New Lower Bound Formulas for Multicolored Ramsey Numbers

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Abstract

We give two lower bound formulas for multicolored Ramsey numbers. These formulas improve the bounds for several small multicolored Ramsey numbers.

1. INTRODUCTION

In this short article we give two new lower bound formulas for edgewise r-colored Ramsey numbers, $R(k_1, k_2, \ldots, k_r)$, $r \geq 3$, defined below. Both formulas are derived via construction.

We will make use of the following notation. Let G be a graph, $V(G)$ the set of vertices of G, and $E(G)$ the set of edges of G. An r-coloring, χ, will be assumed to be an edgewise coloring, i.e. $\chi(G) : E(G) \to \{1, 2, \ldots, r\}$. If $u, v \in V(G)$, we take $\chi(u, v)$ to be the color of the edge connecting u and v in G. We denote by K_n the complete graph on n vertices.

Definition 1.1 *Let $r \geq 2$. Let $k_i \geq 2$, $1 \leq i \leq r$. The number $R = R(k_1, k_2, \ldots, k_r)$ is defined to be the minimal integer such that any edgewise r-coloring of K_R must contain, for some j, $1 \leq j \leq r$, a monochromatic K_{k_j} of color j. If we are considering the diagonal Ramsey numbers, i.e. $k_1 = k_2 = \cdots = k_r = k$, we will use $R_r(k)$ to denote the corresponding Ramsey number.*

The numbers $R(k_1, k_2, \ldots, k_r)$ are well-defined as a result of Ramsey's theorem [Ram]. Using Definition 1.1 we make the following definition.

Definition 1.2 *A Ramsey r-coloring for $R = R(k_1, k_2, \ldots, k_r)$ is an r-coloring of the complete graph on $V < R$ vertices which does not admit any monochromatic K_{k_j} subgraph of color j for $j = 1, 2, \ldots, r$. For $V = R - 1$ we call the coloring a maximal Ramsey r-coloring.*

2. THE LOWER BOUNDS

We start with an easy bound which nonetheless improves upon some current best lower bounds.

Theorem 2.1 *Let $r \geq 3$. For any $k_i \geq 3$, $i = 1, 2, \ldots, r$, we have*

$$R(k_1, k_2, \ldots, k_r) > (k_1 - 1)(R(k_2, k_3, \ldots, k_r) - 1).$$

Proof. Let $\phi(G)$ be a maximal Ramsey $(r-1)$-coloring for $R(k_2, k_3, \ldots, k_r)$ with colors $2, 3, \ldots, r$. Let $k_1 \geq 3$. Define graphs G_i, $i = 1, 2, \ldots, k_1 - 1$, with $|V(G_i)| = |V(G)|$ on distinct vertices (from each other), each with the coloring ϕ. Let H be the complete graph on the vertices $V(H) = \cup_{i=1}^{k_1-1} V(G_i)$. Let $v_i \in G_i$, $v_j \in G_j$ and define $\chi(H)$ as follows:

$$\chi(v_i, v_j) = \begin{cases} \phi(v_i, v_j) & \text{if } i = j \\ 1 & \text{if } i \neq j. \end{cases}$$

We now show that $\chi(H)$ is a Ramsey r-coloring for $R(k_1, k_2, \ldots, k_r)$. For $j \in \{2, 3, \ldots, r\}$, $\chi(H)$ does not admit any monochromatic K_{k_j} of color j by the definition of ϕ. Hence, we need only consider color 1. Since $\phi(G_i)$, $1 \leq i \leq k_1 - 1$, is void of color 1, any monochromatic K_{k_1} of color 1 may only have one vertex in G_i for $1 \leq i \leq k_1 - 1$. By the pigeonhole principle, however, there exists $x \in \{1, 2, \ldots, k_1 - 1\}$ such that G_x contains two vertices of K_{k_1}, a contradiction. □

Examples. Theorem 2.1 implies that $R_5(4) \geq 1372, R_5(5) \geq 7329, R_4(6) \geq 5346$, and $R_4(7) \geq 19261$, all of which beat the current best known bounds given in [Rad].

