A Complete Categorization of When Generalized Tribonacci Sequences Can Be Avoided by Additive Partitions

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Abstract

A set or sequence U in the natural numbers is defined to be avoidable if $\mathbb{N}$ can be partitioned into two sets A and B such that no element of U is the sum of two distinct elements of A or of two distinct elements of B. In 1980, Hoggatt [5] studied the Tribonacci sequence $T = \{t_n\}$ where $t_1 = 1$, $t_2 = 1$, $t_3 = 2$, and $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ for $n \geq 4$, and showed that it was avoidable. Dumitriu [3] continued this research, investigating Tribonacci sequences with arbitrary initial terms, and achieving partial results. In this paper we give a complete answer to the question of when a generalized Tribonacci sequence is avoidable.

1 Introduction

A subset U of the natural numbers is defined to be **avoidable** if there exists a partition of $\mathbb{N}$ into two disjoint sets A and B such that no element of U is the sum of two distinct elements of A or of B; the pair $\{A, B\}$ is sometimes referred to as an **additive partition**. Avoidable sets in $\mathbb{N}$ have been studied for many years, yet very few families of avoidable sets are known. The concept of avoidable sets was pioneered by Alladi, Erdős, and Hoggatt [1] in 1978. They proved that the Fibonacci sequence was avoidable, and achieved results for various special cases of generalized Fibonacci sequences (i.e., sequences $\{f_n\}$ with $f_1, f_2 \in \mathbb{N}$ and $f_k = f_{k-1}+f_{k-2}$ for $k \geq 3$). In 1981, Evans [4] solved yet more cases of the generalized Fibonacci sequence; lastly, in 1993, Shan and Zhu [6] solved a large family of two-term recursive sequences, including generalized Fibonacci sequences. Chow and Long [2] also showed that a family of sequences arising from continued fractions is avoidable.

Hoggatt [5] showed that the Tribonacci sequence $T = \{t_n\}$ was avoidable, where $t_1 = 1, t_2 = 1, t_3 = 2$ and we have

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} \tag{1}$$

for $n \geq 4$. This result suggested that a subfamily of generalized Tribonacci sequences (sequences satisfying recursion (1) with arbitrary initial terms) might be avoidable; in [3], Dumitriu studied generalized Tribonacci sequences and was able to obtain results when the positive integers a, b, and c, the initial terms of the sequence, satisfy $a \leq b \leq c$ with $a + b \geq c$. In this paper we build off Dumitriu's results to determine when an arbitrary generalized Tribonacci sequence is avoidable; all Tribonacci sequences in this paper are assumed to have positive initial terms.

2 Previous Results

Dumitriu [3] achieved two results which form the building blocks of our proofs. First of all, there is the following important lemma, which allows us to construct additive partitions of all of $\mathbb{N}$ from additive partitions of finite sets.

Lemma 1 (Dumitriu [3]). *Given a generalized Tribonacci sequence $T = \{t_n\}$ with initial terms $a \leq b \leq c$, if there exists an additive partition $\{A, B\}$ of $\{1, \ldots, c-1\}$ avoiding T, then this partition can be extended to an additive partition of $\mathbb{N}$ avoiding T.*

We extend this lemma to Tribonacci sequences with arbitrarily ordered initial terms, so that it reads as follows.

Lemma 2. *Given a generalized Tribonacci sequence $T = \{t_n\}$ with initial terms a, b, and c, if there exists an additive partition $\{A, B\}$ of $\{1, \ldots, \max(a, b, c) - 1\}$ avoiding T, then this partition can be extended to an additive partition of $\mathbb{N}$ avoiding T.*

Proof. Examining Dumitriu's proof, we find that only the assumption that $c \geq a, b$ is used. Therefore, if we obtain an additive partition of $\{1, \ldots, a + b + c - 1\}$ avoiding T, we can then use Lemma 1 to extend this partition to a partition of $\mathbb{N}$ avoiding the Tribonacci sequence generated by b, c, and $a + b + c$. As this extended partition will obviously avoid a (because the original partition did, and $a < a + b + c$), we will in fact have generated a partition of $\mathbb{N}$ avoiding T.

