# The Action of the Symmetric Group on a Generalized Partition Semilattice

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**Abstract**

Given an integer $n \geq 2$, and a non-negative integer $k$, consider all affine hyperplanes in $\mathbb{R}^n$ of the form $x_i = x_j + r$ for $i, j \in [n]$ and a non-negative integer $r \leq k$. Let $\Pi_{n,k}$ be the poset whose elements are all nonempty intersections of these affine hyperplanes, ordered by reverse inclusion. It is noted that $\Pi_{n,0}$ is isomorphic to the well-known partition lattice $\Pi_n$, and in this paper, we extend some of the results of $\Pi_n$ by Hanlon and Stanley to $\Pi_{n,k}$.

Just as there is an action of the symmetric group $\mathfrak{S}_n$ on $\Pi_n$, there is also an action on $\Pi_{n,k}$ which permutes the coordinates of each element. We consider the subposet $\Pi^\sigma_{n,k}$ of elements that are fixed by some $\sigma \in \mathfrak{S}_n$, and find its Möbius function $\mu_\sigma$, using the characteristic polynomial. This generalizes what Hanlon did in the case $k = 0$. It then follows that $(-1)^{n-1}\mu_\sigma(\Pi^\sigma_{n,k})$, as a function of $\sigma$, is the character of the action of $\mathfrak{S}_n$ on the homology of $\Pi_{n,k}$.

Let $\Psi_{n,k}$ be this character times the sign character. For $\mathfrak{C}_n$, the cyclic group generated by an $n$-cycle $\sigma$ of $\mathfrak{S}_n$, we take its irreducible characters and induce them up to $\mathfrak{S}_n$. Stanley showed that $\Psi_{n,0}$ is just the induced character $\chi\uparrow_{\mathfrak{C}_n}^{\mathfrak{S}_n}$ where $\chi(\sigma) = e^{2\pi i/n}$. We generalize this by showing that for $k > 0$, there exists a non-negative integer combination of the induced characters described here that equals $\Psi_{n,k}$, and we find explicit formulas. In addition, we show another way to prove that $\Psi_{n,k}$ is a character, without using homology, by proving that the derived coefficients of certain induced characters of $\mathfrak{S}_n$ are non-negative integers.

# 1 Introduction

Given a finite partially ordered set $P$, let $\leq$ denote the partial order, and assume that $P$ has a unique minimal element $\hat{0}$. An *automorphism* $\sigma$ on $P$ is a permutation of the elements of $P$ such that if $x \leq y$, then $\sigma(x) \leq \sigma(y)$. Let $P^\sigma$ be the subposet of $P$ that consists of the elements that are fixed by $\sigma$. If $P$ is a lattice, then so is $P^\sigma$ (For a proof, see page 319 of [9]).

Now we look at one particular lattice. For some positive integer $n$, if we let $\Pi_n$ denote the set of all partitions of the set $[n] = \{1, 2, ..., n\}$, ordered by refinement, then $\Pi_n$ is a lattice. There has been a lot of work on $\Pi_n$, and the action of the symmetric group $\mathfrak{S}_n$ on it. An element of $\mathfrak{S}_n$ permutes the elements of $[n] = \{1, 2, ..., n\}$, and therefore acts as an automorphism on $\Pi_n$. Given $\sigma \in \mathfrak{S}_n$, let $\Pi_n^\sigma$ denote the subposet of $\Pi_n$ of elements that are fixed by $\sigma$.

The *Möbius function* $\mu$ is defined on intervals $[x, y] = \{z\colon\ x \leq z \leq y\}$ of a poset $P$ such that $\mu(x, x) = 1$ for all $x \in P$ and for $x < y$,

$$\sum_{z\in[x,y]} \mu(x, z) = 0.$$

If $P$ has $\hat{1}$, then define $\mu(P)$ to be $\mu(\hat{0}, \hat{1})$. Let $\mu_\sigma$ be the Möbius function of $\Pi_n^\sigma$. In 1981, Hanlon [9, Th. 4] showed that

$$\mu_\sigma(\Pi_n^\sigma) = \begin{cases} \mu(n/d)(-\frac{n}{d})^{d-1}(d-1)! & \text{if } \sigma \text{ is a product of } d \text{ cycles} \\ & \text{of length } n/d \text{ for some } d|n; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Here, $\mu(n/d)$ is the classical number-theoretic Möbius function. In 1982, Stanley [13] used this result and the Lefschetz Fixed Point Theorem (stated in section 3) to show that as a function of $\sigma$, $(-1)^{n-1}\mu_\sigma(\Pi_n^\sigma)$ is the character of a particular representation of $\mathfrak{S}_n$ (Refer to [12, Ch.1] for definitions), its action on the top-dimensional *homology* of $\Pi_n$, which we define in section 3. Let $\sigma$ be an $n$-cycle in $\mathfrak{S}_n$ and let $\mathfrak{C}_n$ be the cyclic subgroup of $\mathfrak{S}_n$ generated by $\sigma$. Let $\chi$ be the (irreducible) character of $\mathfrak{C}_n$ such that $\chi(\sigma) = e^{2\pi i/n}$. Stanley showed that the induced character (defined in [12, §1.12]) $\chi\uparrow_{\mathfrak{C}_n}^{\mathfrak{S}_n}$ equals this homology character times the sign character, which we denote $\Psi_n$. It is appropriate to show that $\Psi_n$ is an induced character from the cyclic group of order $n$ since it is zero for all elements of $\mathfrak{S}_n$ that are not conjugate to any element of $\mathfrak{C}_n$.

In this paper, we extend these results of $\Pi_n$ to a *generalized partition semilattice*, which we now define. We will call it $\Pi_{n,k}$. The partition lattice is isomorphic to a poset of subspaces of $\mathbb{R}^n$ for $n \geq 2$, ordered by reverse inclusion, whose elements are all intersections of the hyperplanes

$$H_{i,j} = \{x \in \mathbb{R}^n : \; x_i = x_j\},$$

for $i, j \in [n]$, with minimal element $\mathbb{R}^n$. In other words, if $i$ and $j$ are in the same block of a partition of $\Pi_n$, then the corresponding subspace of $\mathbb{R}^n$ is contained in $H_{i,j}$.

Now consider the affine hyperplanes

$$H_{i,j,r} = \{x \in \mathbb{R}^n : \; x_i = x_j + r\}.$$

For a non-negative integer $k$, let $\Pi_{n,k}$ be the poset whose elements are all nonempty intersections of the $H_{i,j,r}$ such that $r \in \mathbb{Z}$ and $|r| \leq k$. These sets of hyperplanes are known as the *extended Catalan arrangements* (See after Theorem 2.3 in [14]). The unique minimal element is again the whole space $\mathbb{R}^n$, but there is more than one maximal element if $k > 0$. For an affine subspace $X \in \Pi_{n,k}$, its dimension $\dim(X)$ is equal to the dimension of the linear translation of $X$, the set $\{v - x : \; v \in X\}$ for a particular $x \in X$. So $X$ is maximal if and only if $\dim(X) = 1$.

First, the *characteristic polynomial* of a poset $P$ of affine subspaces of $\mathbb{R}^n$ is given by

$$\lambda_P(t) = \sum_{X \in P} \mu(\hat{0}, X) t^{\dim(X)}.$$

In section 2, we let $P = \Pi_{n,k}$ and consider $P^\sigma$, the subposet of $P$ fixed by some $\sigma \in \mathfrak{S}_n$. We use the characteristic polynomial of $P$ and the paper by Hanlon [9] to show that the Möbius function of this subposet, $\mu_\sigma(P^\sigma)$, is as stated in (4). Then in section 3, we use a result from Stanley's paper [13] to show that the character of the representation of $\mathfrak{S}_n$ acting on the top homology of $P$ is $(-1)^{n-1}\mu_\sigma(P^\sigma)$.