We now look at an off-diagonal bound. This uses and generalizes methods found in [Chu] and [Rob].

Theorem 2.2 *Let $r \geq 3$. For any $3 \leq k_1 < k_2$, and $k_j \geq 3$, $j = 3, 4, \ldots, r$, we have*

$$R(k_1, k_2, \ldots, k_r) > (k_1 + 1)(R(k_2 - k_1 + 1, k_3, \ldots, k_r) - 1).$$

Before giving the proof of this theorem, we have need of the following definition.

Definition 2.3 *We say that the $n \times n$ symmetric matrix*

$$T = T(x_0, x_1, \ldots, x_r) = (a_{ij})_{1 \le i,j \le n}$$

is a Ramsey incidence matrix for $R(k_1, k_2, \ldots, k_r)$ if T is obtained by using a Ramsey r-coloring for $R(k_1, k_2, \ldots, k_r)$, $\chi : E(K_n) \to \{x_1, x_2, \ldots, x_r\}$, as follows. Define $a_{ij} = \chi(i,j)$ if $i \neq j$ and $a_{ii} = x_0$.

From Definition 2.3 we see that an $n \times n$ Ramsey incidence matrix $T(x_0, x_1, \ldots, x_r)$ for $R(k_1, k_2, \ldots, k_r)$ gives rise to an r-colored K_n which does not contain K_{k_i} of color x_i for $i = 1, 2, \ldots, r$.

Proof of Theorem 2.2. We will be using Ramsey incidence matrices to construct an r-colored Ramsey graph on $(k_1+1)(R(k_2-k_1+1, k_3, \ldots, k_r)-1)$ vertices which does not admit monochromatic subgraphs K_{k_i} of color i, $i = 1, 2, \ldots, r$. We start the proof with $R(t,k,l)$ and then generalize to an arbitrary number of colors.

Let $l > t$ and consider a maximal Ramsey 2-coloring for $R = R(k, l-t+1)$. Let $T = T(x_0, x_1, x_2)$ denote the associated Ramsey incidence matrix. Define $A = A^\star = T(0,2,3)$, $B = B^\star = T(3,2,1)$, and $C = T(1,2,3)$, and consider the symmetric $(t+1)(R-1) \times (t+1)(R-1)$ matrix, M, below (so that there are $t+1$ instances of T in each row and in each column). We note that in the definitions of A and $A^\star$ we have the color 0 present. This is valid since, as M is defined in equation (1), the color 0 only occurs on the main diagonal of M and the main diagonal entries correspond to nonexistent edges in the complete graph.

$$M = \begin{array}{ccccccc} A & B^\star & C & C & C & \cdots & C \\ B^\star & A^\star & C & C & C & \cdots & C \\ C & C & A & B & B & \cdots & B \\ C & C & B & A & B & \cdots & B \\ C & C & B & B & A & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & B \\ C & C & B & B & \ldots & B & A \end{array} \tag{1}$$

We will show that M defines a 3-coloring which contains no monochromatic K_t of color 1, no monochromatic K_k of color 2, and, for $l > t$, no monochromatic K_l of color 3, to show that $R(t,k,l) > (t+1)(R(k, l-t+1)-1)$.

Note 1: We will use the phrase *diagonal of* X, where $X = A, A^\star, B, B^\star$, or C, to mean the diagonal of X when X is viewed as a matrix by itself.

Note 2: For ease of reading, we will use (i,j) to represent the matrix entry a_{ij}.

No monochromatic $\mathbf{K_t}$ of color 1. Let $V(K_t) = \{i_1, i_2, \ldots, i_t\}$ with $i_1 < i_2 < \cdots < i_t$, so that we can view $E(K_t)$ as corresponding to the entries in M given by $\cup_{j>k}\ (i_j, i_k)$.

We now argue that not all of these entries can be equal to 1. Assume, for a contradiction, that all entries are equal to 1.

