Identifying codes with small radius in some infinite regular graphs

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Submitted: October 10, 2000; Accepted: March 13, 2002.
MR Subject Classifications: 05C70 (68R10, 94B65)

Abstract

Let $G = (V, E)$ be a connected undirected graph and S a subset of vertices. If for all vertices $v \in V$, the sets $B_r(v) \cap S$ are all nonempty and different, where $B_r(v)$ denotes the set of all points within distance r from v, then we call S an r-identifying code. We give constructive upper bounds on the best possible density of r-identifying codes in four infinite regular graphs, for small values of r.

1 Introduction

Given a connected undirected graph $G = (V, E)$, finite or infinite, we define $B_r(v)$, the *ball* of radius r centred at a vertex $v \in V$, by

$$B_r(v) = \{x \in V : d(x, v) \leq r\},$$

where $d(x, v)$ denotes the number of edges in any shortest path between v and x. Whenever $d(x, v) \leq r$, we say that x and v *r-cover* each other (or simply *cover* if there is no ambiguity). A set of vertices covers a vertex if at least one of its elements does.

We call any nonempty subset S of V a *code* and its elements *codewords*. A code S is called *r-identifying*, or *identifying*, if the sets $B_r(v) \cap S$, $v \in V$, are all nonempty and different. The set $B_r(v) \cap S$ is called the *r-identifying set*, or *identifying set*, of v and will be denoted by $IS_r(v)$ or $IS(v)$. Two vertices which have different identifying sets are said to be *r-separated*, or *separated*.

Remark. For given graph $G = (V, E)$ and integer r, there exists an r-identifying code $S \subseteq V$ if and only if

$$\forall v_1, v_2 \in V\ (v_1 \neq v_2),\ B_r(v_1) \neq B_r(v_2).$$

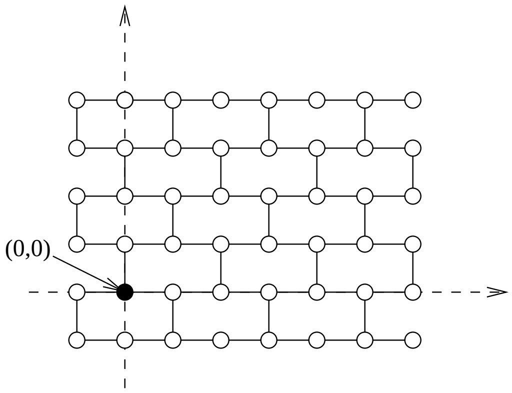

Figure 1: The hexagonal grid (part).

Indeed, if for all $v_1, v_2 \in V$, $B_r(v_1)$ and $B_r(v_2)$ are different, then $S = V$ is r-identifying. Conversely, if for some $v_1, v_2 \in V$, $B_r(v_1) = B_r(v_2)$, then for any code $S \subseteq V$, we have $IS(v_1) = IS(v_2)$. For instance, there is no r-identifying code in a complete graph.

The concept of identifying code was introduced in [15]. It was further studied, for different types of graphs, e.g., in [1]–[4], [6]–[14].

In this paper we will study the following four 2-dimensional infinite grids:
- G_H, the hexagonal grid, with vertex set $V = \mathbb{Z} \times \mathbb{Z}$ and edge set $E_H = \{\{u = (i,j), v\} : u - v \in \{(0, (-1)^{i+j+1}), (\pm 1, 0)\}\}$.
- G_S, the square lattice, with same vertex set and edge set $E_S = \{\{u,v\} : u - v \in \{(0, \pm 1), (\pm 1, 0)\}\}$.
- G_T, the triangular lattice, or square lattice with one diagonal, with same vertex set and edge set $E_T = \{\{u,v\} : u - v \in \{(0, \pm 1), (\pm 1, 0), (1,1), (-1,-1)\}\}$.
- G_K, the square lattice with two diagonals, with same vertex set and edge set $E_K = \{\{u,v\} : u - v \in \{(0, \pm 1), (\pm 1, 0), (1, \pm 1), (-1, \pm 1)\}\}$; we call this graph the king lattice, since on an infinite empty chessboard, the ball of radius r is the set of squares that a king can reach in at most r moves, starting from the centre.

See Figure 1, where we represent the hexagonal grid as a "brick wall", and Figures 2, 3 and 4.

Denote by Q_n the set of vertices $(x,y) \in V = \mathbb{Z} \times \mathbb{Z}$ with $|x| \le n$ and $|y| \le n$. Then we define the *density* of a code S as

$$D(S) = \limsup_{n \to \infty} \frac{|S \cap Q_n|}{|Q_n|}.$$

For a given graph $G = (V, E)$ and a given integer r, we search for r-identifying codes with minimum density, denoted by $D(G, r)$.

The paper is organized as follows: in Section 2, we describe some properties of translations in $\mathbb{Z}^2$, properties which will be necessary to our study of periodic codes in the following section. In Section 4 we describe the heuristics used in our search for good

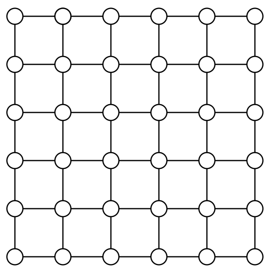

Figure 2: The square lattice (part).

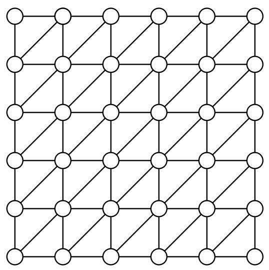

Figure 3: The triangular lattice (part).

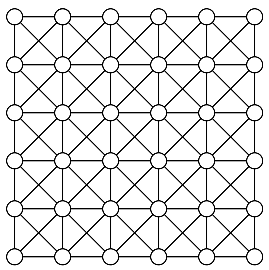

Figure 4: The king lattice (part).

r-identifying codes, in the four grids, for small r. Before we give our results in Section 6 and explicit constructions of identifying codes in Section 7, we survey in Section 5 the best bounds known to us, including from the recent or forthcoming papers [3], [4].

2 Tilings and rectangles

In this section, we show how to associate, to a tiling induced by two translations in $\mathbb{Z}^2$, a rectangle which, in the following section, will be used for generating periodic identifying codes.

We consider two translations of $\mathbb{Z}^2$ of vectors t_1 and t_2, linearly independent. We denote by T the set of translations defined by:

$$T = \{k_1 \times t_1 + k_2 \times t_2 : k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z}\}.$$

We define an equivalence relation on $\mathbb{Z}^2$ by: $P_1 \in \mathbb{Z}^2$ is equivalent to $P_2 \in \mathbb{Z}^2$ if and only if one is obtained from the other by a translation of T. We are interested in the equivalence classes of this relation. We imagine that there is one colour for each equivalence class, and that we colour every point of a given class with the colour of its class: thus any point in $\mathbb{Z}^2$ is coloured; if it is drawn on a plane, one can see a coloured regular infinite tiling, called here *the tiling induced by* t_1 *and* t_2.

