Thue-like sequences and rainbow arithmetic progressions

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Abstract

A sequence $u = u_1u_2...u_n$ is said to be *nonrepetitive* if no two adjacent blocks of u are exactly the same. For instance, the sequence a**bcbc**ba contains a *repetition bcbc*, while *abcacbabcbac* is nonrepetitive. A well known theorem of Thue asserts that there are arbitrarily long nonrepetitive sequences over the set $\{a, b, c\}$. This fact implies, via König's Infinity Lemma, the existence of an infinite ternary sequence without repetitions of any length.

In this paper we consider a stronger property defined as follows. Let $k \geq 2$ be a fixed integer and let C denote a set of colors (or symbols). A coloring $f : \mathbb{N} \to C$ of positive integers is said to be k*-nonrepetitive* if for every $r \geq 1$ each segment of kr consecutive numbers contains a k-term *rainbow* arithmetic progression of difference r. In particular, among any k consecutive blocks of the sequence $f = f(1)f(2)f(3)...$ no two are identical. By an application of the Lovász Local Lemma we show that the minimum number of colors in a k-nonrepetitive coloring is at most $2^{-1}e^{k(2k-1)/(k-1)^2}k^2(k-1)+1$. Clearly at least $k+1$ colors are needed but whether $O(k)$ suffices remains open.

This and other types of nonrepetitiveness can be studied on other structures like graphs, lattices, Euclidean spaces, etc., as well. Unlike for the classical Thue sequences, in most of these situations non-constructive arguments seem to be unavoidable. A few of a range of open problems appearing in this area are presented at the end of the paper.

Keywords: nonrepetitive sequence, rainbow arithmetic progression, chessboard coloring

1 Introduction

In this paper we consider another variant of the nonrepetitive sequences of Thue. A finite sequence $u = u_1u_2...u_n$ of symbols from a set C is called *nonrepetitive* if it does not contain a sequence of the form $xx = x_1x_2...x_mx_1x_2...x_m$, $x_i \in C$, as a subsequence of *consecutive* terms. For instance the sequence $u = abcacbabcbac$ over the set $C = \{a, b, c\}$ is nonrepetitive, while $v = a\mathbf{bcbc}ba$ is not. A striking theorem of Thue [25] asserts that there exist arbitrarily long nonrepetitive sequences built of only three different symbols. Note that this fact implies also the existence of an infinite nonrepetitive sequence over a 3-element set.

Nonrepetitive sequences were rediscovered independently many times in connection with problems appearing in seemingly distant areas of mathematics (see [1], [12], [19], [22]). Their important applications in Combinatorics on Words, Group Theory, Universal Algebra, Number Theory and Dynamical Systems are well known (see [1], [4], [7], [17-20], [23]). Also, a lot of similar concepts were invented leading to new exciting forms of nonrepetitiveness (see [8-16]). Needless to say, a stream of investigations inspired by Thue's discovery seems to expand in ever-widening circles. Let us mention for example a recent graph theoretic variation introduced in [2]. A coloring of the set of edges of a graph G is called *nonrepetitive* if the sequence of colors on any simple path in G is nonrepetitive. The minimum number of colors needed is called the *Thue number* of G and is denoted by $\pi(G)$. For instance, Thue's theorem asserts that $\pi(P_n) = 3$, for all $n \geq 4$, where P_n is the simple path with n edges. It has been proved in [2] that there is an absolute constant c such that $\pi(G) \leq c\Delta^2$ for all graphs G with maximum degree at most Δ. The proof uses the probabilistic method and at the moment no constructive argument is known for any $\Delta \geq 3$.