First, we cannot have two distinct entries in the collection of C's. Assume otherwise and let (i_{j_1}, i_{k_1}) and (i_{j_2}, i_{k_2}) both be in the collection of C's with either $i_{j_1} \neq i_{j_2}$ or $i_{k_1} \neq i_{k_2}$.

Case I. $(i_{j_1} \neq i_{j_2})$ Let $i_{j_1} < i_{j_2}$. Note that the entry 1 occurs only on the diagonal of C. We have two subcases to consider.

Subcase i. $(i_{k_1} = i_{k_2})$ In this subcase, (i_{j_2}, i_{j_1}) is on the diagonal of B, a contradiction.

Subcase ii. $(i_{k_1} \neq i_{k_2})$ In this subcase, one of $(i_{j_1}, i_{k_2}), (i_{j_2}, i_{k_1})$ is not on the diagonal of C, but is in C, a contradiction.

Case II. $(i_{j_1} = i_{j_2}$ and $i_{k_1} \neq i_{k_2})$ Letting $i_{k_1} < i_{k_2}$ forces (i_{k_2}, i_{k_1}) to be on the diagonal of $B^\star$, a contradiction.

The above cases show that we can have at most one entry in the collection of C's.

Next, since A does not contain 1, we must have at least $\binom{t}{2} - 1$ entries in the collection of B's (including $B^\star$). If there exists an entry in $B^\star$ then, since we can have at most one entry in the collection of C's, we must have all of the entries $\cup_{k<j<t} (i_j, i_k)$ in $B^\star$. Since $t \geq 3$, we must have $1 = (i_{t-1}, i_{t-2}) \in A^\star$, a contradiction. Hence, there cannot exist an entry in $B^\star$.

Thus, we must have $\binom{t}{2} - 1$ entries in the collection of B's, but not in $B^\star$. Now, if we assume that (i_{j_1}, i_{k_1}) and (i_{j_2}, i_{k_2}), $i_{j_1} < i_{j_2}$, are both in the same B, then we must have $(i_{j_2}, i_{j_1}) \in A$, a contradiction. Furthermore, we cannot have $i_{j_1} = i_{j_2}$ since this implies that $(i_{k_2}, i_{k_1}) \in A$. Hence, each B contains at most one entry for a total of at most $\binom{t-1}{2}$ entries. Since $\binom{t-1}{2} < \binom{t}{2} - 1$ for $t \geq 3$, we cannot have all entries equal to 1, and hence we cannot have a monochromatic K_t of color 1.

No monochromatic $\mathbf{K_k}$ of color 2. For this case we will use the following lemma.

Lemma 2.3 *Let $S(x_0, x_1, \ldots, x_r)$ be a Ramsey incidence matrix for $R(k_1, k_2, \ldots, k_r)$. Let N be a block matrix defined by instances of S (for example, equation (1)). For $y \geq 3$, let $V(K_y) = \{i_1, i_2, \ldots, i_y\}$ with $i_1 < i_2 < \cdots < i_y$ so that we can associate with $E(K_y)$ the entries of N given by $\cup_{j>k} (i_j, i_k)$. Fix x_f for some $1 \leq f \leq r$. If $x_f = (i_j, i_k)$ for all $1 \leq k < j \leq y$, and x_f as an argument of S is in the same (argument) position, but not the first (argument) position, for all instances of S then $y < k_f$.*

Proof. Let $m = R(k_1, \ldots, k_r) - 1$. By assumption of identical argument positions of x_f in all instances of S, for any entry $(i, j) = x_f$ we must have $(i \pmod{m}, j \pmod{m}) = x_f$. Provided all $(i_j \pmod{m}, i_k \pmod{m})$, $1 \leq k < j \leq y$, are distinct, this would imply that a monochromatic K_y of color f exists in a maximal Ramsey r-coloring for $R(k_1, \ldots, k_r)$, thus giving $y < k_f$.