We say that a subset R of $\mathbb{Z}^2$ is a *rectangle* of *width* $w(R)$ ($w(R) \in \mathbb{N}$) and *height* $h(R)$ ($h(R) \in \mathbb{N}$) if it is defined by:

$$R = \{(i, j) : i \in \mathbb{N}, j \in \mathbb{N}, 0 \leq i < w(R), 0 \leq j < h(R)\}.$$

Consider the following three integers w, h and α:

- w is the minimum integer i, $i \geq 1$, such that the points $(0, 0)$ and $(i, 0)$ are in the same class.

- h is the minimum integer j, $j \geq 1$, such that there exists $i \in \mathbb{Z}$ for which the points $(0, 0)$ and (i, j) are in the same class.

- α is the minimum nonnegative integer such that the points $(0, 0)$ and (α, h) are in the same class.

In other words, if $(0, 0)$ is coloured in green, w gives the position of the first occurrence of green on the right part of the X-axis, h the number of the first line, above the X-axis, where green appears, and α the position on this line of the first occurrence to the right of the Y-axis.

We further denote by $t_{(w,0)}$ the translation of vector $(w, 0)$ and $t_{(\alpha,h)}$ the translation of vector (α, h), and by R the rectangle of width w and height h.

Proposition 1 *We have:*
1) a point (i, j) *and a point* $(i + w, j)$ *are in the same class.*
2) a point (i, j) *and a point* $(i + \alpha, j + h)$ *are in the same class.*
3) all points in R *belong to distinct classes.*
4) for any point $P \in \mathbb{Z}^2$*, there exist a point* P_R *in* R*,* $k \in \mathbb{Z}$ *and* $\ell \in \mathbb{Z}$ *such that* P *is*

obtained from P_R by the translation $k \times t_{(w,0)} + \ell \times t_{(\alpha,h)}$. Moreover, P and P_R are in the same class.

Proof. For 1), observe that the definition of w implies that the translation $t_{(w,0)}$ is in the set T of translations. This result implies that $0 \leq \alpha < w$.

For 2), observe that the definitions of h and α imply that the translation $t_{(\alpha,h)}$ is in the set T of translations.

For 3), suppose that two points (i,j) and (i',j') of R are in the same class. We can assume, without loss of generality, that $j' \geq j$. The point (i',j') is obtained from the point (i,j) by the translation of vector $(i'-i, j'-j)$ which is in the set T of translations. So, the point $(0,0)$ is in the same class as $(i'-i, j'-j)$ and, from 1), as $(i'-i+w, j'-j)$. Note that: $-w < i'-i < w$ and $0 \leq j'-j < h$.

If $i'-i$ is nonnegative, set $i'' = i'-i$, else set $i'' = i'-i+w$. We have $0 \leq i'' < w$ and therefore, the class of $(i'', j'-j) \in R$ is the same as the class of $(0,0)$. If $j'-j = 0$, we have a contradiction with the definition of w. If $j'-j \geq 1$, we have a contradiction with the definition of h.

For 4), if P is a point in $\mathbb{Z}^2$, it is easy to see first that there exists $\ell \in \mathbb{Z}$ such that the ordinate j of the point P' obtained from P by the translation $-\ell \times t_{(\alpha,h)}$ satisfies $0 \leq j < h$; second, that there exists $k \in \mathbb{Z}$ such that the abscissa i of the point P_R obtained from P' by the translation $-k \times t_{(w,0)}$ satisfies $0 \leq i < w$; the ordinate of P_R is equal to the ordinate of P'; P_R is in R and P is obtained from P_R by the translation $k \times t_{(w,0)} + \ell \times t_{(\alpha,h)}$. Using also 1) and 2), we obtain the result. $\triangle$

The above proposition implies that all classes are represented once, and only once, in R and that two points P and P' of $\mathbb{Z}^2$ are in the same class if the corresponding points P_R and P'_R are one and the same; for that, it is necessary and sufficient that P be obtained from P' by a translation equal to $k \times t_{(w,0)} + \ell \times t_{(\alpha,h)}$, $k \in \mathbb{Z}$, $\ell \in \mathbb{Z}$. Consequently, the tiling induced by $t_{(w,0)}$ and $t_{(\alpha,h)}$ is the same as the one induced by t_1 and t_2. We have therefore proved the following theorem.

Theorem 1 *Consider in $\mathbb{Z}^2$ two translations of vectors t_1 and t_2, linearly independent, and the tiling induced by t_1 and t_2. There is a rectangle R, of width w and height h, such that all classes of the tiling are represented once, and only once, in R. Furthermore, there exists also an integer α, $0 \leq \alpha < w$, such that the translations $t_{(w,0)}$ and $t_{(\alpha,h)}$, defined respectively by the vectors $(w,0)$ and (α,h), induce the same tiling.*

In particular, this theorem shows that the number of classes is finite.

3 Periodic codes and tilings

We say that a subset S of $\mathbb{Z}^2$ is *periodic* if there are two translations, linearly independent, leaving S globally invariant. Let us consider a periodic subset S of $\mathbb{Z}^2$ and two corresponding translations. These translations induce a tiling, and S is the union of some of the classes because, if an element of S is in a class, then all this class is included in S. Consider a rectangle R defined as in the above theorem; if we know R, the set $S_R = R \cap S$ and the value of α defined as in the previous section, then S is known.

Conversely, let us choose a rectangle R, of width w and height h, a subset S_R of R and an integer α with $0 \leq \alpha < w$; the set S obtained as the union of the classes of the elements of S_R in the tiling induced by the translations of vectors $(w, 0)$ and (α, h) is a periodic subset of $\mathbb{Z}^2$. This set S is said to be induced by R, α and S_R.

Let us recall that we look for r-identifying codes. We limit here our search to *periodic* r-identifying codes. So, as seen above, if we consider every positive integer w, every positive integer h, every integer α with $0 \leq \alpha < w$ and any subset of the rectangle of width w and height h, we consider every periodic subset of $\mathbb{Z}^2$; if, for each, we check whether it is an r-identifying code, we meet every periodic r-identifying code.

In the cases of king, triangular and square lattices, any translation leaves the lattice globally invariant. But, in the case of the hexagonal grid, a translation of vector $t = (i, j)$ leaves the grid globally invariant if and only if $i + j$ is even; we call such a translation an *even translation*. Now, it is easier to test whether a set corresponding to a tiling is an r-identifying code when the translations of this tiling leave globally invariant the grid. Moreover if a set S is periodic, it is possible to find two even translations leaving S invariant. So, we chose, in the case of the hexagonal grid, to consider only the tilings that are induced by even translations and, therefore, to consider only w, h and α with w and $\alpha + h$ even.