The purpose of this paper is to study higher order nonrepetitiveness involving arithmetic progressions. Let $k \geq 2$ be a fixed integer and let $f : \mathbb{N} \to C$ be a coloring of the set of positive integers. A subset $A \subseteq \mathbb{N}$ is *rainbow* if no two of its elements are of the same color. A coloring f is called *k-nonrepetitive* if for every $r \geq 1$ each segment of kr consecutive numbers contains k-term rainbow arithmetic progression of difference r. In particular, this property implies that no two of any k consecutive blocks in the sequence $f = f(1)f(2)f(3)...$ are the same. Note that the existence of k-nonrepetitive colorings of $\mathbb{N}$ with finite number of colors is not *a priori* clear for any $k > 2$. We will prove it, by an application of the Lovász Local Lemma in Section 2. More precisely, we show that

$$T(k) \leq 2^{-1}e^{k(2k-1)/(k-1)^2}k^2(k-1) + 1,$$

where $T(k)$ is the minimum number of colors needed. On the other hand, it is easy to see that $T(k) \geq k + 1$, and by the result of Thue we know that $T(2) = 3$. It might be even the case that there is a constant c (not depending on k) such that $T(k) = k + c$, for all $k \geq 2$.

Another avoidance property involving arithmetic progressions has been recently introduced by Currie and Simpson [10] in connection with nonrepetitive colorings of lattice points of the n-dimensional Euclidean space. A sequence u is said to be *nonrepetitive up to mod k* if it is nonrepetitive on every arithmetic progression of difference $r = 1, 2, ..., k$. For instance, the sequence $u = \mathbf{a}b\mathbf{c}a\mathbf{d}b\mathbf{c}d\mathbf{b}a\mathbf{c}d\mathbf{a}d\mathbf{b}a\mathbf{d}b$ is nonrepetitive up to *mod* 2 since it is nonrepetitive itself and each of its two subsequences with even and odd indices, respectively, is nonrepetitive, too. It is easy to see that the minimum number $M(k)$ of symbols in an infinite such sequence is at least $k+2$. It was demonstrated in [10] that $M(k) = k+2$ for $k = 2,3$ and there is some numerical evidence that this holds for all $k \geq 1$. Note that, unlike for the numbers $T(k)$, the fact that $M(k)$ are finite for all $k \geq 1$, follows from an obvious recursive construction. However, the resulting bound on $M(k)$ is of enormously large order $O(k!)$. In Section 3 we give a probabilistic upper bound on $M(k)$ which is much better, yet far from the expected.

In Section 4 we make a short excursion to the continuous world and look for rainbow copies of subsets of Euclidean spaces. We show that there are k-colorings of $\mathbb{R}^n$ such that every set of k points has a rainbow translated copy of itself lying within arbitrarily small distance. This property extends the result of Bean, Ehrenfeucht and McNulty [4] on *square-free* colorings of the real line. Since the proof uses transfinite induction, the question of constructiveness arises naturally. One ingenious example of an explicit coloring of the line provided by Rote [24] will be presented. The last section of the paper collects a variety of open problems for future consideration.

2 Probabilistic bound on $T(k)$

In this section we prove that all numbers $T(k)$ are finite by applying the probabilistic method. It will be convenient to use the following formulation of the Lovász Local Lemma, which is equivalent to the standard asymmetric version (see [3]).

Lemma 1 *(The Local Lemma; Multiple Version) Let $A_1, A_2, ..., A_n$ be events in an arbitrary probability space and let $G = (V, E)$ be a related dependency graph, where $V = \{A_1, A_2 ..., A_n\}$ and A_i is mutually independent of all the events $\{A_j : A_iA_j \notin E\}$, for each $1 \leq i \leq n$. Let $V = V_1 \cup V_2 \cup ... \cup V_m$ be a partition such that all events $A_i \in V_r$ have the same probability p_r, $r = 1, 2, ..., m$. Suppose that there are real numbers $0 \leq x_1, x_2, ..., x_m < 1$ and $\Delta_{rs} \geq 0$, $r, s = 1, 2, ..., m$ such that the following conditions hold:*

- *$p_r \leq x_r \prod_{s=1}^{m} (1 - x_s)^{\Delta_{rs}}$ for all $r = 1, 2, ..., m$,*
- *for each $A_i \in V_r$ the size of the set $\{A_j \in V_s : A_iA_j \in E\}$ is at most Δ_{rs}, for all $r, s = 1, 2, ..., m$.*

Then $\Pr(\bigcap_{i=1}^{n} \bar{A}_i) > 0$.

We use this Lemma in the proof of the following theorem.

