Irreducible coverings by cliques and Sperner's theorem

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Abstract

In this note it is proved that if a graph G of order n has an irreducible covering of its vertex set by $n-k$ cliques, then its clique number $\omega(G) \leq k+1$ if $k=2$ or 3 and $\omega(G) \leq \binom{k}{\lfloor k/2 \rfloor}$ if $k \geq 4$. These bounds are sharp if $n \geq k+1$ (for $k=2$ or 3) and $n \geq k + \binom{k}{\lfloor k/2 \rfloor}$ (for $k \geq 4$).

Key Words: clique, irreducible covering, antichain, Sperner's theorem

1 Definitions and preliminary results

For a graph G having vertex set $V(G)$ and edge set $E(G)$ a clique is a subset of vertices inducing a complete subgraph of G which is maximal relative to set inclusion. The clique number of G, denoted $\omega(G)$, is the size of a largest clique in G [1]. A k-clique is a clique containing k vertices. A family of different cliques $c_1, c_2, \ldots, c_s$ of G is a covering of G by cliques if $\bigcup_{i=1}^{s} c_i = V(G)$. A covering C of G consisting of s cliques $c_1, \ldots, c_s$ of G will be called an irreducible covering of G if the union of any $s-1$ cliques from C is a proper subset of $V(G)$. This means that there exist s vertices $x_1, \ldots, x_s \in V(G)$ that are uniquely covered by cliques of C, i.e., $x_i \notin \bigcup_{\substack{k=1 \\ k \neq i}}^{s} c_k$ for every $1 \leq i \leq s$.

If $G = K_{p,q}$, every clique of G is an edge and an irreducible covering by edges of $K_{p,q}$ consists of a set of vertex-disjoint stars, some centered in the part with p vertices and others in the part with q vertices of $K_{p,q}$, which cover together all vertices of $K_{p,q}$. Some properties of the numbers $N(p,q)$ of all irreducible coverings by edges of $K_{p,q}$ were deduced in [8] and the exponential generating function of these numbers was given in [9]. Also, by denoting $I(n, n-k)$ the maximum number of irreducible coverings of the vertices of an n-vertex graph by $n-k$ cliques, in [8] it was shown that $\lim_{n\to\infty} I(n, n-k)^{1/n} = \alpha(k)$, where $\alpha(k)$ is the greatest number of cliques a graph with k vertices can have.

The problem of determining $\alpha(k)$ was solved by Miller and Muller [2] and independently

by Moon and Moser [3]. Furthermore, $I(n, n-2) = 2^{n-2} - 2$ and the extremal graph (unique up to isomorphism) coincides with $K_{2,n-2}$ for every $n \geq 4$. In [10] it was proved that for sufficiently large n, $I(n, n-3) = 3^{n-3} - 3 \cdot 2^{n-3} + 3$, and the extremal graph is (up to isomorphism) $K_{3,n-3}$, the second extremal graph being $K_{3,n-3} - e$.
There is a class of algorithms which yield all irreducible coverings for the set-covering problem, an example of an algorithm in this class being Petrick's algorithm [5]. This algorithm was intensively used for obtaining the minimal disjunctive forms of a Boolean function using prime implicants of the function or for minimizing the number of states of an incompletely specified Mealy type automaton A by finding a closed irreducible covering of the set of states of A by "maximal compatible sets of states", which are cliques in the graph of compatible states of A [4,7], since every minimum covering is an irreducible one. The chromatic number $\chi(G)$ of G equals the minimum number of cliques from an irreducible covering by cliques of the complementary graph $\overline{G}$.

2 Main result

We will evaluate the clique number $\omega(G)$ when G of order n has an irreducible covering by $n - k$ cliques.

Theorem 2.1 *Let $k \geq 2$. If the graph G of order n has an irreducible covering by $n - k$ cliques, then $\omega(G) \leq k + 1$ if $k = 2$ or 3 and $\omega(G) \leq \binom{k}{\lfloor k/2 \rfloor}$ if $k \geq 4$. Moreover, these bounds are sharp for every $n \geq k + 1$ if $k = 2$ or 3 and $n \geq k + \binom{k}{\lfloor k/2 \rfloor}$ if $k \geq 4$.*