4 Description of the heuristic

The heuristic uses the previous study and tries to answer the following question: for given integer r, rectangle R (of width w and height h), integer α with $0 \leq \alpha < w$, and integer $c \leq |R|$, is there a subset S_R of R, of cardinality c, such that R, α and S_R induce a (periodic) r-identifying code S ?

We call *solution* any subset S_R of R with cardinality c. The goal is to find a solution for which the corresponding set S is an r-identifying code. We define an objective function, f, by associating a value to a solution. For this, we consider the sets R_1, R_2 and R_3 defined by:

$$R_1 = \{P \in \mathbb{Z}^2 : \text{ there exists } P' \in R \text{ with } d(P, P') \leq r\};$$
$$R_2 = \{P \in \mathbb{Z}^2 : \text{ there exists } P' \in R \text{ with } d(P, P') \leq 2 \times r\};$$
$$R_3 = \{P \in \mathbb{Z}^2 : \text{ there exists } P' \in R \text{ with } d(P, P') \leq 3 \times r\},$$

where d is the distance corresponding to the considered grid.

We compute the sets $S_{R_1} = S \cap R_1$, $S_{R_2} = S \cap R_2$ and $S_{R_3} = S \cap R_3$, and $f(S_R) =$ the number of points in R not r-covered by S + the number of pairs $\{P, P'\}$, $P \in S_{R_2}$,

$P' \in S_{R_2}$, such that at least one of the points P and P' is in R, and P and P' are not r-separated.

One can remark that:
- a vertex in R is r-covered if and only if it is covered by a codeword in S_{R_1};
- a pair $\{P, P'\}$, $P, P' \in R_2$, such that at least one of the points P and P' is in R, is r-separated if and only if it is r-separated by a codeword in S_{R_3};
- the set S is an r-identifying code if and only if $f(S_R) = 0$.

We applied three methods, a systematic method, a descent method and a kind of noising method [5]. To describe these methods, we consider that we have a set of c tokens and that these tokens are put on the vertices of R to define the subset S_R.

The systematic method consists in trying all possibilities for the c places of the tokens. This method spends too much time except if c is very small.

In the descent method, we start with a random choice of c distinct places for the tokens. Then, we consider the first token and try to move it, without moving the others, by computing the place in R for which the objective function is minimum; when the first token is re-placed, we perform the same work with another token, and successively with all tokens; when it is finished, we try again with the first token and so on. We stop the process when it is not possible to improve the objective function by moving one token, or when the objective function is equal to zero.

The noising-like method, called further noising method, successively considers the tokens as in the descent method; for each token, we have two possibilities which occur at random: the token is moved either to its best place, or to a random place; the probability that the token is moved at random decreases from an initial value (typically 0.2 or 0.3) down to zero. The process ends when the objective function is equal to zero or when a fixed amount of moves (typically 300 times the number of vertices in R) has been performed.

We observed that the noising method is much more efficient than repeated descents when the instance is not too small.

Each of the three methods has been incorporated in loops to begin to try successively every rectangle R, every parameter α and every value of c such that the density $\dfrac{c}{|R|}$ is at least the minimum bound we know of for the considered problem, and at most the maximum bound. According to the size of the instance, we decided on a maximum value for $|R|$ and for CPU time (some hours or some days).

5 Lower and upper bounds

We gather various known lower and upper bounds on the cardinality of an r-identifying code S, in the case of the four grids G_H, G_S, G_T and G_K.

5.1 Lower bounds

From [15], we have, for an r-identifying code S in a regular graph $G = (V, E)$:

$$|S| \geq \frac{2|V|}{B_r + 1},$$

where B_r denotes the size of a ball of radius r (independent of its centre). For our infinite grids, this yields

$$D(G, r) \geq \frac{2}{B_r + 1}, \tag{1}$$

where we have simply to replace B_r by the right expression, depending on which grid we consider: the volume of a ball of radius r equals

$\frac{3}{2}r^2 + \frac{3}{2}r + 1$ in the hexagonal grid;
$2r^2 + 2r + 1$ in the square lattice;
$3r^2 + 3r + 1$ in the triangular lattice;
$(2r + 1)^2$ in the king lattice.

Another general result was obtained in [9], improving on (1) when r grows: for the square, triangular and king lattices,

$$D(G, r) \geq \frac{1}{4r + 2},$$

and for the hexagonal grid,

$$D(G_H, r) \geq \frac{1}{4r + 4}.$$

The square lattice case was improved in [14]:

$$D(G_S, r) \geq \frac{2}{7r + 4}.$$

Then improvements are given in a recent paper [3]:

$$D(G_H, r) \geq \frac{2}{5r + 3} \quad \text{for } r \text{ even}, \tag{2}$$

$$D(G_H, r) \geq \frac{2}{5r + 2} \quad \text{for } r \text{ odd}, \tag{3}$$

$$D(G_S, r) \geq \frac{3}{8r + 4}, \tag{4}$$

$$D(G_T, r) \geq \frac{2}{6r + 3}. \tag{5}$$

For $r = 1$, *ad hoc* methods are usually more efficient, and the inequality

$$D(G_S, 1) \geq 15/43 \tag{6}$$

is stated in [7] and proved in [9]. Also, from [8]:

$$D(G_H, 1) \geq 16/39. \tag{7}$$

Finally, for the king lattice,

$$D(G_K, r) \geq \frac{1}{4r} \quad \text{for } r > 1 \tag{8}$$

is proved in the forthcoming paper [4], and in [10] we have the inequality

$$D(G_K, 1) \geq 2/9. \tag{9}$$

5.2 Upper bounds

Upper bounds are proved by construction. From [15], we have

$$D(G_T, 1) \leq 0.25, \tag{10}$$

a value which meets the lower bound (1), since $B_1 = 7$ in the triangular lattice. In [11], [6], [10] and [14] we have, respectively:

$$D(G_H, 1) \leq 3/7, \tag{11}$$

$$D(G_S, 1) \leq 0.35, \tag{12}$$

$$D(G_K, 1) \leq 4/17, \tag{13}$$

$$D(G_S, 2) \leq 5/29. \tag{14}$$

General constructions, working for all values of r, can be found in [14]:

$$D(G_S, r) \leq \frac{2}{5r} \quad \text{for } r \text{ even}, \tag{15}$$

$$D(G_S, r) \leq \frac{2r}{5r^2 - 2r + 1} \quad \text{for } r \text{ odd}, \tag{16}$$

and in [3]:

$$D(G_H, r) \leq \frac{8r - 8}{9r^2 - 16r} \quad \text{for } r \equiv 0 \mod 4, \tag{17}$$

$$D(G_H, r) \leq \frac{8}{9r - 25} \quad \text{for } r \equiv 1 \mod 4, \tag{18}$$

$$D(G_H, r) \leq \frac{8}{9r - 34} \quad \text{for } r \equiv 2 \mod 4, \tag{19}$$

$$D(G_H, r) \leq \frac{8r - 16}{(r-3)(9r - 43)} \quad \text{for } r \equiv 3 \mod 4; \tag{20}$$

$$D(G_T, r) \leq \frac{1}{2r + 4} \quad \text{for } r \equiv 0 \mod 4,$$

$$D(G_T, r) \leq \frac{1}{2r+2} \quad \text{for } r \equiv 1, 2 \text{ or } 3 \mod 4.$$

For small values of r, these general constructions can often be beaten.