Theorem 1 *Let $k \geq 2$ be a fixed integer. For every $\varepsilon > 0$ there exists a coloring of $\mathbb{N}$ with at most $ck^{\frac{1}{\varepsilon}+1}(k-1)+1$ colors, where c is a constant depending on k and ε, such that for every $r \geq 1$, each segment of size at least $(k-1)r + \varepsilon r$ contains a k-term rainbow arithmetic progression of difference r.*

Proof. Let $N \geq 1$ be fixed and consider a random coloring of the set $\{1, 2, ..., N\}$ with C colors, where C will be specified later. Denote $f(r) = \lceil \varepsilon r \rceil$ and $L_r = (k-1)r + f(r)$, for $r \geq 1$. Let $R \subseteq \{1, 2, ..., N\}$ be a segment of L_r consecutive integers and let $A(R)$ denote the event that R does not contain a rainbow arithmetic progression of length k and difference r. Consider a dependency graph $G = (V, E)$ for all the events $A(R)$ and set $V_r = \{A(R) : R \text{ is a segment of length } L_r\}$. Thus,

$$p_r = \frac{\left(C^k - C(C-1)...(C-k+1)\right)^{f(r)}}{C^{kf(r)}} \leq C^{-f(r)} \binom{k}{2}^{f(r)}.$$

Since a segment of length L_r intersects at most $L_r + L_s - 1$ segments of length L_s, we may take $\Delta_{rs} = (k-1)(r+s) + f(r) + \varepsilon s$. Let $x_s = k^{-s}$. Note that $k \geq 2$ implies $(1 - x_s) \geq e^{-2x_s}$ for all $s \geq 1$. Hence the Local Lemma applies provided

$$C^{-f(r)} \binom{k}{2}^{f(r)} \leq x_r \prod_s e^{-2x_s \Delta_{rs}},$$

which transforms to

$$C^{-f(r)} \leq 2^{f(r)} k^{-r-f(r)} (k-1)^{-f(r)} \prod_s e^{-2k^{-s} \Delta_{rs}}$$

and next to

$$C \geq 2^{-1} k^{\frac{1}{\varepsilon}+1} (k-1) \exp\left(2 \sum_s k^{-s} \frac{\Delta_{rs}}{f(r)}\right).$$

Finally, since $\sum_{s=1}^{\infty} k^{-s}(1+s) = (2k-1)(k-1)^{-2}$ we have

$$\begin{aligned} \sum_{s=1}^{\infty} k^{-s} \frac{\Delta_{rs}}{f(r)} &= \sum_{s=1}^{\infty} k^{-s} \frac{(k-1)(r+s) + f(r) + \varepsilon s}{f(r)} \\ &\leq \sum_{s=1}^{\infty} k^{-s} \left(\frac{k-1}{\varepsilon} + 1\right)(1+s) = \frac{2k-1}{(k-1)^2} \left(\frac{k-1}{\varepsilon} + 1\right), \end{aligned}$$

and the assertion holds if only $C \geq 2^{-1} \exp\left(\frac{4k-2}{k-1}\left(\frac{1}{\varepsilon} + \frac{1}{k-1}\right)\right) k^{\frac{1}{\varepsilon}+1}(k-1)$. As N may be arbitrarily large this completes the proof. ∎

For $\varepsilon = 1$ we get the following upper bound on the numbers $T(k)$ defined in the Introduction.

Corollary 1 *For all $k \geq 2$*

$$T(k) \leq 2^{-1} e^{k(2k-1)/(k-1)^2} k^2 (k-1) + 1.$$

In the same way one may obtain the following n-dimensional version of Theorem 1.