Let $\Psi_{n,k}$ be this character times the sign character, so $\Psi_{n,k} = (-1)^{d-1}\mu_\sigma(P)^\sigma)$. In sections 4 and 5, we show that $\Psi_{n,k}$ can be expressed as a non-negative integer combination of the characters of $\mathfrak{S}_n$ that are induced from irreducible characters of $\mathfrak{C}_n$, as in (15). First, we show that the induced characters in this sum are a basis for all induced characters from $\mathfrak{C}_n$. Then the main result in section 4 is that $\Psi_{n,k}$ is a sum of induced characters from $\mathfrak{C}_d$ for each $d|n$. In section 5, we find an explicit expression for $\Psi_{n,k}$ in terms of these induced characters, also proving some concepts from number theory which we use along the way.

In the last section, we prove separately that the coefficients are non-negative integers, using the formula derived in Lemma 11, which gives us a way to prove that $\Psi_{n,k}$ is a character without proving that it is a homology character.

There is a lot more one can do on the subject of $\Pi_{n,k}$. For example, Christos Athanasiadis in his Ph.D. thesis [1] used the Möbius Inversion Formula to find the characteristic polynomial of numerous affine hyperplane arrangements, including this one [1, Th. 5.1]. Also, Julie Kerr in her Ph.D. thesis [10] discusses the poset obtained by adding a unique maximal element to $\Pi_{n,k}$. Although it becomes a lattice, its characteristic polynomial does not in general factor linearly as it does for $\Pi_{n,k}$. But its top-dimensional homology is isomorphic to a direct sum of copies of the algebra $\mathbb{C}\mathfrak{S}_n$, known as the *regular representation* of $\mathfrak{S}_n$. There is also additional work on the poset $\Pi_{n,k}$ in [7].

# 2 The Möbius Function of $\Pi^{\sigma}_{n,k}$

We first state [1, Th. 5.1]. This generalizes the characteristic polynomial of the well-known partition lattice, which is the case $k = 0$.

**Theorem 1.** *The characteristic polynomial of* $\Pi_{n,k}$ *is given by*

$$\lambda_{\Pi_{n,k}}(t) = t(t - nk - 1)(t - nk - 2)...(t - n(k + 1) + 1). \tag{2}$$

We now extend some more results of the partition lattice $\Pi_n$ to $\Pi_{n,k}$, first from Hanlon's paper [9]. Given any poset $P$ with a unique minimal element $\hat{0}$, let $\mathrm{Max}(P)$ denote the set of maximal elements of $P$ and let

$$\mu(P) = \sum_{x \in \mathrm{Max}(P)} \mu(\hat{0}, x). \tag{3}$$

Now let $P = \Pi_{n,k}$ and consider the action of the symmetric group $\mathfrak{S}_n$ on $P$, permuting the coordinates of the elements. We consider the subposet $P^{\sigma}$, which consists of the elements of $P$ that are fixed by a permutation $\sigma \in \mathfrak{S}_n$, meaning whenever $X \in P^{\sigma}$ and $X \subseteq H_{i,j,r}$, then $X \subseteq H_{\sigma(i),\sigma(j),r}$. Note that if $\varepsilon$ is the identity permutation, then $P^{\varepsilon} = P$.

Let $\mu_{\sigma}$ denote the Möbius function in $P^{\sigma} = \Pi^{\sigma}_{n,k}$. The goal in this section is to prove that

$$\mu_{\sigma}(P^{\sigma}) = \begin{cases} \mu(n/d)(-\frac{n}{d})^{d-1}\binom{(k+1)d-1}{d-1}(d-1)! & \text{if } \sigma \text{ is a product of } d \text{ cycles} \\ & \text{of length } n/d \text{ for some } d|n; \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

This is the Möbius function of $P^{\sigma}$, defined as in (3). It generalizes Hanlon's result for $k = 0$, stated in (1). If $\sigma, \tau \in \mathfrak{S}_n$, then one can verify the isomorphism $P^{\sigma} \cong P^{\tau\sigma\tau^{-1}}$. Hence, viewed as a function of $\sigma$, $\mu_{\sigma}(P^{\sigma})$ is a class function on $\mathfrak{S}_n$. It is in fact, up to a sign, a character of $\mathfrak{S}_n$, as we will soon see.

In order to find $\mu_{\sigma}(P^{\sigma})$, we find the sum of the Möbius functions of each maximal interval of $P^{\sigma}$. The methods we use here are in many cases very similar to those used by

Hanlon, with a slightly different poset. We state a well-known theorem that we will use here. Suppose we are given a finite lattice $L = [\hat{0}, \hat{1}]$. For some $x \in L$, define $\mathrm{Comp}(x)$ to be the set of *complements* of $x$ in $L$, i.e., $\mathrm{Comp}(x) = \{y \in L \colon x \wedge y = \hat{0} \text{ and } x \vee y = \hat{1}\}$. Then Crapo's Complementation Theorem [5, Th. 4] says that for any $x \in L$,

$$\mu(L) = \sum_{\substack{y,z \in \mathrm{Comp}(x) \\ y \leq z}} \mu(\hat{0}, y)\mu(z, \hat{1}), \tag{5}$$

and if some element of $L$ has no complements, then $\mu(L) = 0$.

So we need to show that $[\hat{0}, X]^\sigma$ is a lattice for each $X \in \mathrm{Max}(P^\sigma)$. By [18, Prop. 3.1], since every element of $P$ is an intersection of affine hyperplanes from a given set, it is a geometric semilattice. Thus each maximal interval $[\hat{0}, X]$ in $P$ is a lattice, and by the first paragraph of section 1, $[\hat{0}, X]^\sigma$ is a lattice too. So Crapo's Theorem applies here. Now we determine which element we use in Equation (5).

For each $\sigma \in \mathfrak{S}_n$, it can be verified that

$$\mathrm{Max}(P^\sigma) = \mathrm{Max}(P) \cap P^\sigma, \tag{6}$$

which is mentioned in the proof of [10, Th. 2.1]. For $\sigma$, let

$$\sigma = \sigma_1 \sigma_2 ... \sigma_d \tag{7}$$

be the decomposition of $\sigma$ into disjoint cycles. For $i = 1, ..., d$, let $C_i$ be the *support* of the cycle $\sigma_i$, that is, the set of all numbers from the cycle $\sigma_i$. It will be convenient to extend Hanlon's definition of the *hinge* of $\Pi_n$ [9, p. 324], the partition which puts each cycle of $\sigma$ into its own block. Here, we want to extend it to any $k \geq 0$, so that it is an intersection of affine hyperplanes. In $\Pi_{n,0}$ the element that corresponds to the hinge of $\Pi_n$ is the intersection of all $H_{j,l}$ such that $j$ and $l$ are in the same $C_i$. So this will be the hinge of $\Pi_{n,0}$. The following lemma shows that for any $k$, only certain hyperplanes in $\Pi_{n,k}$ can contain the hinge.

**Lemma 2.** *Suppose two numbers $i$ and $j$ are in the same cycle of $\sigma$, and for some $Z \in P^\sigma$, $Z$ is contained in $H_{i,j,r}$ for some $r$. Then $r = 0$.*

**Proof.** Suppose $Z \in P^\sigma$ and $\sigma_1$ is one of the disjoint cycles of $\sigma$ as in (7), with length $m \geq 2$. Suppose without loss of generality that $i, j \in C_1$ and $Z \subseteq H_{i,j,r}$. Then there exists an $s$ such that $\sigma^s(i) = j$, so let $\tau = \sigma^s$. Then $Z \subseteq H_{i,\tau(i),r}$ and then $Z \subseteq H_{\tau^\omega(i),\tau^{\omega+1}(i),r}$, since for each integer $\omega$, $P^\sigma \subseteq P^{\sigma^\omega}$. This means for any $z \in Z$, $z_i = z_{\tau^l(i)} + lr$, so $z_i = z_{\tau^m(i)} + mr = z_i + mr$, since $\tau^m$ fixes all elements of $C_1$. Therefore $r = 0$ and $Z \subseteq H_{i,j}$. $\square$