Comparing lower and upper bounds when r goes to infinity, we see that

$$2/5r \lesssim D(G_H, r) \lesssim 8/9r, 3/8r \lesssim D(G_S, r) \lesssim 2/5r, 1/3r \lesssim D(G_T, r) \leq 1/2r.$$

Finally, for the king lattice, it is a remarkable fact that the minimum density is known for all values of r: for $r = 1$, inequality (9) is now met with equality, since we found a 1-identifying code with density 2/9 (see Section 7.1, Figure 5). For $r \geq 1$, it is proved in the forthcoming [4] that $D(G_K, r) \leq 1/4r$, which, together with (8), proves that $D(G_K, r) = 1/4r$ for $r > 1$.

6 Results

We list below some of the new upper bounds obtained by our heuristics. Lower bounds are given for comparision, including those coming from [3], [4]. An empty box in the "new upper bounds" column means that we only found again an earlier bound.

king lattice			
r	lower bounds	new upper bounds	previous upper bounds
1	$2/9 \approx 0.2222$ (9)	$2/9 \approx 0.2222$	$4/17 \approx 0.2353$ (13)
2	$1/8 = 0.125$ (8)	$1/8 = 0.125$	1
3	$1/12 \approx 0.0833$ (8)	$1/12 \approx 0.0833$	1
4	$1/16 = 0.0625$ (8)	$1/16 = 0.0625$	1

triangular lattice			
r	lower bounds	new upper bounds	previous upper bounds
1	$1/4 = 0.25$ (1)		$1/4 = 0.25$ (10)
2	$2/15 \approx 0.1333$ (5)	$1/6 \approx 0.1667$	1
3	$2/21 \approx 0.0952$ (5)	$2/17 \approx 0.1177$	1
4	$2/27 \approx 0.0741$ (5)	$1/12 \approx 0.0833$	1
5	$2/33 \approx 0.0606$ (5)	$1/13 \approx 0.0769$	1
6	$2/39 \approx 0.0513$ (5)	$1/14 \approx 0.0714$	1

square lattice			
r	lower bounds	new upper bounds	previous upper bounds
1	$15/43 \approx 0.3488$ (6)		$7/20 = 0.35$ (12)
2	$3/20 = 0.15$ (4)		$5/29 \approx 0.1724$ (14)
3	$3/28 \approx 0.1071$ (4)	$1/8 = 0.125$	$3/20 = 0.15$ (16)
4	$1/12 \approx 0.0833$ (4)	$8/85 \approx 0.0941$	$1/10 = 0.1$ (15)
5	$3/44 \approx 0.0682$ (4)	$2/25 = 0.08$	$5/58 \approx 0.0862$ (16)
6	$3/52 \approx 0.0577$ (4)	$3/46 \approx 0.0652$	$1/15 \approx 0.0667$ (15)
7	$1/20 = 0.05$ (4)		$7/116 \approx 0.0603$ (16)

hexagonal grid			
r	lower bounds	new upper bounds	previous upper bounds
1	$16/39 \approx 0.4102$ (7)		$3/7 \approx 0.4286$ (11)
2	$2/11 \approx 0.1818$ (1)	$4/19 \approx 0.2105$	1
3	$2/17 \approx 0.1176$ (3)	$1/6 \approx 0.1667$	1
4	$2/23 \approx 0.0870$ (2)	$1/9 \approx 0.1111$	1
5	$2/27 \approx 0.0741$ (3)	$4/35 \approx 0.1143$	1
6	$2/33 \approx 0.0606$ (2)	$1/11 \approx 0.0909$	1
7	$2/37 \approx 0.0541$ (3)	$1/12 \approx 0.0833$	1
8	$2/43 \approx 0.0465$ (2)	$1/13 \approx 0.0769$	1

7 Some code constructions

Constructions with the densities listed in the "new upper bounds" columns of the Tables of Section 6 are given here in two ways: we specify the parameters w, h and α, together with the set $S_R = C \cap R$, and we also draw the corresponding (partial) graphs, where codewords are in black. In some figures, dotted lines will indicate tiles with a more practical shape than the one provided by w and h.

7.1 The king lattice

$r = 1$:
density $= 4/18 \approx 0.2222$,
$w = 6, h = 3, \alpha = 3$,
$S_R = \{(0,0), (4,0), (2,1), (5,2)\}$.
See Figure 5.

$r = 2$:
density $= 1/8 = 0.125$,
$w = 4, h = 2, \alpha = 2$,
$S_R = \{(0,0)\}$.
See Figure 6.

$r = 3$:
density $= 1/12 \approx 0.0833$,
$w = 6, h = 2, \alpha = 2$,
$S_R = \{(0,0)\}$.
See Figure 7.

$r = 4$:
density $= 1/16 = 0.0625$,
$w = 8, h = 2, \alpha = 2$,
$S_R = \{(0,0)\}$.
See Figure 8.

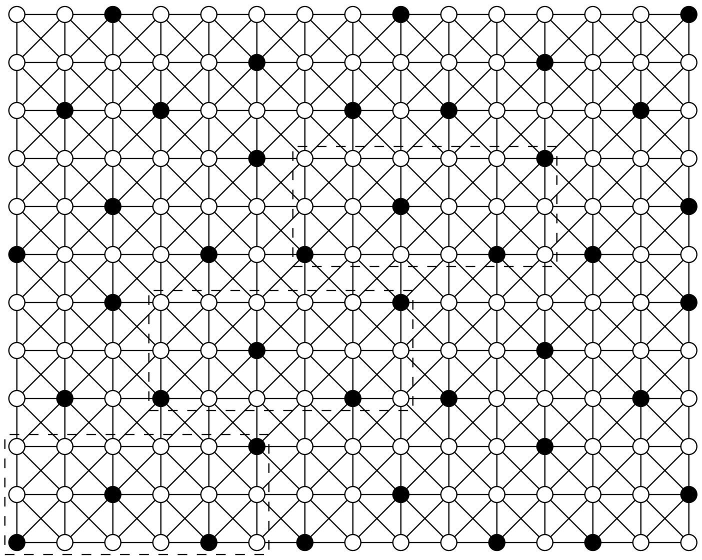

Figure 5: A 1-identifying code for the king lattice.