Theorem 2 *Let $S \subset \mathbb{Z}^n$ be a fixed finite set and let $\mathcal{A} = \{u + rS : u \in \mathbb{Z}^n, r \in \mathbb{N}\}$ be the collection of all integral affine copies of S. Then for any $\varepsilon > 0$ there is a finite coloring of $\mathbb{Z}^n$ such that each set $X \in \mathcal{A}$ has a rainbow translated copy $v + X$ satisfying $\|v\| \leq \varepsilon r$.*

3 Nonrepetitive chessboard colorings and the bound on $M(k)$

An interesting new type of nonrepetitive colorings appeared in a recent paper of Currie and Simpson [10]. Consider a coloring of an $N \times N$ chessboard such that the sequence of colors traced by any *singular* Queen's move is nonrepetitive. It is not hard to see that for $N > 4$ at least five colors are needed, but to prove that this number suffices for all N is not an easy task. Currie and Simpson achieve it by constructing arbitrarily long sequences over five symbols which are nonrepetitive up to *mod* 3 (see Introduction). In fact, if $a = a_1 a_2 ... a_{3N-2}$ is nonrepetitive on all arithmetic progressions of differences $r = 1, 2, 3$ then the coloring

a_1	a_2	a_3	...
a_3	a_4	a_5	...
a_5	a_6	a_7	...
...	...	...	...

satisfies the required condition. Similarly, sequences nonrepetitive up to *mod* $(2^n - 1)$ give solutions to the analogous n-dimensional chessboard coloring problem, in which the Queen moves along the lines of slopes determined by vertices of the unit n-cube (see [10]). Clearly, at least 2^n colors are needed and it looks like $2^n + 1$ should suffice. In fact, some numerical experiments suggest that $M(k) = k + 2$, where, as in the Introduction, $M(k)$ denotes the minimum number of symbols necessary to construct arbitrarily long sequences nonrepetitive up to *mod* k.

The following linear upper bound on $M(k)$ is obtained, similarly as before, by an application of the Local Lemma.

Theorem 3 *For every* $k \geq 2$

$$M(k) \leq k e^{8k^2/(k-1)^2} + 1.$$

Proof. We will proceed as in the proof of Theorem 1 starting with preparations for the Local Lemma. Consider a random C-coloring of the set $\{1, 2, ..., N\}$ with N arbitrarily large. Let R denote any arithmetic progression of length $2r$ and positive difference $t \leq k$ contained in $\{1, 2, ..., N\}$. Let $A(R)$ denote the bad event that the first half of R is colored the same as the second. Put $V_r = \{A(R) : R \text{ is a progression of } 2r \text{ terms}\}$. Thus $p_r = C^{-r}$. Since an arithmetic progression of length $2r$ intersects at most $4rs$ progressions of length $2s$ with fixed difference t we may take $\Delta_{rs} = 4krs$. Let $x_s = k^{-s}$ and note that $(1 - x_s) \geq e^{-2x_s}$ for all $s \geq 1$. Hence the Local Lemma applies provided

$$C^{-r} \leq x_r \prod_s e^{-2x_s \Delta_{rs}},$$

that is

$$C^{-r} \leq k^{-r} \prod_s e^{-2k^{-s} 4krs}.$$

Taking both sides to the power of $-1/r$ gives

$$C \geq k \exp\left(8k \sum_s k^{-s} s\right).$$

Since $\sum_{s=1}^{\infty} k^{-s} s = \frac{k}{(k-1)^2}$ the proof is completed. ■

Chessboard colorings may be generalized to arbitrary lattices, and even to arbitrary discrete sets of points in $\mathbb{R}^n$. Let P be a discrete set of points and let L be a fixed set of lines in $\mathbb{R}^n$. A coloring of P is *nonrepetitive* (with respect to L) if no sequence of consecutive points on any $l \in L$ is colored repetitively. For a point $p \in P$ let $i(p)$ denote the number of lines from L incident with p and let $I = I(P, L) = \max\{i(p) : p \in P\}$ be the *maximum incidence* of the configuration (P, L). Once again we apply the Local Lemma to get the following result for configurations with bounded $I(P, L)$.