Let $d = \max(a, b, c)$. We proceed as follows:

Case 1. $a+b+c \leq 2d$. In this case, we extend the partition $\{A, B\}$ of $\{1, \ldots, d-1\}$ avoiding T to one of $\{1, \ldots, a + b + c - 1\}$ by, for $d \leq x < a + b + c$, placing x in the opposite set from $a+b+c-x$ (which is in A or B unless $a+b+c=2d$ and $x=d$, in which case we can place x in either set.) Clearly, the resulting partition $\{C, D\}$ will avoid a, b, c, and $a+b+c$; also, $t_6 = 2a+3b+4c$ is too large to be expressed as the sum of two terms in $\{1, \ldots, a+b+c-1\}$. Therefore, we need only check that $\{C, D\}$ avoids $t_5 = a+2b+2c$. Suppose not; then without loss of generality, we have $y, z \in C$ with $y+z = a+2b+2c$. Then unless y or z is equal to $(a+b+c)/2$, $a+b+c-y$ and $a+b+c-z$ are both in D, yet their sum is equal to $2a+2b+2c-(y+z) = a$, contradicting the fact that D avoids a. If $y = (a+b+c)/2$, $a+b+c$ is even and $z = (a+3b+3c)/2$. However, since in this case we must have $a \le b+c$, we conclude that $z \ge a+b+c$, a contradiction. Therefore, $\{C, D\}$ avoids T as desired.

Case 2. $a+b+c > 2d$. In this case, we define:

$$\begin{aligned}
S_1 &= \{d, d+1, \ldots, \left\lfloor \frac{a+b+c}{2} \right\rfloor\} \\
S_2 &= \{\left\lfloor \frac{a+b+c}{2} \right\rfloor + 1, \ldots, a+b+c-d\} \\
E &= \{x \mid a+b+c-d < x < a+b+c, a+b+c-x \in B\} \\
F &= \{x \mid a+b+c-d < x < a+b+c, a+b+c-x \in A\}
\end{aligned}$$

We then let $C = A \cup S_1 \cup E$ and $D = B \cup S_2 \cup F$. Clearly as before $\{C, D\}$ avoids a, b, c, and $a+b+c$, and $t_6 = 2a+3b+4c$ is too large to be expressed as the sum of two elements of C or D. Suppose we had two elements $x, y \in C$ (respectively D) with $x+y = t_5 = a+2b+2c$. Then x and y must both be in E (respectively F) , for as $x < a+b+c$, we have $y > b+c \ge a+b+c-d$ as $d \ge a$. However, this then implies that $a+b+c-x$ and $a+b+c-y$ are both in B (respectively A), which is impossible because their sum is $2a+2b+2c-(x+y) = a$ and $\{A, B\}$ avoids a. Therefore $\{C, D\}$ avoids T. □

We also note the following easy lemma.

Lemma 3. *If $a > 2b+2c$, any partition of the set $A = \{1, \ldots, a-1\}$ avoiding $\{a, b, c, a+b+c\}$ also avoids $a+2b+2c$.*

Proof. Let $\{E, F\}$ be such a partition of A. Without loss of generality, suppose for purposes of contradiction that we have $x+y = a+2b+2c$ with x, y distinct elements of E. As $\{E, F\}$ avoids $a+b+c$, $a+b+c-x$ and $a+b+c-y$ (which are both in A as $2b+2c < x, y < a$) are distinct elements of F. However, their sum is then equal to $2a+2b+2c-(x+y) = a$, contrary to the assumption that $\{E, F\}$ avoids a. □

Therefore, any partition of $A = \{1, \ldots, \max(a, b, c) - 1\}$ avoiding $\{a, b, c, a+b+c\}$ in fact avoids all of T, as t_n is too large to be expressed as the sum of two elements of A for $n \ge 6$, and $t_5 = a+2b+2c$ is too large except in the case $a > 2b+2c$, which is covered by Lemma 3. We make the following definition.

Definition. *The symmetric Boolean variable $P(a,b,c)$ is defined to be equal to 1 if there exists a partition of $\{1,\ldots,\max(a,b,c)-1\}$ avoiding $\{a,b,c,a+b+c\}$, and 0 otherwise.*

Now, Lemma 2 allows us to conclude the following important equivalence theorem.

Theorem 1. *The Tribonacci sequence with initial terms $t_1 = a$, $t_2 = b$, and $t_3 = c$ is avoidable if and only if $P(a,b,c) = 1$.*

The importance of this theorem is that it reduces the problem of determining when a Tribonacci sequence generated by three initial terms is avoidable to a symmetric one. Because $P(a,b,c)$ is a symmetric function, it suffices to determine its values for $a \le b \le c$. We note that if a, b and c are all odd, then we can take as a partition of $\mathbb{N}$ the sets $A = \{2k\}$, $B = \{2k+1\}$, so in this case $P(a,b,c) = 1$.

Dumitriu also achieved the following result, which we will use as the base case of our argument.

Theorem 2 (Dumitriu [3]). *Say $t_1 = a, t_2 = b, t_3 = c$ with a,b,c not all odd. If $a < b < c < a+b$, then $P(a,b,c) = 1$ if and only if either:*

1. *a and b are even, c is odd, and $c \ge a + \frac{b}{2}$, or*
2. *b and c are even, a is odd, and $a \le \frac{c}{2}$, or*
3. *a and c are even, b is odd, and either*
   - *$2b = a + c$, or*
   - *$c \ge b + \frac{a}{2}$ and $a \le \frac{c}{2}$.*

We also need the following supplementary result to cover the special case where we have equality instead of one of Dumitriu's inequalities.