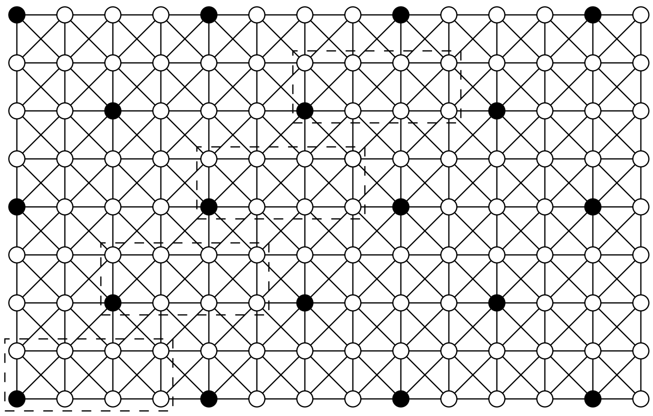

Figure 6: A 2-identifying code for the king lattice.

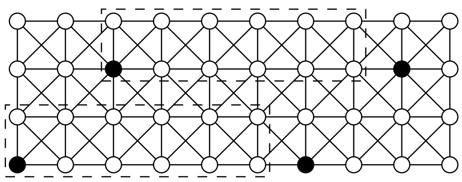

Figure 7: A 3-identifying code for the king lattice.

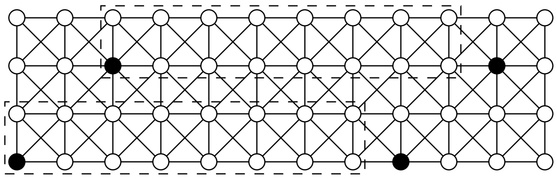

Figure 8: A 4-identifying code for the king lattice.

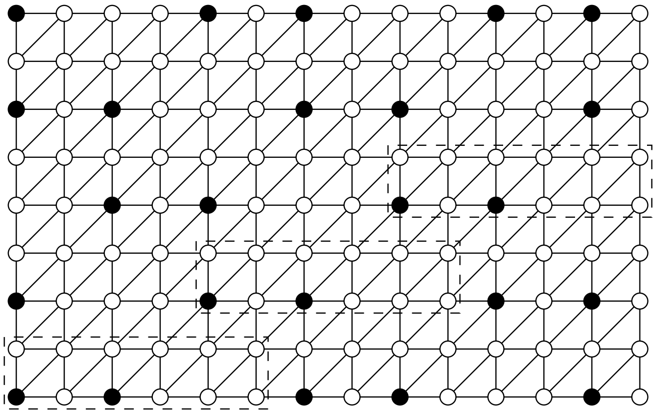

Figure 9: A 2-identifying code for the triangular lattice.

7.2 The triangular lattice

$r = 2$:
density $= 2/12 \approx 0.1667$,
$w = 6, h = 2, \alpha = 4$,
$S_R = \{(0,0),(2,0)\}$.
See Figure 9.

$r = 3$:
density $= 10/85 \approx 0.1177$,
$w = 85, h = 1, \alpha = 9$,
$S_R = \{(0,0),(4,0),(19,0),(23,0),(36,0),(40,0),(53,0),(55,0),(68,0),(72,0)\}$.
See Figure 10.

$r = 4$:
density $= 1/12 \approx 0.0833$,
$w = 6, h = 2, \alpha = 4$,
$S_R = \{(0,0)\}$.
See Figure 11.

$r = 5$:
density $= 3/39 \approx 0.0769$,
$w = 39, h = 1, \alpha = 17$,
$S_R = \{(0,0),(8,0),(19,0)\}$.
See Figure 12.

$r = 6$:
density $= 2/28 \approx 0.0714$,
$w = 14, h = 2, \alpha = 4$,
$S_R = \{(0,0),(6,0)\}$.
See Figure 13.

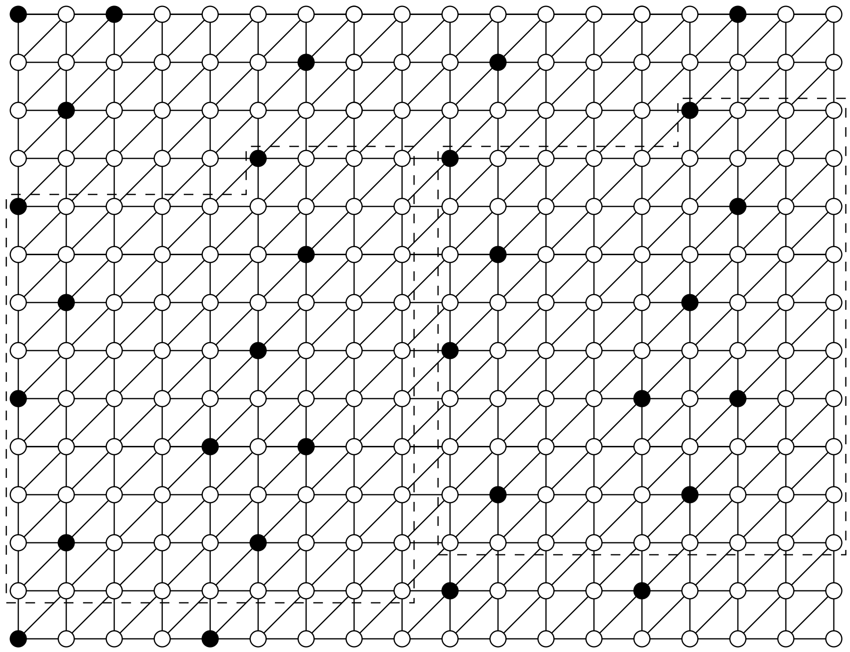

Figure 10: A 3-identifying code for the triangular lattice.

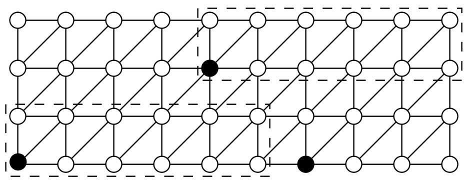

Figure 11: A 4-identifying code for the triangular lattice.

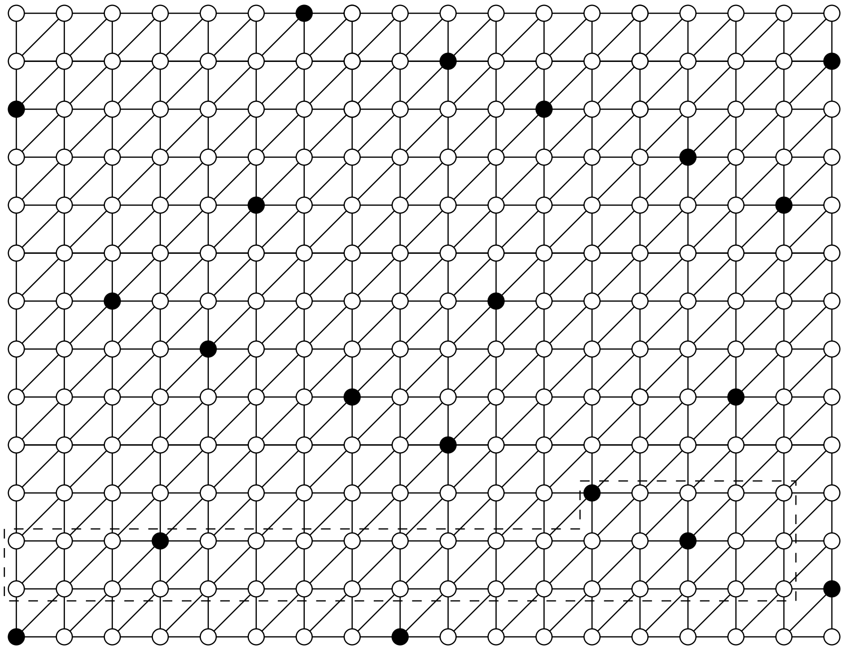

Figure 12: A 5-identifying code for the triangular lattice.