This proves that no nontrivial extension of the hinge is possible for $\Pi_{n,k}$. So define the *hinge* $h^\sigma$ of $P^\sigma$ to be the intersection of all $H_{j,l}$ for which $j$ and $l$ are in the same cycle of $\sigma$. For an element $Y \in \Pi_{n,k}$, define $\pi(Y)$ to be the partition that corresponds to $Y$. In other words, if $Y \subseteq H_{i,j,r}$ for some $r$, then $i$ and $j$ are in the same block of $\pi(Y)$. Therefore, each $C_i$ is a block of $\pi(h^\sigma)$. Then $\dim(h^\sigma)$ is the number of blocks of $\pi(h^\sigma)$ and the number of cycles of $\sigma$. For example, if $\sigma = (1,2,3)(4,5)(6) \in \mathfrak{S}_6$, then $h^\sigma = H_{1,2} \cap H_{1,3} \cap H_{2,3} \cap H_{4,5}$ in $\mathbb{R}^6$, and $\dim(h^\sigma) = 3$. Notice that by Lemma 2, $h^\sigma \leq X$ for all $X \in \mathrm{Max}(P^\sigma)$, and that $h^\sigma$ is the greatest lower bound of all the maximal elements of $P^\sigma$. This is the element that whose complements we will find in order to prove the main result of this section. Now we are ready to prove one case of (4).

**Theorem 3.** $\mu_\sigma(P^\sigma) = 0$ *unless all disjoint cycles of* $\sigma$ *have the same length.*

**Proof.** We prove this by showing that for a given $X \in \mathrm{Max}(P^\sigma)$, $h^\sigma$ has no complements in $[\hat{0}, X]^\sigma$. If $\sigma$ is an $n$-cycle, then all cycles of $\sigma$ have the same length, and we do not consider that here. Otherwise, given any two blocks $B_1$ and $B_2$ of $\pi(h^\sigma)$, a given element $Z \in P^\sigma$ is a complement of $h^\sigma$ in $[\hat{0}, X]^\sigma$ only if there exists one element from each of the two blocks, say $i \in B_1$ and $j \in B_2$, such that $Z \subseteq H_{i,j,r}$ for some $r$. We need to show that if any two blocks of $\pi(h^\sigma)$ are not the same size, or equivalently, if any two cycles are not the same length, then there is an element less than $Z$ that is not $\hat{0}$ and is also less than $h^\sigma$.

Suppose we pick out two cycles from $\sigma$ that have different lengths. We can assume that $\sigma_1 = (1, ..., m)$ and $\sigma_2 = (m+1, ..., m+b)$, as defined in (7), and $m < b$. In order for $Z$ to be a complement of $h^\sigma$ in $[\hat{0}, X]^\sigma$, $Z$ must be contained in some $H_{1,j,r}$ for some $r$ and for $m+1 \leq j \leq m+b$. So assume without loss of generality that $Z \subseteq H_{1,m+1,r}$, so then $Z \subseteq H_{s,m+s,r}$ for all $s = 1, ..., m$. Let $g = \gcd(m, b)$. Then $g < b$ and $Z \subseteq H_{m+1,m+g+1}$. Let $Y$ be the intersection of all hyperplanes $H_{s-g,s}$ for $m+g < s \leq m+b$. Then $Z \geq Y$ and $\pi(Y)$ is a refinement of $\pi(h^\sigma)$, so since by Lemma 2, only hyperplanes $H_{i,j}$ can contain $h^\sigma$, $h^\sigma \geq Y$ too.

Therefore, $h^\sigma \wedge Z \geq Y > \hat{0}$, so $Z$ is not a complement of $h^\sigma$ in $[\hat{0}, X]^\sigma$. Since we chose an arbitrary $X \in \mathrm{Max}(P^\sigma)$, we have proved that $h^\sigma$ has no complements in any $[\hat{0}, X]^\sigma$. Thus $\mu_\sigma(\hat{0}, X) = 0$ for all $X \in \mathrm{Max}(P^\sigma)$ and therefore, $\mu_\sigma(P^\sigma) = 0$ unless all cycles of $\sigma$ have the same length. □

Now we will find $\mu_\sigma(P^\sigma)$ for the other case of (4), if $\sigma$ is a product of $d$ cycles of length $n/d$. To do this, we may assume that

$$\sigma = (1, 2, ..., j)(j+1, ..., 2j) \cdots (n-j+1, ..., n),$$

where $j = n/d$. Again, for each $X \in \mathrm{Max}(P^\sigma)$, we use complements of the hinge $h^\sigma$ in $[\hat{0}, X]^\sigma$ and equation (5). If $C \in \mathrm{Comp}(h^\sigma)$ in $[\hat{0}, x]^\sigma$, then $C \not\subseteq H_{\omega_1,\omega_2,r}$ for any $\omega_1, \omega_2 \in [j]$ and any $r$, and

$$C \subseteq H_{1,sj+i_s,r_s} \tag{8}$$

for $s = 1, 2, ..., d-1$, and $r_s$ and $i_s \in [j]$ that depend on $s$. Note that $\dim(C) = j$ for all such $C$, so no two complements are comparable to each other. This will be used later to simplify (5). We now state the other case that we will prove, but we need a few lemmas first. Many of the lemmas here are similar to parts of [9, Lemma 6].

**Theorem 4.** *$\mu_\sigma(P^\sigma) = \mu(n/d)(-\frac{n}{d})^{d-1}\binom{(k+1)d-1}{d-1}(d-1)!$ if $\sigma$ is a product of $d$ disjoint cycles of length $n/d$.*

**Lemma 5.** *Given $X \in \mathrm{Max}(P^\sigma)$, if $C \in \mathrm{Comp}(h^\sigma)$ in $[\hat{0}, X]^\sigma$, then $[C, X]^\sigma \cong \mathfrak{D}_j$, the lattice of divisors of $j$. Thus, $\mu_\sigma(C, X) = \mu(n/d)$.*

**Proof.** For any point in an affine subspace from $[C, X]^\sigma$, whatever equality is in the coordinates $1, 2, ..., j$, the same equality holds for corresponding coordinates from the other blocks of $\pi(h^\sigma)$, depending on $i_s$ in (8). At the bottom element $C$ of the interval $[C, X]^\sigma$, for any $i_1, i_2 \in [j]$, $C \not\subseteq H_{i_1,i_2,r}$ for any $r$. At the maximal element, $X \subseteq H_{i_1,i_2}$ by Lemma 2. So $[C, X]^\sigma$ here is isomorphic to $[C, \hat{1}]^\sigma$ in the case $k = 0$. Since [9, Lemma 6c] says that $[C, \hat{1}]^\sigma \cong \mathfrak{D}_{n/d}$, we are done. □

**Lemma 6.** *There exists a one-to-one correspondence between the maximal elements of $P^\sigma$ and the maximal elements of $\Pi_{d,k}$.*

**Proof.** If $d = 1$, then $P^\sigma$ has only one maximal element, and $|\Pi_{1,k}| = 1$. If $d \geq 2$, then by Lemma 2, if $X \in \mathrm{Max}(P^\sigma)$, then $X \subseteq H_{j(i-1)+\omega, j(i-1)+\omega+1}$ for all $i = 1, ..., d$ and all $\omega \in [j-1]$. Then the $X \in \mathrm{Max}(P^\sigma)$ such that $X \subseteq H_{j(\omega_1-1)+1, j(\omega_2-1)+1,r} \subseteq \mathbb{R}^n$ corresponds to the $Y \in \mathrm{Max}(\Pi_{d,k})$ such that $Y \subseteq H_{\omega_1,\omega_2,r} \subseteq \mathbb{R}^d$, and vice-versa. So this correspondence is a bijection. □

**Lemma 7.** *Given a maximal element $X \in P^\sigma$, let $Y$ be its corresponding maximal element in $\Pi_{d,k}$, as described in Lemma 6. If $d \geq 2$, then for all $C \in \mathrm{Comp}(h^\sigma)$ in $[\hat{0}, X]^\sigma$, $[\hat{0}, C]^\sigma \cong [\hat{0}, Y]_{\Pi_{d,k}}$. If $d = 1$, then $C = \hat{0}$ is the only complement. Thus $\mu_\sigma(\hat{0}, C)$ is constant for all $C \leq X$.*