It remains to show that all $(i_j \pmod{m}, i_k \pmod{m})$, $1 \leq k < j \leq y$, are distinct. Assume not and consider (i_{j_1}, i_{k_1}) and (i_{j_2}, i_{k_2}) with either $i_{j_1} \neq i_{j_2}$ or $i_{k_1} \neq i_{k_2}$.

Case I. ($i_{j_1} \neq i_{j_2}$) Let $i_{j_1} < i_{j_2}$. Since $i_{j_1} \equiv i_{j_2} \pmod{m}$ this implies that (i_{j_2}, i_{j_1}) must be on the diagonal of some instance of S, a contradiction, since the first argument denotes the diagonal, and all entries are not on the diagonal of any instance of S.

Case II. ($i_{k_1} \neq i_{k_2}$) Let $i_{k_1} < i_{k_2}$. As in Case I, this implies that (i_{k_2}, i_{k_1}) must be on the diagonal of some instance of S, a contradiction. □

Applying Lemma 2.3 with $N = M$, $S = T$, and $f = 2$ we see that we cannot have a monochromatic K_k of color 2.

No monochromatic $\mathbf{K_l}$ of color 3. Let $V(K_l) = \{i_1, i_2, \ldots, i_l\}$ with $i_1 < i_2 < \cdots < i_l$, so that we can view $E(K_l)$ as corresponding to the entries in M given by $\cup_{j>k} \ (i_j, i_k)$. We now argue that not all of these entries can be equal to 3. Suppose, for a contradiction, that all of these entries are equal to 3.

If there are no entries in the collection of B's (including $B^\star$), then by Lemma 2.3 (with $N = M$, $S = T$, and $f = 3$) we must have $l < l - t + 1$, a contradiction. Hence, there exists an entry in some B or $B^\star$.

Next, note that 3 only occurs on the diagonals of B and $B^\star$. Thus, we cannot have (i_{j_1}, i_{k_1}) and (i_{j_2}, i_{k_2}), $i_{j_1} < i_{j_2}$, both be in the same B or the same $B^\star$, for otherwise (i_{j_2}, i_{k_1}) is not on the diagonal of B or $B^\star$, a contradiction. Hence, each B and $B^\star$ contains at most one entry.

Consider the complete subgraph K_{l-t+1} of K_l on the vertices $\{i_2, i_3, \ldots, i_{l-t+2}\}$, so that we can view $E(K_{l-t+1})$ as corresponding to the entries in M given by $\cup_{l-t+2 \geq j > k \geq 2} \ (i_j, i_k)$. By construction, none of these entries are in the collection of B's and $B^\star$'s. To see this, note that we may have $(i_k, i_1) \in B^\star$ for at most one $2 \leq k \leq t$ and we may have $(i_k, i_j) \in B$ for each $l - (t-2) + 1 \leq k \leq l$ for at most one $1 \leq j < k$ (i.e. one entry in each of the bottom $t-2$ rows of M). Hence, none of the edges of K_{l-t+1} on $\{i_2, \ldots, i_{l-t+2}\}$ are associated with an entry in B or $B^\star$.

Applying Lemma 2.3 (with $N = M$, $S = T$, and $f = 3$) we get $l - t + 1 < l - t + 1$, a contradiction. Thus, no monochromatic K_l of color 3 exists.

The full theorem. To generalize the above argument to an arbitrary number of colors we change the definitions of A, $A^\star$, B, $B^\star$, and C; $A = A^\star = T(0, 2, 3, 4, 5, \ldots, r)$, $B = B^\star = T(3, 2, 1, 4, 5, \ldots, r)$, $C = T(1, 2, 3, 4, 5, \ldots, r)$. To see that there is no monochromatic K_{k_j} of color j for $j = 4, 5, \ldots, r$, see the argument for no monochromatic K_k of color 2 above. □

Example. Theorem 2.2 implies that $R(3,3,3,11) \geq 437$, beating the previous best lower bound of 433 as given in [Rad].

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