Large equiangular sets of lines in Euclidean space

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In memory of Norman J. Pullman (1931-1999)

Abstract

A construction is given of $\frac{2}{9}(d+1)^2$ equiangular lines in Euclidean d-space, when $d = 3 \cdot 2^{2t-1} - 1$ with t any positive integer. This compares with the well known "absolute" upper bound of $\frac{1}{2}d(d+1)$ lines in any equiangular set; it is the first known constructive lower bound of order d^2 .

For background and terminology we refer to Seidel [3]. The standard method for obtaining a system of equiangular lines in Euclidean space is as follows. Let G be a graph, with Seidel adjacency matrix S, i.e. $S_{xy} = -1$ if vertices x and y are adjacent, $S_{xy} = 1$ if x and y are distinct and non-adjacent, $S_{xx} = 0$ for all x. Letting θ denote the smallest eigenvalue of S, we see that $M := I - \frac{1}{\theta}S$ is positive semidefinite of rank $d = n - m$ where n is the number of vertices and m is the eigenvalue multiplicity of θ. Hence M is representable as the Gram matrix of n unit vectors $x_1, ..., x_n$ in real d-space, with $< x_i, x_j >= \pm\frac{1}{\theta}$ whenever i and j are distinct. Thus the lines (1-dimensional subspaces) spanned by these x_i's have constant pairwise angle arccos $(\frac{1}{\theta})$.

It is not hard to see that the above process is reversible, so that finding a large equiangular set of lines in Euclidean space amounts to finding a graph whose Seidel adjacency matrix has smallest eigenvalue of large multiplicity.

Theorem. For each $d = 3 \cdot 2^{2t-1} - 1$, with t any positive integer, there exists an equiangular set of $\frac{2}{9}(d+1)^2$ lines in Euclidean d-space.

In order to describe the graphs relevant to this construction, we need to recall some terms and facts from the theory of quadratic forms over $GF(2)$; a convenient reference is [1], which contains everything we need here as well as some pointers to earlier literature.

Let V be a vector space over $GF(2)$. If $Q : V \to GF(2)$ is a quadratic form, then its polarization $B(x, y) := Q(x + y) + Q(x) + Q(y)$ is an alternating bilinear form. Note that B can be non-singular only if V has even dimension; so we will assume that $\dim(V) = 2t$ for some positive integer t. If Q polarizes to a non-singular B, then Q must be of one of two types $\chi(Q) = \pm 1$, where Q has exactly $2^{2t-1} + \chi(Q)2^{t-1}$ zeroes. Next, let $\{B_1, B_2, ..., B_r\}$ be a set of alternating bilinear forms on V; if $B_i + B_j$ is non-singular for all $i \neq j$ then the set is called non-singular. It is not hard to show that a non-singular set has $r \leq 2^{2t-1}$; when equality holds it is called a Kerdock set. Such maximal non-singular sets do exist for all t.

We may now describe the graphs occurring in our construction of equiangular lines. Let K be a Kerdock set of alternating forms on V, where $\dim(V) = 2t$ as above. The graph G_t will have as vertex-set all pairs (B, Q) where B belongs to K and Q polarizes to B. Two vertices (B, Q) and (B', Q') are declared adjacent precisely when $B \neq B'$ and $\chi(Q + Q') = -1$. Note that G_t is one of the two non-trivial relations in what is called the Cameron-Seidel 3-class association scheme in [1]. The eigenvalues of the Seidel adjacency matrix $S(G_t)$ are as follows:

$\theta_1 = 2^{3t-1} + 2^{2t} - 2^t - 1$; multiplicity one.

$\theta_2 = 2^{3t-1} - 2^t - 1$; multiplicity $2q - 1$ where $q := 2^{2t-1}$.

$\theta_3 = 2^{2t} - 2^t - 1$; multiplicity $q - 1$.

$\theta_4 = -2^t - 1$; multiplicity $(q - 1)(2q - 1)$.

The foregoing spectral information can be derived from the (dual) eigenmatrix Q on page 326 of [2], by setting $n = 2^{2t}$, $r = 2^{2t-1}$, $a = 2^{t+1}$, $\theta = 2^t - 1$ and $\tau = -2^t - 1$ in that paper; the adjacency eigenvalues of G_t are then given by the fourth column of Q and the corresponding multiplicities by the first row of the P-matrix. Also please note that the Seidel matrix S and ordinary adjacency matrix A are related by $S = J - I - 2A$.

We now have the following situation. The eigenvalue $\theta = \theta_4$ is the smallest eigenvalue of $S(G_t)$ and it has very large multiplicity. Indeed the rank of $M = I - \frac{1}{\theta}S$ is $d = 3q - 1$ and the graph has $2q^2 = \frac{2}{9}(d+1)^2$ vertices. From the standard procedure sketched earlier, we thus obtain an equiangular set of $\frac{2}{9}(d + 1)^2$ lines in Euclidean d-space, whenever $d = 3q - 1 = 3 \cdot 2^{2t-1} - 1$ for some positive integer t. This completes the presentation and verification of our construction, or in other words, the proof of our theorem.

The graphs G_t have already been known for over twenty-five years. It is perhaps surprising that their relevance to equiangular lines was not noticed before. A likely reason is that, generally speaking, the best constructions seem to come from regular two-graphs where the Seidel adjacency matrix has just two distinct eigenvalues; for example the absolute upper bound of $\frac{1}{2}d(d + 1)$ can only be achieved by a regular two-graph. But so far (cf. [3], p.884) constructions using regular two-graphs have yielded nothing better asymptotically than a constant times $d\sqrt{d}$.

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References

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