**Proof.** If $d = 1$, then since $h^\sigma$ is the maximal element, $\hat{0}$ is its only complement. If $d \geq 2$, then we must find a bijection between the elements of $[\hat{0}, C]^\sigma \subseteq P^\sigma$ for a given $C \in \mathrm{Comp}(h^\sigma)$ in $[\hat{0}, X]^\sigma$ and $[\hat{0}, Y] \subseteq \Pi_{d,k}$. Suppose $C$ is as in (8), and assume without loss of generality that $i_s = 1$ for all $s$. Then for all $l = 1, ..., j$, $C \subseteq H_{l,sj+l,r_s}$. Given $Z \in [\hat{0}, C]^\sigma$, if $Z \not\subseteq H_{(\omega_1-1)i+1,(\omega_2-1)i+1,r}$ for any $r$, then this corresponds to the element $Z' \in [\hat{0}, y]$ such that $Z' \not\subseteq H_{\omega_1,\omega_2,r}$ for any $r$. If $Z \subseteq H_{(\omega_1-1)i+1,(\omega_2-1)i+1,r}$, then the corresponding $Z' \subseteq H_{\omega_1,\omega_2,r}$. This correspondence can be defined similarly the other way, $Z' \mapsto Z$, so $[\hat{0}, C]^\sigma \cong [\hat{0}, Y]$. Thus $\mu_\sigma(\hat{0}, C) = \mu_{\Pi_{d,k}}(\hat{0}, Y)$ for all complements $C$ of $h^\sigma$ in $[\hat{0}, X]$. □

**Lemma 8.** *Given $X \in \mathrm{Max}(P^\sigma)$, $h^\sigma$ has $(\frac{n}{d})^{d-1}$ complements in $[\hat{0}, X]^\sigma$.*

**Proof.** If $d = 1$, then $h^\sigma$ is the maximal element of $P^\sigma$, so $\hat{0}$ is its only complement. If $d > 1$, then for each $s = 1, 2, ..., d-1$, $i_s$, as described in (8), has $n/d$ possible values, all independent of each other. So the number of complements of $h^\sigma$ is $(\frac{n}{d})^{d-1}$ for $d \geq 1$. □

**Proof of Theorem 4.** Let $C_X$ be some complement of $h^\sigma$ in $[\hat{0}, X]^\sigma$ for each $X \in \mathrm{Max}(P^\sigma)$. Thus:

$$\begin{aligned}
\sum_{X \in \mathrm{Max}(P^\sigma)} \mu_\sigma(\hat{0}, X) &= \sum_X \sum_{\substack{C \in \mathrm{Comp}(h^\sigma) \\ \text{in } [\hat{0},X]^\sigma}} \mu_\sigma(\hat{0}, C)\mu_\sigma(C, X) && (9)\\
&= \mu(n/d) \sum_X \sum_C \mu_\sigma(\hat{0}, C) && (10)\\
&= \mu(n/d) \left(\frac{n}{d}\right)^{d-1} \sum_X \mu_\sigma(\hat{0}, C_X) && \text{(Lemmas 7 and 8)}\\
&= \mu(n/d) \left(\frac{n}{d}\right)^{d-1} \sum_{Y \in \mathrm{Max}(\Pi_{d,k})} \mu_{\Pi_{d,k}}(\hat{0}, Y) && \text{(Lemma 7)}\\
&= \mu(n/d) \left(\frac{n}{d}\right)^{d-1} (-1)^{d-1}(d-1)! \binom{(k+1)d-1}{d-1} && (11)\\
&= \mu(n/d) \left(-\frac{n}{d}\right)^{d-1} \binom{(k+1)d-1}{d-1} (d-1)!
\end{aligned}$$

Equation (9) holds by Crapo's Complementation Theorem. Ordinarily, the sum would be over all $C, C' \in \mathrm{Comp}(h^\sigma)$ such that $C \leq C'$. But no two complements of $h^\sigma$ are comparable, as mentioned right before the statement of this theorem. So the sum is just over all $C \in \mathrm{Comp}(h^\sigma)$.

Equation (10) is true by Lemma 5. Also, $\bigcup_X \{C \in \mathrm{Comp}(h^\sigma) \text{ in } [C, X]^\sigma\}$ has to be a disjoint union. Suppose $C \in \mathrm{Comp}(h^\sigma)$ in both $[\hat{0}, X]^\sigma$ and $[\hat{0}, Y]^\sigma$ for $X \neq Y$. Then there exist $i$ and $j$ such that $X \subseteq H_{i,j,r_1}$ and $Y \subseteq H_{i,j,r_2}$, where $r_1 \neq r_2$. If we let $Z = X \wedge Y$, then $i$ and $j$ are in different blocks of $\pi(Z)$, and $Z \not\subseteq H_{i',j',r}$ for any $r$ and for any $i'$ in the same block as $i$ of $\pi(Z)$ and any $j'$ in the same block as $j$, since $X \subseteq H_{i',j',r_1}$ and $Y \subseteq H_{i',j',r_2}$. So $Z$ cannot be greater than any complement of $h^\sigma$ in $[\hat{0}, X]^\sigma$ or in $[\hat{0}, Y]^\sigma$. But $C \leq X, Y$, which means $C \leq X \wedge Y = Z$, a contradiction. So it is a disjoint union.

To get the result (11), find $\mu(\Pi_{d,k})$ by extracting the coefficient of $t$ in the characteristic polynomial (2). □

# 3 A Homology Character from $\mu_\sigma(P^\sigma)$

Again, let $P = \Pi_{n,k}$. Now we define an integer-valued function $\Psi_{n,k}$ on $\mathfrak{S}_n$ given by

$$\Psi_{n,k}(\sigma) = (-1)^{d-1}\mu_\sigma(P^\sigma), \tag{12}$$

where $d$ is the number of cycles of $\sigma$. Note that the cycles do not all have to be the same length; if they are not, then $\mu_\sigma(P^\sigma) = 0$, so $\Psi_{n,k}(\sigma) = 0$.

We now prove that $\Psi_{n,k}(\sigma)$ is, up to a sign, the character afforded by a linear action of $\mathfrak{S}_n$ on a suitable homology. In fact, the character is $(-1)^{n-1}\mu_\sigma(P^\sigma)$, and $\Psi_{n,k}$ is this character times the sign character of $\mathfrak{S}_n$. We use the methods of Stanley in [13].

Let $Q$ be a poset with $\hat{0}$ and $\hat{1}$, and let $\bar{Q} = Q \setminus \{\hat{0}, \hat{1}\}$. We follow the notation of [16]. The *order complex* $\Delta(\bar{Q})$ is the abstract simplicial complex whose vertex set is $\bar{Q}$ and whose $r$-dimensional faces are all chains of the form $x_0 < x_1 < \cdots < x_r$ in $\bar{Q}$. The dimension of $\Delta(Q)$ is the largest possible value of $r$ for any chain in $\bar{Q}$.

Now $n$ is the number of elements in the largest chain in $Q$, which means $r \le n-3$. So for $r = 0, ..., n-3$, $C_r(\bar{Q})$ is defined to be the vector space over $\mathbb{C}$ whose basis is the $r$-dimensional faces of $\Delta(\bar{Q})$. Also, $C_{-1}(\bar{Q})$ is the one-dimensional vector space generated by the null chain. For all other $r$, $C_r(\bar{Q}) = 0$. For $r = -1, 0, ..., n-3$, the map $\partial_r \colon C_r(\bar{Q}) \longrightarrow C_{r-1}(\bar{Q})$ is a linear map called the *boundary map*, defined as

$$\partial_r(y_0 < y_1 < \cdots < y_r) = \sum_{i=0}^{r} (-1)^i (y_0 < y_1 < \cdots < \hat{y}_i < \cdots < y_r),$$

where $\hat{y}_i$ means that $y_i$ is deleted. The *homology* of $Q$ for each $r$ is

$$H_r(\bar{Q}) = \ker \partial_r / \operatorname{im} \partial_{r+1}. \tag{13}$$

Now suppose $P$ is a poset with least element $\hat{0}$ and maybe more than one maximal element. For any $X \in P$, let $Q^X = [\hat{0}, X]$. We now define a boundary map the same way as above, except that we include the maximal element $X$ in the chains, following the definition in [4, §5]. So $\partial_r$ defined above corresponds to $\partial_{r+1}$ here. We also define the homology the same way as in (13). The $r$-dimensional homology for the boundary map on the order complex is known as the *Whitney homology*, denoted $H_r^W(P)$, which was first defined in [2].