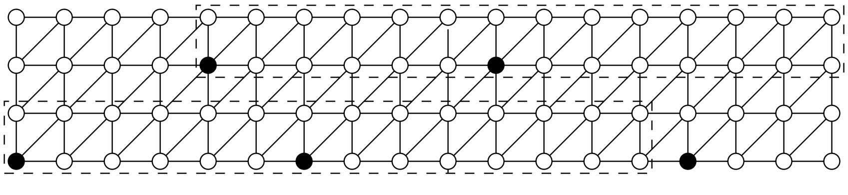

Figure 13: A 6-identifying code for the triangular lattice.

7.3 The square lattice

$r = 3$:
density $= 7/56 = 0.125$,
$w = 56, h = 1, \alpha = 10$,
$S_R = \{(0,0), (7,0), (14,0), (21,0), (28,0), (35,0), (42,0)\}$.
See Figure 14.

$r = 4$:
density $= 8/85 \approx 0.0941$,
$w = 85, h = 1, \alpha = 38$,
$S_R = \{(0,0), (2,0), (12,0), (21,0), (33,0), (54,0), (66,0), (75,0)\}$.
See Figure 15.

$r = 5$:
density $= 2/25 = 0.08$,
$w = 25, h = 1, \alpha = 7$,
$S_R = \{(0,0), (2,0)\}$.
See Figure 16.

$r = 6$:
density $= 3/46 \approx 0.0652$,
$w = 46, h = 1, \alpha = 8$,
$S_R = \{(0,0), (17,0), (20,0)\}$.
See Figure 17.

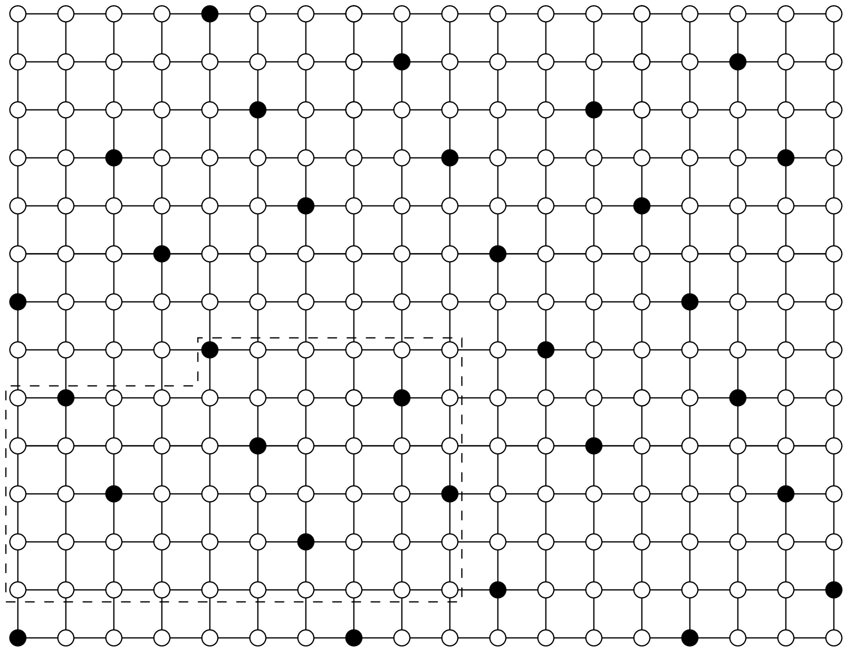

Figure 14: A 3-identifying code for the square lattice.

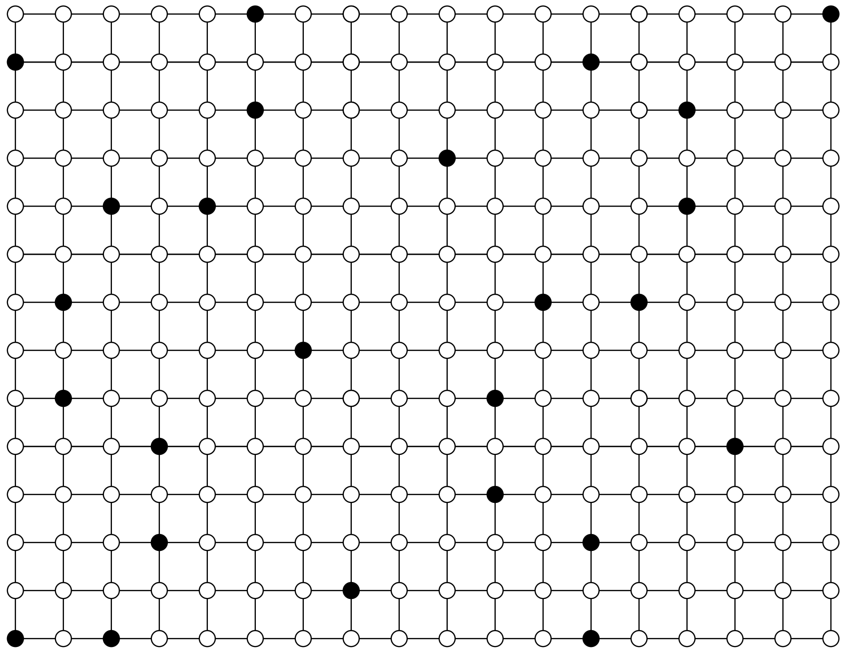

Figure 15: A 4-identifying code for the square lattice.

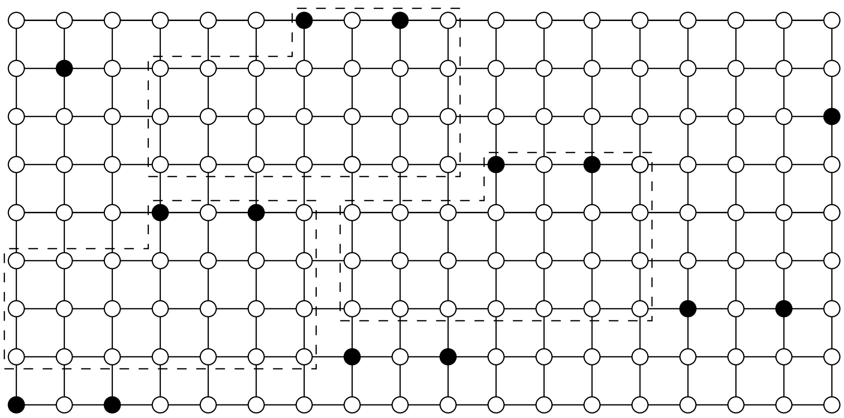

Figure 16: A 5-identifying code for the square lattice.