Proof: Let $C = \{c_1, \ldots, c_{n-k}\}$ be an irreducible covering by $n-k$ cliques of G. It follows that there are $n - k$ vertices $x_1, \ldots, x_{n-k} \in V(G)$ such that $x_i \in c_i \setminus \bigcup_{j \neq i} c_j$ for every $i = 1, \ldots, n-k$. Denoting $X = \{x_1, \ldots, x_{n-k}\}$ and $Y = V(G) \setminus X$ one has $|Y| = k$. Each clique c_i consists of x_i and a subset of Y. For every subset $A \subseteq Y$ let $X_A \subseteq X$ be defined by

$$X_A = \{x_i \in X \, : \, c_i = \{x_i\} \cup A\}.$$

It is clear that if $x_i, x_j \in X_A$ then $x_i x_j \notin E(G)$ since otherwise $A \cup \{x_i, x_j\}$ induces a complete subgraph in G whose vertex set contains strictly c_i and c_j, which contradicts the definition of a clique. Similarly, if $x_i \in X_A$, $x_j \in X_B$ and $A \subset B$ it follows that $x_i x_j \notin E(G)$ since otherwise $A \cup \{x_i, x_j\}$ induces a complete subgraph in G, thus contradicting the hypothesis that c_i is a clique.
This implies that each clique c in G has the form $\{t_1, \ldots, t_s\} \cup \bigcap_{i=1}^{s} A_i$ for some $s \geq 2$, where $X_{A_i} \neq \emptyset$, $t_i \in X_{A_i} \subset X$ for every $1 \leq i \leq s$ and $\{A_1, \ldots, A_s\}$ is an antichain in the poset of subsets of Y, or c induces a maximal complete subgraph with vertex set included in $Y \cup \{x_i\}$ for some $1 \leq i \leq n - k$.
We will show for the first case that

$$\max_{s \geq 2} \max_{\{A_1, \ldots, A_s\}} (s + |\bigcap_{i=1}^{s} A_i|) = \binom{k}{\lfloor k/2 \rfloor}, \tag{1}$$

where the second maximum in the left-hand side of (1) is taken over all antichains of length $s \geq 2$, $\{A_1, \dots, A_s\}$ in the poset of subsets of Y ($|Y| = k \geq 2$), ordered by inclusion. The proof of (1) is by double inequality. If we choose $\{A_1, \dots, A_s\}$ to be the family of all $\lfloor k/2 \rfloor$-subsets of Y we have $s = \binom{k}{\lfloor k/2 \rfloor}$ and $\bigcap_{i=1}^{s} A_i = \emptyset$, whence

$$\max_{s \geq 2} \max_{\{A_1, \dots, A_s\}} (s + |\bigcap_{i=1}^{s} A_i|) \geq \binom{k}{\lfloor k/2 \rfloor}.$$

On the other hand, let $B = \bigcap_{i=1}^{s} A_i$ and $r = |B|$. Since $s \geq 2$ and $\{A_1, \dots, A_s\}$ is an antichain, it follows that $r \leq k - 2$. By deleting elements of B from $A_1, \dots, A_s$ we get an antichain in the poset of subsets of $Y \backslash B$ ($|Y \backslash B| = k - r$), ordered by inclusion. By Sperner's theorem [6] it follows that

$$\max_{\{A_1, \dots, A_s\}} (s + |\bigcap_{i=1}^{s} A_i|) \leq \binom{k-r}{\lfloor (k-r)/2 \rfloor} + r$$

and the last expression is less than or equal to $\binom{k}{\lfloor k/2 \rfloor}$ for every $k \geq 2$ and
$0 \leq r \leq k - 1$ and (1) is proved. Since any maximal complete subgraph in $Y \cup \{x_i\}$ can have at most $k + 1$ vertices, it follows that

$$\omega(G) \leq \max\,(k + 1, \binom{k}{\lfloor k/2 \rfloor}),$$

i.e., $\omega(G) \leq k + 1$ if $k = 2$ or 3 and $\omega(G) \leq \binom{k}{\lfloor k/2 \rfloor}$ if $k \geq 4$.

If $k = 2$ or $k = 3$ we can consider a graph G consisting of $n - k$ cliques of size $k + 1$ each having a k-clique in common; then G has order n, an irreducible covering by $n - k$ cliques and $\omega(G) = k + 1$.
If $k \geq 4$ we define a graph G of order $n \geq k + \binom{k}{\lfloor k/2 \rfloor}$ possessing an irreducible covering by $n - k$ cliques and $\omega(G) = \binom{k}{\lfloor k/2 \rfloor}$ as follows: Take a complete graph K_k and other $n - k$ vertices $x_1, \dots, x_{n-k}$. Let $A_1, \dots, A_p$ with $p = \binom{k}{\lfloor k/2 \rfloor}$ be all subsets of $V(K_k)$ of cardinality $\lfloor k/2 \rfloor$. Since $n - k \geq p$, there is a partition $X = X_1 \cup \ldots \cup X_p$ into p classes of $X = \{x_1, \dots, x_{n-k}\}$. Now join by an edge each vertex $x \in X_i$ to each vertex $y \in A_i$ for every $1 \leq i \leq p$ and add edges between some pairs of vertices in X such that X induce a complete multipartite graph whose parts are $X_1, \dots, X_p$. This graph has an irreducible covering by $n - k$ cliques, clique number $\omega(G) = p = \binom{k}{\lfloor k/2 \rfloor}$ and $\prod_{i=1}^{p} |X_i|$ cliques with p vertices.

□

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