A poset with $\hat{0}$ and $\hat{1}$ is *Cohen-Macaulay* if every interval $I$ has $H_r(I) = 0$ whenever $r \ne \dim(\Delta(I))$. As mentioned earlier, $P = \Pi_{n,k}$ is a geometric semilattice. If we add a unique maximal element to the poset $P = \Pi_{n,k}$, then it is a geometric lattice and therefore Cohen-Macaulay by Theorem 4.1 in [6]. So the homology of each maximal interval in $P$ is concentrated in dimension $n-3$. Therefore, by Theorem 5.1 in [4], the Whitney homology of $P$ is

$$H_{n-2}^W(P) = \bigoplus_{X \in \mathrm{Max}(P)} H_{n-3}(\bar{Q}^X). \tag{14}$$

We use the Lefschetz Fixed Point Theorem (See [13, §1], and it appears that it was first stated in [3]). A version of it states that if the homology of a poset $Q$ is concentrated in a single dimension $r$, then the character of the action of a group $G$ on the homology of $Q$ is $(-1)^r$ times the Möbius function of the poset. Note that in many papers, the dimension of the homology of the null chain is 0. In that case, the character is $(-1)^{r+1}$ times the Möbius function. The sign depends on this.

The action of $\mathfrak{S}_n$ on $P$ induces a canonical action on both the homology and the Whitney homology of $P$. For a Cohen-Macaulay poset $Q$, let $\beta^Q\colon \mathfrak{S}_n \to \mathrm{End}(H(Q))$ be the representation with $\mathfrak{S}_n$ action on $H(Q)$, the non-trivial top-dimensional homology on $Q$, following the notation of [13]. For any linear representation $\alpha$ of $\mathfrak{S}_n$, let $\langle \alpha, \tau \rangle$ be the character of $\alpha$ evaluated at $\tau \in \mathfrak{S}_n$. Then here we have

$$\left\langle \beta^{Q^X}, \sigma \right\rangle = (-1)^{n-1}\mu_\sigma(\hat{0}, X)$$

for any $X \in \mathrm{Max}(P^\sigma)$. Therefore, if we let $\beta^P$ be the representation with $\mathfrak{S}_n$-action on $H^W(P) = H^W_{n-2}(P)$, then using (14) and (6),

$$\begin{aligned} \left\langle \beta^P, \sigma \right\rangle &= \sum_{X \in \mathrm{Max}(P^\sigma)} \left\langle \beta^{Q^X}, \sigma \right\rangle \\ &= \sum_{X \in \mathrm{Max}(P^\sigma)} (-1)^{n-1}\mu_\sigma(\hat{0}, X) = (-1)^{n-1}\mu_\sigma(P^\sigma). \end{aligned}$$

This is the character of the action of $\mathfrak{S}_n$ on $H^W(P)$. If we combine this with (12), we get $\Psi_{n,k}(\sigma) = (-1)^{n-d}\left\langle \beta^P, \sigma \right\rangle$ and hence the following result.

**Theorem 9.** *$\Psi_{n,k}$ is the product of the sign character of $\mathfrak{S}_n$ and the character afforded by the action of $\mathfrak{S}_n$ on the Whitney homology of $\Pi_{n,k}$.*

Note that $(-1)^{n-d} = -1$ only if $n$ is even and $d$ is odd. If $4|n$, then $\mu(n/d) = 0$. Therefore, $\Psi_{n,k}$ is the homology character, and is *self-conjugate* unless $n \equiv 2 \pmod 4$. (This is an extension of [13, Lemma 7.3].)

# 4 $\Psi_{n,k}$ is an Induced Character

Now we know that $\Psi_{n,k}$ is a character, and it is zero for elements of $\mathfrak{S}_n$ that are not conjugate to any elements of $\mathfrak{C}_n$. So a good direction to go now is to prove that $\Psi_{n,k}$ can be represented as a sum of induced characters $\chi \uparrow_{\mathfrak{C}_n}^{\mathfrak{S}_n}$, where $\chi$ is an irreducible character of $\mathfrak{C}_n$. By [13, Lemma 7.2], $\Psi_{n,0}$ is simply the value of the induced character $\psi \uparrow_{\mathfrak{C}_n}^{\mathfrak{S}_n}$, where $\psi$ evaluated at a given $n$-cycle that generates $\mathfrak{C}_n$ is $e^{2\pi i/n}$. We now extend this result to $\Psi_{n,k}$ for $k > 0$.

Let $\sigma$ be an $n$-cycle of $\mathfrak{S}_n$ in $\mathfrak{C}_n$. Let $\zeta = e^{2\pi i/n}$ and for $s = 1, ..., n$, let $\chi_{s,n}$ be the character of $\mathfrak{C}_n$ such that $\chi_{s,n}(\sigma) = \zeta^s$. Then it is known that the $\chi_{s,n}$ are the irreducible characters of $\mathfrak{C}_n$. Let $\chi^\star_{s,n}$ be the induced character $\chi_{s,n} \uparrow^{\mathfrak{S}_n}_{\mathfrak{C}_n}$. If $\gcd(d,n) = \gcd(s,n)$, then it can be verified that $\chi^\star_{s,n} = \chi^\star_{d,n}$, so the goal in this section and the next is to prove that

$$\Psi_{n,k} = \sum_{s|n} a^s_{n,k} \chi^\star_{s,n} \tag{15}$$

for non-negative integers $a^s_{n,k}$. In this section, the main theorem shows that $\Psi_{n,k}$ is an induced character. In the next section, we calculate all the coefficients. It is well-known that if $\zeta$ is a primitive $n$-th root of unity, then

$$\sum_{i=1}^{n} \zeta^{ir} = \begin{cases} n & \text{if } n|r; \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

First, we prove the following lemma.

**Lemma 10.** *The $\chi^\star_{d,n}$ for $d|n$ are linearly independent.*

**Proof.** For $d|n$, let $\lambda_{d,n} = \sum_{i=1}^{n/d} \chi_{di,n}$. Then $\lambda_{d,n} = 1 \uparrow^{\mathfrak{C}_n}_{\mathfrak{C}_d}$, because if we evaluate it at some $\sigma^r$, then we get (16), so $\lambda_{d,n}(\sigma^r) = 0$ unless $\frac{n}{d}|r$, in which case it is $\frac{n}{d}$. Let

$$\nu_{d,n} = \sum_{s|d} \mu(d/s) s \lambda_{s,n}.$$

Given $r$, let $g = \gcd(n,r)$. Then $\lambda_{s,n}(\sigma^r) = 0$ unless $\frac{n}{s}|r$, or equivalently, $\frac{n}{g}|s$, in which case it is $\frac{n}{s}$. So

$$\begin{aligned} \nu_{d,n}(\sigma^r) &= \sum_{s|d} \mu(d/s) s \lambda_{s,n}(\sigma^r) = \sum_{s|d,\, \frac{n}{g}|s} \mu(d/s) s \lambda_{s,n}(\sigma^r) \\ &= n \sum_{s|d,\, \frac{n}{g}|s} \mu(d/s) = n \sum_{s|\frac{dg}{n}} \mu(s) = \begin{cases} n & \text{if } g = \frac{n}{d}; \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

since for any positive integer $m$, $\sum_{d|m} \mu(d) = \delta_{1,m}$. Inducing this up to $\mathfrak{S}_n$,