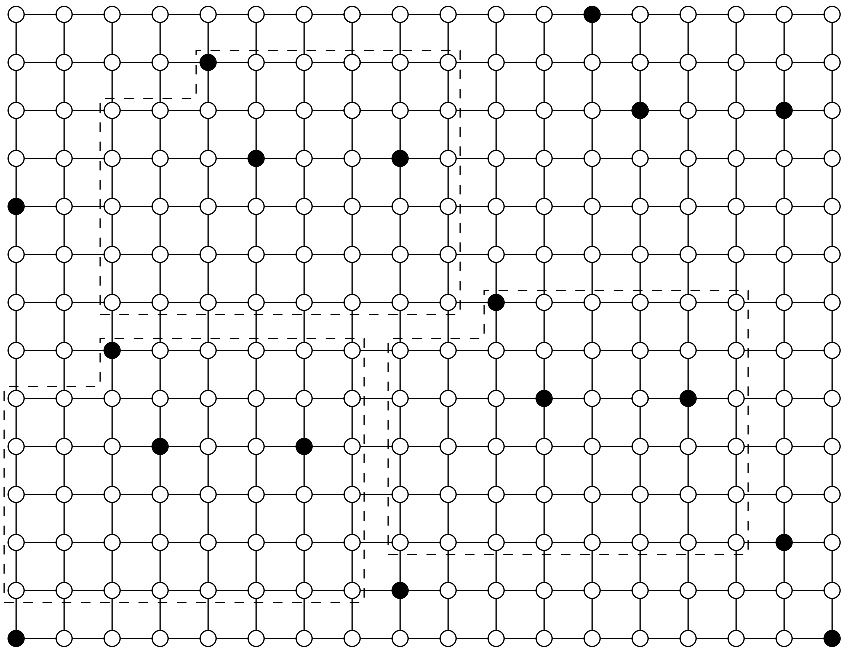

Figure 17: A 6-identifying code for the square lattice.

7.4 The hexagonal grid

$r = 2$:
density $= 8/38 \approx 0.2105$,
$w = 38, h = 1, \alpha = 15$,
$S_R = \{(0,0), (6,0), (10,0), (13,0), (18,0), (21,0), (25,0), (31,0)\}$.
See Figure 18.

$r = 3$:
density $= 1/6 \approx 0.1667$,
$w = 6, h = 1, \alpha = 3$,
$S_R = \{(0,0)\}$.
See Figure 19.

$r = 4$:
density $= 2/18 \approx 0.1111$,
$w = 6, h = 3, \alpha = 3$,
$S_R = \{(0,0), (0,2)\}$.
See Figure 20.

$r = 5$:
density $= 8/70 \approx 0.1143$,
$w = 70, h = 1, \alpha = 19$,
$S_R = \{(1,0), (16,0), (19,0), (23,0), (26,0), (33,0), (58,0), (61,0)\}$.
See Figure 21.

$r = 6$:
density $= 8/88 \approx 0.0909$,
$w = 88, h = 1, \alpha = 21$,
$S_R = \{(1,0), (11,0), (18,0), (25,0), (28,0), (35,0), (42,0), (52,0)\}$.
See Figure 22.

$r = 7$:
density $= 4/48 \approx 0.0833$,
$w = 48, h = 1, \alpha = 11$,
$S_R = \{(1,0), (6,0), (19,0), (24,0)\}$.
See Figure 23.

$r = 8$:
density $= 2/26 \approx 0.0769$,
$w = 26, h = 1, \alpha = 7$,
$S_R = \{(0,0), (15,0)\}$.
See Figure 24.

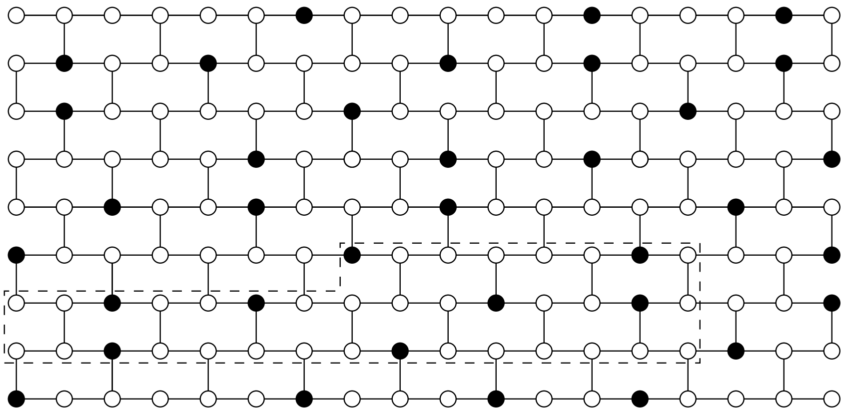

Figure 18: A 2-identifying code for the hexagonal grid.

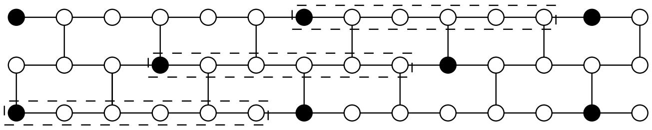

Figure 19: A 3-identifying code for the hexagonal grid.

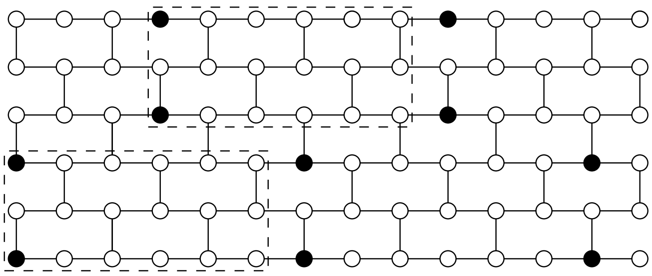

Figure 20: A 4-identifying code for the hexagonal grid.