Lemma 4. *Say $t_1 = a, t_2 = b, t_3 = c$ with $a \le b \le c \le a+b$. If either $a = b$, $b = c$, or $a + b = c$, then $P(a,b,c) = 1$.*

Proof. We only need to show that there exists an additive partition of $\{1,\ldots,c-1\}$ avoiding $\{a,b,c\}$, because $a+b+c$ is too large to be written as the sum of two elements smaller than c. If $a = b$ or $b = c$, then we need only to find a partition of $\{1,\ldots,c-1\}$ avoiding $\{a,c\}$; if $a+b = c$, we need to find a partition of $\{1,\ldots,a+b-1\}$ avoiding $\{a,b,a+b\}$. However, the existence of both partitions is proven by Shan and Zhu in [6]. □

Using these results as our base case, we will recursively determine whether a given generalized Tribonacci sequence is avoidable.

3 Generalized Tribonacci Sequences

We have already established the value of $P(a, b, c)$ in the case where $a \leq b \leq c \leq a + b$. In this section, we prove the following complementary theorem.

Theorem 3. *If $a \leq b \leq c$ and $a + b < c$, then a partition of $\{1, \ldots, c-1\}$ avoiding a, b, c and $a + b + c$ exists if and only if a partition of $\{1, \ldots, \max(c - a - b, b) - 1\}$ avoiding $c - a - b, a, b$ and c exists.*

Proof. We have two cases to consider:

Case 1. $c \leq 3(a + b)$. In this case, let $A = \{1, \ldots, \lfloor \frac{a+b+c}{2} \rfloor\}$, and $B = \{\lfloor \frac{a+b+c}{2} \rfloor + 1, \ldots, c-1\}$.

Claim 1. *A partition of $A \cup B$ avoiding $\{a, b, c, a+b+c\}$ exists if and only if a partition of A avoiding $\{c - a - b, a, b, c\}$ exists.*

Proof. Given such a partition $\{X, Y\}$ of $A \cup B$, we create a partition of A by restricting the partition of $A \cup B$. The only thing remaining to check is that no two elements in the same set of the partition $\{X \cap A, Y \cap A\}$ sum to $c - a - b$. Without loss of generality, suppose there exist distinct elements $w, x \in X \cap A$ with $w + x = c - a - b$. Because $\{X, Y\}$ avoids c, unless x or w is equal to $c/2$, it follows that $c - x$ and $c - w$ (which lie in $A \cup B$) must be in Y, but then $(c - x) + (c - w) = a + b + c$, contrary to the assumption that $\{X, Y\}$ avoids $a + b + c$. Now, consider the case where, without loss of generality, $x = c/2$. In this case, we note that since $w > 0$ and $w + x = c - a - b$, $c > 2b$. Consequently, we have $x > a, b$, so we need not worry about x summing to a or b or *a priori* c. Therefore, if necessary, we can simply modify our partition by placing x in whichever of $X \cap A$ and $Y \cap A$ the element $w = c - a - b - x$ is not in.

Conversely, given a partition $\{X, Y\}$ of A avoiding $c-a-b, a, b$, and c, we can extend this partition to a partition $\{X', Y'\}$ of $A \cup B$ by placing each $x \in B$ in the set opposite to $c - x$ (which is in A.) The resulting partition still avoids c by construction, and still avoids a and b as the elements being added are all at least $(a+b+c+1)/2 > a+b$. So we need only check that the partition of $A \cup B$ avoids $a+b+c$; however, if without loss of generality we have distinct $w, x \in X'$ with $w + x = a + b + c$, then unless w or x is equal to $c/2$, $c - x$ and $c - w$ are distinct positive integers in Y' whose sum is $c - a - b$. Because $c - a - b \leq (a + b + c)/2$ as $c \leq 3(a + b)$, they are therefore both in A and thus in Y, contradicting the fact that $\{X, Y\}$ is a partition of A avoiding $c - a - b$. As before, we must consider the special case where $x = c/2$. However, as $w < c$, it follows that $c > 2b$, so as before we can modify our partition if necessary by placing x in the opposite set from $a + b + c - x$. □

Now, to complete the proof of Case 1 of the theorem, we need only prove the following claim.