$$\nu^\star_{d,n}(\sigma^r) = \nu_{d,n} \uparrow^{\mathfrak{S}_n}_{\mathfrak{C}_n} (\sigma^r) = \begin{cases} \frac{n!\phi(d)}{|\mathrm{ccl}_{\mathfrak{S}_n}(\sigma^r)|} & \text{if } \gcd(r,n) = \frac{n}{d} \text{ (or equivalently, } \sigma^r \sim \sigma^{n/d}); \\ 0 & \text{otherwise,} \end{cases}$$

which, if not zero, is the number of $\tau \in \mathfrak{S}_n$ such that $\tau\sigma^r\tau^{-1} \in \mathfrak{C}_n$. Here, $\mathrm{ccl}_{\mathfrak{S}_n}(\sigma)$ is the conjugacy class of $\sigma$ in $\mathfrak{S}_n$. From the $\nu^\star_{d,n}$, we can get the standard basis of class functions that are zero in all classes that do not contain an element of $\mathfrak{C}_n$. Since each $\nu^\star_{d,n}$ can be expressed as a linear combination of the $\chi^\star_{d,n}$, and vice-versa, and the $\nu^\star_{d,n}$ are linearly independent for $d|n$, we can conclude that the $\chi^\star_{d,n}$ are linearly independent too, since there are the same number of $\chi^\star_{d,n}$ as there are $\nu^\star_{d,n}$. □

Thus the $\chi^*_{d,n}$ for $d|n$ are a basis for the induced characters from $\mathfrak{C}_n$ up to $\mathfrak{S}_n$. By Theorem 3 and the last part of the proof of Lemma 10, $\Psi_{n,k}$ can be expressed as a linear combination, as in (15). We now need to show that the $a^s_{n,k}$ are non-negative integers by finding formulas for them. We first determine $a^n_{n,k}$, which can be found by simply using inner products. Then we find the $a^s_{n,k}$ in the next section.

**Lemma 11.** $a^n_{n,k} = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)\binom{(k+1)d-1}{d-1}$ *for* $n \geq 1$.

**Proof.** For $n > 1$, we show that $a^n_{n,k} = \langle \Psi_{n,k}, 1_{\mathfrak{S}_n}\rangle_{\mathfrak{S}_n}$. Using Frobenius Reciprocity [12, Th. 1.12.6],

$$\langle \Psi_{n,k}, 1_{\mathfrak{S}_n}\rangle_{\mathfrak{S}_n} = \sum_{d|n} a^d_{n,k}\left\langle \chi^\star_{d,n}, 1_{\mathfrak{S}_n}\right\rangle_{\mathfrak{S}_n} = \sum_{d|n} a^d_{n,k}\left\langle \chi_{d,n}, 1_{\mathfrak{C}_n}\right\rangle_{\mathfrak{C}_n} = a^n_{n,k}.$$

The last equality is by orthogonality of irreducible characters. Thus the only time $1_{\mathfrak{S}_n}$ appears in the sum (15) is when $d = n$. So for an $n$-cycle $\sigma$,

$$\begin{aligned} a^n_{n,k} = \langle \Psi_{n,k}, 1_{\mathfrak{S}_n}\rangle &= \frac{1}{n!}\sum_{\tau\in\mathfrak{S}_n}\Psi_{n,k}(\tau) \\ &= \frac{1}{n!}\sum_{d|n}\left|\mathrm{ccl}_{\mathfrak{S}_n}(\sigma^d)\right|\ \Psi_{n,k}(\sigma^d) && \text{(Theorem 3)} \\ &= \frac{1}{n!}\sum_{d|n}\left(\frac{n!}{(n/d)^d d!}\right)\mu\left(\frac{n}{d}\right)\left(\frac{n}{d}\right)^{d-1}\binom{(k+1)d-1}{d-1}(d-1)! && \text{(Theorem 4)} \\ &= \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)\binom{(k+1)d-1}{d-1}. \end{aligned}$$

For $n = 1$, it is clear that $\Psi_{1,k} \equiv 1$ and $a^1_{1,k} = 1$. $\square$

Since $\Psi_{n,k}$ is a character of $\mathfrak{S}_n$, it follows by the first assertion in the proof of Lemma 11 that $a^n_{n,k}$ is a non-negative integer. For $d|n$, let $\mathfrak{C}_{n/d} = \left\langle\sigma^d\right\rangle$, and let $\chi_{s,\frac{n}{d}}$ for $s \leq \frac{n}{d}$ be an irreducible character for $\mathfrak{C}_{n/d}$ such that $\chi_{s,\frac{n}{d}}(\sigma^d) = \zeta^{sd}$ for $1 \leq s \leq \frac{n}{d}$. Now let $\chi^{\mathrm{reg}}_{d,n} = \chi_{1,\frac{n}{d}}\uparrow^{\mathfrak{S}_n}_{\mathfrak{C}_{n/d}}$. Notice that $\chi^{\mathrm{reg}}_{n,n}$ is the regular character for $\mathfrak{S}_n$ and $\chi^{\mathrm{reg}}_{1,n} = \chi^\star_{1,n}$. Also, note that the $\chi^{\mathrm{reg}}_{d,n}$ need not be linearly independent. Now we use the next lemma to prove the main result of this section, that $\Psi_{n,k}$ is an induced character from $\mathfrak{C}_n$, because each $\chi^{\mathrm{reg}}_{d,n}$ is.

**Lemma 12.** *Let $\sigma$ be an $n$-cycle in $\mathfrak{S}_n$. Then we have the following identity:*

$$\frac{1}{d}\chi^{\mathrm{reg}}_{d,n}(\sigma^r) = \begin{cases}\chi^\star_{1,n}(\sigma^r) & \textit{if } d|r \\ 0 & \textit{otherwise.}\end{cases}$$

**Proof.** We know that $\frac{1}{d}\chi^{\mathrm{reg}}_{d,n}(\sigma^r) = \frac{1}{d}\chi_{1,\frac{n}{d}} \uparrow^{\mathfrak{S}_n}_{\mathfrak{C}_{n/d}} (\sigma^r)$ and $\chi^{\star}_{1,n} = \chi_{1,n} \uparrow^{\mathfrak{S}_n}_{\mathfrak{C}_n}$. Since induction of representations is transitive, it is enough to show that

$$\frac{1}{d}\chi_{1,\frac{n}{d}} \uparrow^{\mathfrak{C}_n}_{\mathfrak{C}_{n/d}} (\sigma^r) = \begin{cases} \chi_{1,n}(\sigma^r) & \text{if } d|r \\ 0 & \text{otherwise.} \end{cases}$$

$d|r$ if and only if $\sigma^r \in \mathfrak{C}_{n/d}$, and then both sides are $\zeta^r$. We induce both sides up to $\mathfrak{S}_n$, and we are done. □

**Theorem 13.** $\Psi_{n,k} = \sum_{d|n} a^d_{d,k}\chi^{\mathrm{reg}}_{d,n}$

**Proof.** Let $\mathfrak{H}_{k,s} = \binom{(k+1)s-1}{s-1}$. We know that

$$\Psi_{n,k}(\tau) = \begin{cases} \mu(d)(\frac{n}{d})^{d-1}(d-1)!\mathfrak{H}_{k,d} & \text{if } \tau \text{ is a product of } d \text{ cycles} \\ & \text{of length } n/d \text{ for some } d|n; \\ 0 & \textit{otherwise.} \end{cases}$$

Again assume that $d|n$ and $\sigma$ is an $n$-cycle such that $\mathfrak{C}_n = \langle \sigma^d \rangle$.

$$\sum_{r|n} a^r_{r,k}\chi^{\mathrm{reg}}_{r,n}\left(\sigma^d\right) = \sum_{r|n}\left(\frac{1}{r}\sum_{s|r}\mu\left(\frac{r}{s}\right)\mathfrak{H}_{k,s}\right)\chi^{\mathrm{reg}}_{r,n}\left(\sigma^d\right) \qquad \text{(Lemma 11)}$$

$$= \sum_{r|n}\frac{1}{r}\chi^{\mathrm{reg}}_{r,n}\left(\sigma^d\right)\sum_{s|r}\mu\left(\frac{r}{s}\right)\mathfrak{H}_{k,s} = \chi^{\star}_{1,n}(\sigma^d)\sum_{r|d}\sum_{s|r}\mu\left(\frac{r}{s}\right)\mathfrak{H}_{k,s}$$