Theorem 4 *Let (P, L) be a configuration of points and lines in $\mathbb{R}^n$ with finite maximum incidence $I \geq 2$. If $C \geq Ie^{(8I^2+8I-4)/(I-1)^2}$ then there is a nonrepetitive C-coloring of P with respect to L.*

Proof. We will proceed as before. Consider a random C-coloring of the set P. For a segment $R \subset l$ of $2r$ consecutive points of some line $l \in L$, let $A(R)$ denote the event that R is colored repetitively. Put $V_r = \{A(R) : R \text{ is a segment of } 2r \text{ points}\}$. Thus $p_r = C^{-r}$. In the worst case each point of R is incident with I lines, hence we may take $\Delta_{rs} = 2r + 2s + 4rsI$. Set $x_s = I^{-s}$ and note that $(1 - x_s) \geq e^{-2x_s}$ for all $s \geq 1$. Hence the Local Lemma applies provided

$$C^{-r} \leq x_r \prod_s e^{-2x_s \Delta_{rs}},$$

that is

$$C^{-r} \leq I^{-r} \prod_s e^{-2I^{-s}(2r+2s+4rsI)}.$$

Further transformations give

$$C \geq I \prod_s e^{4I^{-s}(1+s/r+2sI)} = I \exp\left(\sum_s 4I^{-s}(1 + \frac{s}{r} + 2sI)\right).$$

Since the maximum sum of the last series is attained for $r = 1$, we conclude that there is a desired C-coloring of P, provided

$$C \geq I \exp\left(4 \sum_{s=1}^{\infty} I^{-s}(1 + s + 2sI)\right) = Ie^{(8I^2+8I-4)/(I-1)^2}.$$

The proof is completed. ■

4 Rainbow sets in colored Euclidean spaces

In a seminal paper on avoidable patterns [4] Bean, Ehrenfeucht and McNulty introduced *square-free colorings* of real and rational numbers as a continuous version of nonrepetitive sequences of Thue. The following theorem extends their result by allowing longer arithmetic progressions as well as arbitrarily small $\varepsilon > 0$. Our proof goes along the same lines and uses transfinite induction.

Theorem 5 *For every $k \geq 2$ there exists a coloring of $\mathbb{R}$ with k colors such that for every $\varepsilon > 0$ and every $\rho > 0$, each closed interval of length $(k-1)\rho + \varepsilon$ contains a k-term rainbow arithmetic progression of difference ρ.*

Proof. Let S be the set of all pairs $\{A, \varepsilon\}$ where $\varepsilon > 0$ and $A = \{a + \rho i : i = 0, 1, ..., k-1\}$ is a k-term arithmetic progression of real numbers of positive difference $\rho > 0$. Clearly, S is of cardinality 2^ω. Put the elements of S into a well order and define a coloring of $\mathbb{R}$ inductively as follows. At each step $\alpha < 2^\omega$ take a pair $\{A, \varepsilon\}$ labeled by α and choose a translated copy $B = t + A$ of A, $0 \leq t < \varepsilon$, no point of which has been colored so far. Such a copy must exist since the interval $[0, \varepsilon)$ contains 2^ω elements while the cardinality of the set of points colored before the step α is at most $k\alpha < 2^\omega$. So, one may color each point of B differently which, by transfinite induction, completes the proof.

■

An interesting question appearing here is that of finding explicit coloring function. For $k = 2$ it has been done by Rote [24] as follows. Consider a coloring function $f : \mathbb{R} \to \{a, b\}$ defined by $f(x) = a$ if $\ln |x| \in \mathbb{Q}$, and $f(x) = b$ otherwise. To show that it satisfies the desired property we invoke a famous result from algebraic number theory, known as the Lindemann-Weierstrass theorem. It asserts that the equation

$$a_1 e^{b_1} + ... + a_n e^{b_n} = 0$$

can not hold, if $a_1, ..., a_n$ are non-zero algebraic numbers and $b_1, ..., b_n$ are pairwise distinct algebraic numbers. Now, let $\varepsilon > 0$ be arbitrarily small and assume that $0 < x < y$. If the colors of x and y agree then choose t_1 so that $x + t_1 = e^{q_1}$, $0 \leq t_1 < \varepsilon$ and $q_1 \in \mathbb{Q}$. Thus $f(x + t_1) = a$ and if at the same time $f(y + t_1) = a$, i.e., $y + t_1 = e^{q_2}$ for another rational number $q_2 \neq q_1$, pick t_2 so that $t_1 < t_2 < \varepsilon$ and $x + t_2 = e^{q_3}$, for some $q_3 \in \mathbb{Q}$. If it would happened again that $y + t_2 = e^{q_4}$ is a rational power of e, then we will get $e^{q_1} - e^{q_2} = e^{q_3} - e^{q_4}$, with all q_i different rational numbers, which contradicts the Lindemann-Weierstrass theorem. So, the interval $[x, y + \varepsilon)$ must contain a rainbow translated copy of the pair $\{x, y\}$.

