2E_5 + 35E^8E_3E_4 - 210E^9E^2{}_2E_4 - 280E^9E_2E^2{}_3 + 1260E^{10}E^3{}_2E_3 - 945E^{11}E^5{}_2. \tag{2.31}$$

Expressions of $z^{(k)}(0)$ and $h^{(k)}(0)$ are listed again in table I. By developing $z(v)$ by a Maclaurin Series of the following form:

$$z(v) = \sum_{k=0}^{\infty} \frac{z^k(0)}{k!}v^k \ , \tag{2.32}$$

After expanding the series we get the following result:

$$z(v) = \frac{z^0(0)}{0!}v^0 + \frac{z^{(1)}(0)}{1!}v^1 + \cdots + \frac{z^{(12)}(0)}{12!}v^{12} + \cdots + R(v) \ , \tag{2.33}$$

where $R(v)$ is the remainder defined by the series. Expression (2.33) can be therefore written as:

$$z(v) = -\sigma\{\sqrt{2\pi}v + \frac{(\sqrt{2\pi}v)^3}{3!} + 7\frac{(\sqrt{2\pi}v)^5}{5!} + 127\frac{(\sqrt{2\pi}v)^7}{7!} + 4369\frac{(\sqrt{2\pi}v)^9}{9!} + 318493\frac{(\sqrt{2\pi}v)^{11}}{11!} + 20493907\frac{(\sqrt{2\pi}v)^{13}}{13!} + R(v)\}. \tag{2.34}$$

Using equation (2.2), equation (2.34) can be written in terms of $p(u)$ as:

$$z(p(u)) = -\sigma\{\sqrt{2\pi}(p-1/2) + \frac{(\sqrt{2\pi}(p-1/2))^3}{3!} + 7\frac{(\sqrt{2\pi}(p-1/2))^5}{5!} + 127\frac{(\sqrt{2\pi}(p-1/2))^7}{7!} + 4369\frac{(\sqrt{2\pi}(p-1/2))^9}{9!} + 318493\frac{(\sqrt{2\pi}(p-1/2))^{11}}{11!} + 20493907\frac{(\sqrt{2\pi}(p-1/2))^{13}}{13!} + R(p)\}. \tag{2.35}$$

Finally, the distribution for $\mu(p)$, p, $h(\mu(p))$ and $\omega(p)$ are given by the following two expressions, respectively:

$$h(\mu(p)) = \frac{1}{\sqrt{2\pi}\sigma_\mu} \exp\left(- \frac{(L - \sigma I(p) - m)^2}{2\sigma^2_\mu} \right), \tag{2.36}$$

where $z(p) = -\sigma I(p)$, and $I(p)$ are terms resulting from the expansion in equation (2.35). Then if $d\mu(p)/dp = -\sigma'(p)$, it can be written that:

$$\begin{aligned} \omega(p) &= \frac{1}{\sqrt{2\pi}\sigma_\mu} \exp\left[- \frac{(L - m - \sigma I(p))}{\sqrt{2}\sigma} \right]^2 \cdot (\sigma'(p)) \\ &= \frac{I'(p)}{\sqrt{2\pi}(\sigma_\mu / \sigma)} \exp\left[- \frac{L - m}{\sigma} - I(p) \right]^2 I(2\sigma^2_\mu / \sigma^2) \qquad 0 < p < 1. \end{aligned} \tag{2.37}$$

3. Concluding Remarks.

The primary objective of this work has been described to derive a distribution of the fraction defective based on the lot mean (μ). From our analysis of this model we were also able to verify that $\omega(p)$ is in fact a probability distribution function (p.d.f.). An intensive use of $\omega(p)$ is made in order to derive the expected value of the fraction defective given multiple observations in the lot, the variance of the fraction defect (p) and many other components that are of use in the industry (Jalbout [5]). An operational control (O.C.) curve can be constructed based on this model presented and by adjusting the parameters one may also be able to construct a chart based on p to dispose of the lot. Applications of this work are numerous and among the most interesting are investigations into more chemically involved processes. [6].

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