(Lemma 12)

$$= \chi^{\star}_{1,n}(\sigma^d)\sum_{s|d}\mathfrak{H}_{k,s}\sum_{r|d,s|r}\mu\left(\frac{r}{s}\right) = \chi^{\star}_{1,n}(\sigma^d)\sum_{s|d}\mathfrak{H}_{k,s}\delta_{d,s} \qquad (17)$$

$$= \mu(n/d)(n/d)^{d-1}(d-1)!\mathfrak{H}_{k,d} \qquad (18)$$

The second equality of (17) holds because given $s$,

$$\sum_{r|d,s|r}\mu\left(\frac{r}{s}\right) = \sum_{q|\frac{d}{s}}\mu(q) = \delta_{d,s}.$$

For equation (18), [13, Lemma 7.2] proves that $\chi^{\star}_{1,n}(\sigma^d) = \Psi_{n,0}(\sigma^d)$. Thus we have proved that the right-hand side is equal to a value that we know is $\Psi_{n,k}$. □

Given a real number $x$, let $\langle x\rangle$ be the nearest integer to $x$. Recall that the irreducible characters of $\mathfrak{S}_n$ are the $\chi^\lambda$ for each partition $\lambda$ of $n$, and $f^\lambda$ is the degree of $\chi^\lambda$. We now have a way to decompose $\Psi_{p,k}$ into irreducible characters of $\mathfrak{S}_p$ for any odd prime $p$. The following corollary is an extension of a result by Stanley.

**Corollary 14.** *Given an odd prime $p$, and a partition $\lambda$ of $p$,*

$$\left\langle \Psi_{p,k}, \chi^\lambda \right\rangle_{\mathfrak{S}_n} = a^p_{p,k} f^\lambda + \left\langle f^\lambda/p \right\rangle.$$

**Proof.** By Theorem 13,

$$\Psi_{p,k} = a^1_{1,k}\chi^*_{1,p} + a^p_{p,k}\chi^{\text{reg}}_{p,p}.$$

Therefore, since $a^1_{1,k} = 1$ for all $k$, we get

$$\left\langle \Psi_{p,k}, \chi^\lambda \right\rangle_{\mathfrak{S}_n} = a^p_{p,k}\left\langle \chi^{\text{reg}}_{p,p}, \chi^\lambda \right\rangle + \left\langle \chi^*_{1,p}, \chi^\lambda \right\rangle = a^p_{p,k} f^\lambda + \left\langle f^\lambda/p \right\rangle.$$

The first summand is well-known about the regular character of $\mathfrak{S}_n$ (For example, see Theorem 1.10.1 of [12]). The second summand is by Corollary 7.4 of [13]. □

# 5 The Coefficients $a^s_{n,k}$

By Theorem 13, $\Psi_{n,k}$ is an induced character from $\mathfrak{C}_n$ up to $\mathfrak{S}_n$. For the sake of completeness, we now determine the coefficients $a^s_{n,k}$. This also makes it more clear how to decompose $\Psi_{n,k}$ into irreducibles if $n$ is not prime, since there are references that have the decomposition of induced characters from $\mathfrak{C}_n$. See for Example [15, §3], where it is done in terms of standard Young tableaux. By the next lemma, each $a^s_{n,k}$ is a sum of non-negative integers $a^d_{d,k}$ for each $d$ such that $\chi^\star_{d,n}$ appears in the expression of $\chi^{\text{reg}}_{s,n}$. All we need to do now is determine what $a^s_{n,k}$ is for $s$ properly dividing $n$.

**Lemma 15.** *For $d|n$, $\chi^{\text{reg}}_{d,n} = \sum_{i=0}^{d-1} \chi^\star_{i\frac{n}{d}+1,n}$.*

*Sketch of Proof.* It is enough to show that $\chi_{1,\frac{n}{d}} \uparrow^{\mathfrak{C}_n}_{\mathfrak{C}_{n/d}} = \sum_{i=0}^{d-1} \chi_{i\frac{n}{d}+1,n}$. Just evaluate both sides at $\sigma^r$, and then induce up to $\mathfrak{S}_n$. □

This shows that for a given $k$, $a^s_{n,k}$ can be expressed in terms of the numbers $a^d_{d,k}$. So let $A^s_n$ be the infinite sequence $\{a^s_{n,k}\}^\infty_{k=0}$. Since the expression of $a^s_{n,k}$ in terms of the numbers $a^d_{d,k}$ is independent of $k$, this is a way to write these numbers so that we do not have to write $k$ all the time. We need to find a formula for these sequences, now that we know the $A^d_d$ sequences for $d|n$. Define the sum of two sequences and scalar multiplication of a sequence the usual way. By theorem 13, $a^d_{d,k}$ is the coefficient of $\chi^{\text{reg}}_{d,n}$ in $\Psi_{n,k}$, so by Lemma 15, it appears in the coefficient of $\chi^\star_{(i-1)\frac{n}{d}+1,n}$ for $i = 1, 2, ..., d$, given $k$. Also, if $s = \gcd(g,n)$ for some $g$, then $\chi^\star_{s,n} = \chi^\star_{g,n}$, so each $A^s_n$ sequence can be written as a non-negative integer combination of the $A^d_d$ sequences for which $a^d_{d,k}$ appears in the coefficient of $\chi^\star_{s,n}$, and that is for each $d$ such that there exists an $r \equiv 1 \pmod{\frac{n}{d}}$ with $\gcd(r,n) = s$. In this case, $\gcd(s, \frac{n}{d}) = \gcd(r, n, \frac{n}{d}) = 1$ and $s|d$. Our goal is to find how many of these $r$ exist up to modulo $n$, and that will be the coefficient of $A^d_d$ in $A^s_n$. First we prove a lemma from number theory that we will use.

**Lemma 16.** *Given positive integers $a, b, t$ such that $\gcd(a,b) = 1$, there exists an integer $u$ such that $0 \le u < t$ and $\gcd(ua+b, t) = 1$.*

**Proof.** We form an increasing sequence $a_0, a_1, ..., a_t$ of integers that are pairwise relatively prime, or *coprime*. Let $a_0 = a$ and $a_1 = b$. For $m \ge 1$, let

$$a_{m+1} = a_m + a_0 a_1 ... a_{m-1}.$$

We prove that these numbers are coprime by induction. First, it is given that $a_0$ and $a_1$ are relatively prime. Now assuming that $a_0, ..., a_m$ are coprime, we use the Euclidean algorithm to prove that $a_{m+1}$ is relatively prime to all $a_i$ for $i = 0, 1, ..., m$. Now $\gcd(a_m, a_{m+1}) = \gcd(a_m, a_0 ... a_{m-1})$, and for $i < m$,

$$\gcd(a_i, a_{m+1}) = \gcd(a_i, a_m).$$

By hypothesis, both are 1, so the numbers in the sequence are all coprime. To prove that the $a_m$ are of the form $u'a+b$ for some integer $u'$ and $m \ge 1$, we know that it holds for $m = 1$; it is $0a+b$. Now assume that it holds for some $m \ge 1$ and prove it for $m+1$. We need to show that $a|(a_{m+1} - b)$. We know that $a_{m+1} - b = a_m - b + a_0 ... a_{m-1}$. We also know that $a$ divides $a_m - b$ by hypothesis, and $a$ divides $a_0 ... a_{m-1}$ because $a = a_0$. This proves that $a_1, ..., a_t$ are $t$ coprime numbers, all congruent to $b$ modulo $a$, and therefore there must be at least one of these numbers that is relatively prime to $t$. Suppose $a_i$ is one such number. Let $u' = \frac{a_i - b}{a}$. In other words, $a_i = u'a + b$. Then there exists a unique expression $u' = u''t + u$ such that $0 \le u < t$. It follows that $\gcd(ua+b, t) = 1$. □