Similarly as Theorem 5 one may prove the following, rather counter intuitive, property.

Theorem 6 *For any integer $n \geq 1$ and any cardinal $k \leq \aleph_0$ there is a k-coloring of the space $\mathbb{R}^n$ such that given any set $S \subset \mathbb{R}^n$ of cardinality k and any $\varepsilon > 0$ there is a vector t, with $\|t\| < \varepsilon$, such that $t + S$ is rainbow.*

Finding an explicit coloring with the above property seems hopeless.

5 Final discussion

5.1 Thue-like problems and the probabilistic method

The probabilistic method has already been applied in two earlier results establishing strong avoidance properties for infinite binary sequences. One of them is the theorem of Beck [5] asserting, for any $\varepsilon > 0$, the existence of an infinite binary sequence in which two identical blocks of length $n > n_0(\varepsilon)$ are separated by at least $(2-\varepsilon)^n$ terms. The other is an exercise in the book of Alon and Spencer [3], and says that for any $\varepsilon > 0$ there is an infinite binary sequence in which every two adjacent blocks of length $m > m_0(\varepsilon)$ differ in at least $(\frac{1}{2} - \varepsilon)m$ places. Both results relay on the Lovász Local Lemma and both imply the existence of infinite nonrepetitive sequences over some finite set of symbols, although the resulting number of symbols is much bigger than three. Theorem 1 however does not seem to follow directly from any of the above results.

5.2 The numbers $T(k)$ and $M(k)$

In view of some numerical experiments it looks like the cubic upper bound provided in Theorem 1 is far from the true order of $T(k)$. In fact, it seems plausible that $T(k) = O(k)$. More risky would be to expect that the following conjecture is true.

Conjecture 1 *$T(k) \leq k + c$ for some absolute constant c.*

Obviously, $T(k) \geq k+1$, and by the result of Thue we know that $T(2) = 3$. Moreover, it was demonstrated in [6] that there is an infinite sequence over four symbols such that no two out of any three consecutive blocks are the same. Thus, one could even suspect (as we did in [13]) that $T(k) = k + 1$ for all $k \geq 2$. However, an exhaustive computer search has shown that $T(3) > 4$, destroying this supposition.

Concerning the numbers $M(k)$ the situation looks different. Up to $k = 100$ the computer was able to generate very long sequences nonrepetitive up to *mod* k on $k + 2$ symbols. Also, the Theorem 3 gives a relatively better estimate which supports, at least heuristically, the following conjecture.

Conjecture 2 *$M(k) = k + 2$ for all $k \geq 1$.*

5.3 Nonrepetitive map coloring

Let the fields of an $N \times N$ chessboard be colored so that no non-self-intersecting *walk* formed by a sequence of moves of the Queen is repetitive. The result of [2] guarantees the existence of a finite number Q of colors that suffices for such a coloring, no matter how large is N. In fact, chessboard fields can be considered as vertices of a graph with adjacency relation defined in the obvious way. Since the maximum degree of this graph is at most 8 we have $Q \leq \pi_v(8)$, where $\pi_v(n)$ denotes the maximum *vertex* Thue number among all graphs G with $\Delta(G) \leq n$. It would be nice to know the exact value of Q as well as its n-dimensional analogues, say $Q(n)$. This time we have no intuition concerning the possible growth of the sequence $Q(n)$.

Similar questions may be posed for more general "boards" with fields of other geometrical shapes which may even have different sizes. The most intriguing general problem could be stated, in the spirit of graph theoretic tradition, as follows. Let M be a planar map and let $P = F_1F_2...F_n$ be any sequence of distinct faces of M each pair of consecutive ones sharing an edge. A coloring of M is *nonrepetitive* if no such sequence P in M is colored repetitively. Now, is there a natural number N such that any planar map has a nonrepetitive N-coloring?

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