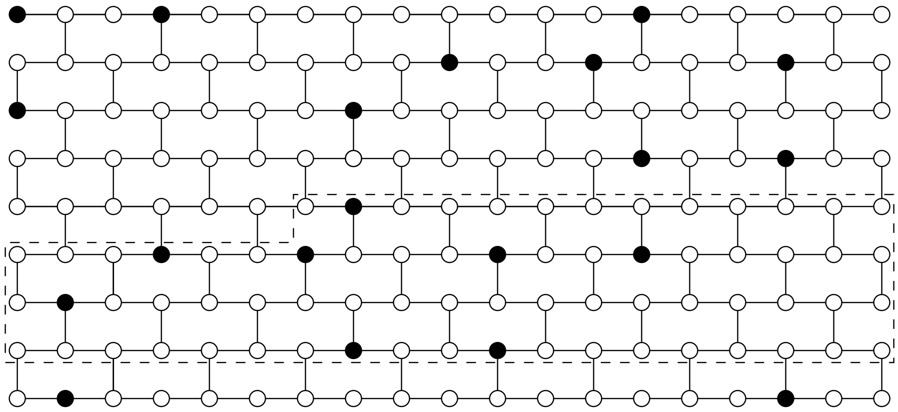

Figure 21: A 5-identifying code for the hexagonal grid.

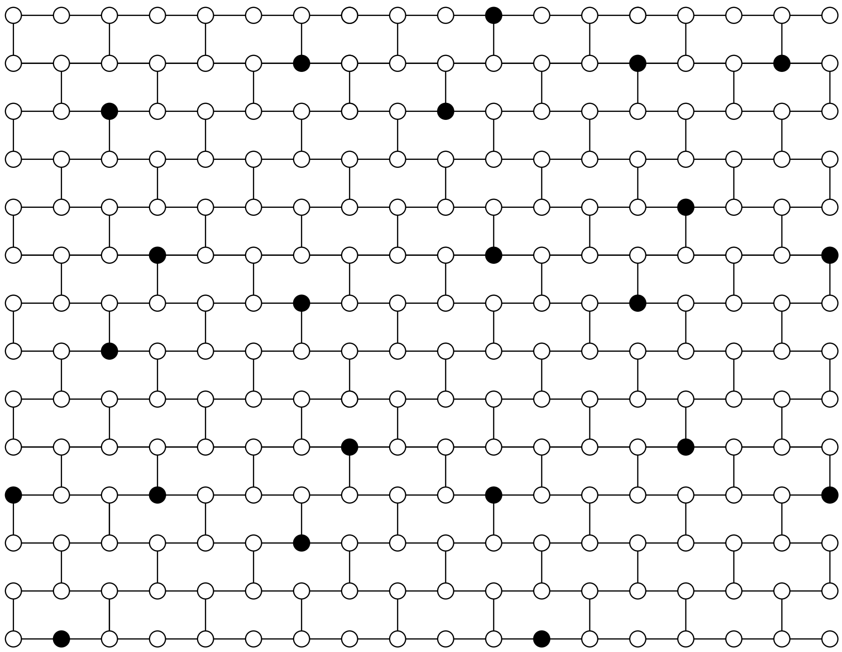

Figure 22: A 6-identifying code for the hexagonal grid.

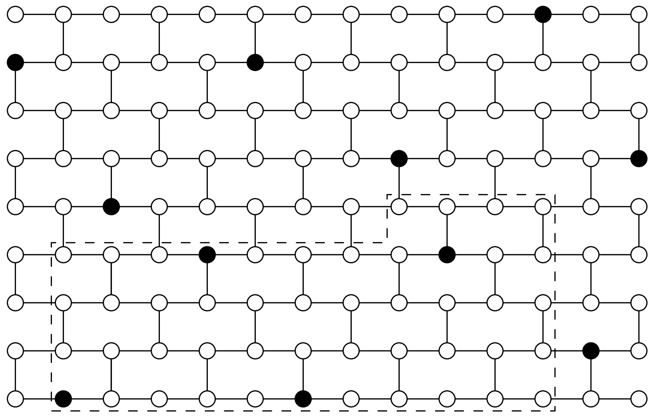

Figure 23: A 7-identifying code for the hexagonal grid.

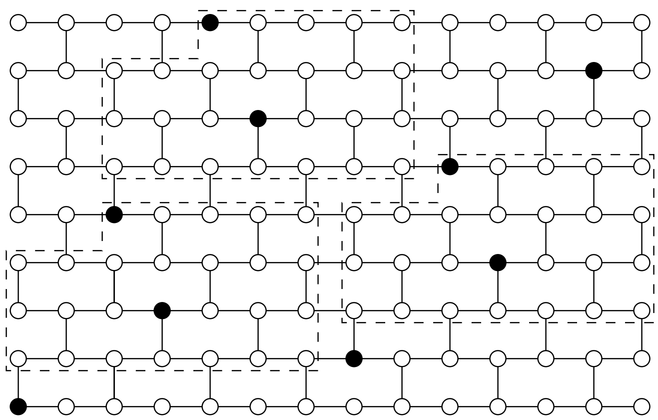

Figure 24: An 8-identifying code for the hexagonal grid.

Still for the hexagonal grid, we add some upper bounds slightly better than those of the general constructions (17)–(20). For each, we give a construction, which has been checked by our software.
$r = 9$: density $= 1/14$, $w = 14, h = 1, \alpha = 5, S_R = \{(0,0)\}$.
$r = 10$: density $= 1/14$, $w = 14, h = 1, \alpha = 5, S_R = \{(0,0)\}$.
$r = 11$: density $= 1/16$, $w = 16, h = 1, \alpha = 5, S_R = \{(0,0)\}$.
$r = 12$: density $= 1/16$, $w = 16, h = 1, \alpha = 5, S_R = \{(0,0)\}$.
$r = 13$: density $= 1/18$, $w = 18, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 14$: density $= 1/18$, $w = 18, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 15$: density $= 1/18$, $w = 18, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 16$: density $= 1/18$, $w = 18, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 17$: density $= 1/22$, $w = 22, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 18$: density $= 1/22$, $w = 22, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 19$: density $= 1/22$, $w = 22, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 20$: density $= 1/22$, $w = 22, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 21$: density $= 1/26$, $w = 26, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 22$: density $= 1/26$, $w = 26, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 23$: density $= 1/28$, $w = 28, h = 1, \alpha = 5, S_R = \{(0,0)\}$.
$r = 24$: density $= 1/28$, $w = 28, h = 1, \alpha = 5, S_R = \{(0,0)\}$.
$r = 25$: density $= 1/30$, $w = 30, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 26$: density $= 1/30$, $w = 30, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 27$: density $= 1/32$, $w = 32, h = 1, \alpha = 5, S_R = \{(0,0)\}$.
$r = 28$: density $= 1/32$, $w = 32, h = 1, \alpha = 5, S_R = \{(0,0)\}$.
$r = 29$: density $= 1/34$, $w = 34, h = 1, \alpha = 3, S_R = \{(0,0)\}$.
$r = 30$: density $= 1/34$, $w = 34, h = 1, \alpha = 3, S_R = \{(0,0)\}$.

These results suggest general constructions with densities $1/(r+5)$ for odd r and $1/(r+4)$ for even r, which would be weaker however than (17)–(20) for growing r.

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