Claim 2. *Let $d = max(c - a - b, b)$. Then a partition of the set A avoiding $\{c - a - b, a, b, c\}$ exists if and only if a partition of $\{1, \ldots, d-1\}$ avoiding $\{c - a - b, a, b, c\}$ exists.*

Proof. Noting that $d-1 < \lfloor \frac{a+b+c}{2} \rfloor$, one direction is trivial: if we have a partition of A, we can restrict it to $\{1, \ldots, d-1\}$ and it will still avoid $\{c-a-b, a, b, c\}$. For the other direction, we can simply take the partition on $\{1, \ldots, d-1\}$ and extend it by placing x in the opposite set from $c-x$. The partition then obviously avoids c; also, as no number larger than $d-1$ can be added to a natural number to get a, b, or $c-a-b$, this partition also avoids a, b, and $c-a-b$. $\square$

Case 2. $c > 3(a+b)$. In this case, we let $A = \{1, \ldots, c-a-b-1\}$ and $B = \{c-a-b, \ldots, c-1\}$. Again, we claim that a partition of A avoiding $\{c-a-b, a, b, c\}$ exists if and only if a partition of $A \cup B$ avoiding $\{a, b, c, a+b+c\}$ exists.

If we have a partition of $A \cup B$ avoiding $\{a, b, c, a+b+c\}$, we can restrict it to a partition of A avoiding $\{c-a-b, a, b, c\}$; if x and y are in one set of the partition summing to $c-a-b$, then $c-x$ and $c-y$ must be in the other set of the partition and $(c-x)+(c-y) = 2c-(x+y) = a+b+c$, an impossibility. In the other direction, we extend the partition $\{X, Y\}$ of A to a partition $\{X', Y'\}$ of $A \cup B$ by placing $x \in B$ in the opposite set from $c-x$. We need only check that the resulting partition avoids $a+b+c$. Suppose we have distinct $w, x \in X'$ (respectively Y')with $w+x = a+b+c$. Then unless x or w is equal to $c/2$, $c-x$ and $c-w$ are distinct elements of Y' (respectively X') summing to $c-a-b$, and hence are in A; therefore, they are in Y (respectively X), contradicting the fact that $\{X, Y\}$ avoids $c-a-b$. Furthermore, if $x = c/2$, as in Case 1 we can modify our partition if necessary by placing x in the opposite set of the partition from $c+a+b-x$, as again $x > a, b$. $\square$

We can now completely solve the question of when a Tribonacci sequence is avoidable via a method of descent. Since $P(a,b,c)$ is a symmetric function, it suffices to establish the answer for $a \le b \le c$. For $a+b \ge c$, Theorem 2 and Lemma 4 establish when $P(a,b,c) = 1$. Meanwhile, if we have $a+b < c$, Theorem 3 states that $P(a,b,c) = P(a,b,c-a-b)$; note that a, b, and $c-a-b$ are all odd if and only if a, b, and c are. Then, using this recursion as well as the symmetry of P, for any triple (a,b,c) not all odd, we can reduce $P(a,b,c)$ to a case covered in Theorem 2 or Lemma 4. And, as stated before, if a, b, and c are all odd, $P(a,b,c) = 1$.

4 Open Questions

The author would like to thank the referee for suggesting the following two questions.

Question 1. *In this paper, we have only considered Tribonacci sequences with positive initial terms. One can ask the same question for initial terms not all positive; in this case, the negative terms of the Tribonacci sequence place no constraint on a potential additive partition $\{A, B\}$ of $\mathbb{N}$. Which such Tribonacci sequences are avoidable?*

Question 2. *Given the recursion outlined in this paper, is there a simple non-recursive way to categorize avoidable Tribonacci sequences?*

In addition, one can proceed to the next case, that of sequences satisfying $t_n = t_{n-1} + t_{n-2} + t_{n-3} + t_{n-4}$ for $n \geq 5$, and beyond. It is worth noting that as the depth of the recursion increases, the ratio between the terms approaches 2; furthermore, any sequence satisfying $t_n > 2t_{n-1} - 1$ for all n is certainly avoidable by direct construction of the form outlined in this paper. Therefore, one would expect that the main roadblock to constructing additive partitions avoiding such a recursive sequence would lie in the initial terms. Indeed, Theorem 1 is true for deeper recursions with an almost identical proof, so, as in the case of Tribonacci sequences, determining avoidability boils down to determining whether an appropriate additive partition of a finite set exists.

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References

[1] K. Alladi, P. Erdős, V.E. Hoggatt, Jr., "On additive partitions of integers," *Discrete Math.* **23** (1978), 201-211

[2] T.Y. Chow and C.D. Long, "Additive partitions and continued fractions," *Ramanujan J.* **3** (1999), 55-72

[3] I. Dumitriu, "On generalized Tribonacci sequences and additive partitions," *Discrete Math.* **219** (2000), 65-83.

[4] R.J. Evans, "On additive partitions of sets of positive integers," *Discrete Math.* **36** (1981), 239-245

[5] V.E. Hoggatt, Jr., "Additive partitions of the positive integers," *Fibonacci Quart.* **18** (1980), 220-226

[6] Z. Shan and P.-T. Zhu, "On (a,b,k)-partitions of positive integers," *Southeast Asian Bull. Math* **17** (1993), 51-58