The number of $r$ up to modulo $n$ such that $r \equiv 0 \pmod{s}$ is $n/s$, and the number of such $r$ with $\gcd(r,n) = s$ is $\phi(n/s)$. Now we find how many of these $r$ have $r \equiv 1 \pmod{\frac{n}{d}}$. Let $t_1$ and $t_2$ be units in the ring $\mathbb{Z}/(\frac{n}{d})$, $t_1$ chosen so that there definitely exists an $r$ among these numbers such that $r \equiv t_1 \pmod{\frac{n}{d}}$ ($t_1$ can be $s$, for example). Suppose that $r_1, ..., r_w$ are the numbers mod $n$ such that

$$r_i \equiv t \left(\mathrm{mod}\, \frac{n}{d}\right) \quad \text{and} \quad \gcd(r_i, n) = s. \tag{19}$$

for $t = t_1$. In $\mathbb{Z}/(n)$, $t_2$ is not necessarily a unit, but by Lemma 16, there exists a $u$ such that $\gcd(t_2 + u(\frac{n}{d}), n) = 1$. Then $(t_2 + u(\frac{n}{d}))r_1, ..., (t_2 + u(\frac{n}{d}))r_w$ are all congruent to $t_1 t_2$ mod $n/d$ and distinct mod $n$, since the $r_i$ were all multiplied by a unit of $\mathbb{Z}/(n)$. This proves that there are at least $w$ of them. To prove there are exactly $w$, consider $t_2^{-1}$ in $\mathbb{Z}/(\frac{n}{d})$ and suppose there are $w'$ numbers, $w' \geq w$. By Lemma 16, there exists a $u'$ such that $\gcd(t_2^{-1} + u'(\frac{n}{d}), n) = 1$. Then the numbers $(t_2^{-1} + u'(\frac{n}{d}))(t_2 + u(\frac{n}{d}))r_i$ are all congruent to $t_1$ mod $n/d$, and there are at least $w'$ such numbers. Since there are $w$ of them, $w = w'$. So we can let $t_2 = t_1^{-1}$, and then we have $w$ numbers that are congruent to 1 mod $n/d$. Thus for each unit $t$ mod $n/d$, there are $w$ numbers up to mod $n$ that satisfy (19), so one out of every $\phi(n/d)$ of the numbers $r$ is congruent to 1 mod $n/d$. Therefore,

$$A_n^s = \sum_d \frac{\phi(n/s)}{\phi(n/d)} A_d^d, \tag{20}$$

summing over all $d$ such that $a_{d,k}^d$ appears in the coefficient of $\chi_{s,n}^\star$.

**Example.** Suppose $n = 72$ and $s = 3$. Then the values of $d$ that would have nonzero coefficients in (20) are $d = 9, 18, 36, 72$. We do $d = 9$ in detail. Let $t_1 = 3$ in $\mathbb{Z}/(8)$. The values of $r$ such that $\gcd(r, 72) = 3$ are $3, 15, 21, 33, 39, 51, 57, 69$, and those $r$ such that $r \equiv 3 \pmod 8$ are 3 and 51, so $w = 2$. Then $t_1^{-1} \equiv 3 \pmod 8$, but 3 is not a unit in $\mathbb{Z}/(72)$. However, by adding 8 ($u = 1$), we get 11, which is a unit. So multiply 3 and 51 by 11 to get 33 and 57, respectively, mod 72. These are the numbers $r_i$ described in (19) for $t = 1$. Therefore, the coefficient of $A_9^9$ in $A_{72}^3$ is 2 and $\phi(n/s) = \phi(24) = 8$ and $\phi(n/d) = \phi(8) = 4$, which verifies (20) here. The expression for $A_{72}^3$ in terms of the $A_d^d$ is

$$A_{72}^3 = 2A_9^9 + 4A_{18}^{18} + 8A_{36}^{36} + 8A_{72}^{72}.$$

There is another equivalent condition on the $d$ that can be used. With this, we will have a quicker way of obtaining which values of $d$ have nonzero terms in the sum. We have the following lemma.

**Lemma 17.** *Given integers $s$ and $d$ that divide $n$, there exists an $r \equiv 1 \pmod d$ such that $\gcd(r, n) = s$ if and only if $\gcd(s, d) = 1$.*

**Proof.** To prove ($\Longrightarrow$), if such an $r$ exists, then since $s|r$, $\gcd(s, d) = 1$.

To prove ($\Longleftarrow$), if $\gcd(s, d) = 1$, then it follows that $s|\frac{n}{d}$, so it's enough to show there exists an $r \equiv 1 \pmod d$ such that $\gcd(r, \frac{n}{d}) = s$. There exists a $t$ such that $0 < t < d$ and $st \equiv 1 \pmod d$. This equation also holds if $t$ is replaced by $id + t$ for $i = 0, 1, ..., \frac{n}{d} - 1$. By Lemma 16, $\gcd(id + t, \frac{n}{d}) = 1$ for at least one of these $i$, and then we let $r = s(id + t)$. □

Now in order to find which $d$ have nonzero summands, we only need to find those divisors of $n$ that are relatively prime to $s$, and then divide $n$ by them. Thus we have the result.

**Theorem 18.** *For $s|n$ and some integer $k \geq 0$,*

$$a^s_{n,k} = \sum_{\substack{d|n \\ \gcd(s,\frac{n}{d})=1}} \frac{\phi(n/s)}{\phi(n/d)} a^d_{d,k} = \sum_d \frac{\phi(n/s)}{d\phi(n/d)} \sum_{t|d} \mu\left(\frac{d}{t}\right) \binom{(k+1)t-1}{t-1}.$$

# 6 Another Proof that $\Psi_{n,k}$ is a Character

It is also possible to prove that $\Psi_{n,k}$ is a character of $\mathfrak{S}_n$ without using homology of posets, as we did in section 3. In section 4, the only time the homology result was used was when it was found that $\Psi_{n,k}$ is a character. From this, it was concluded that the $a^d_{d,k}$ are all non-negative integers because of Lemma 11. By Theorem 13,

$$\Psi_{n,k} = \sum_{d|n} a^d_{d,k} \chi^{\text{reg}}_{d,n}.$$

If we can show that $a^n_{n,k}$ is a non-negative integer for each $n$, then we will have shown that $\Psi_{n,k}$ is a sum of characters of $\mathfrak{S}_n$, and therefore, a character itself. By Lemma 11,

$$a^n_{n,k} = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \binom{(k+1)d-1}{d-1}. \tag{21}$$

**Lemma 19.** *For positive integers $n$ and $r$, and $k = r-1$, $a^n_{n,k}$ is an integer.*

**Proof.** Let $p$ be a prime divisor of $n$, and let $s$ be the positive integer such that $p^s \parallel n$. Then

$$\begin{aligned} na^n_{n,k} &= \sum_{d|\frac{n}{p^s}} \left(\mu\left(\frac{n}{p^{s-1}d}\right)\binom{p^{s-1}rd-1}{p^{s-1}d-1} + \mu\left(\frac{n}{p^s d}\right)\binom{p^s rd-1}{p^s d-1}\right) \\ &= \sum_{d|\frac{n}{p^s}} \mu\left(\frac{n}{p^s d}\right)\left(\binom{p^s rd-1}{p^s d-1} - \binom{p^{s-1}rd-1}{p^{s-1}d-1}\right), \end{aligned}$$

since the other summands are definitely zero. We show that each summand is congruent to 0 mod $p^s$.

$$\begin{aligned} \binom{p^s rd-1}{p^s d-1} &= \frac{(p^s rd-1)(p^s rd-2)\cdots(p^s(r-1)d+1)}{(p^s d-1)(p^s d-2)\cdots 3\cdot 2\cdot 1} \\ &= A\frac{(p^s rd-p)(p^s rd-2p)\cdots(p^s(r-1)d+p)}{(p^s d-p)(p^s d-2p)\cdots 3p\cdot 2p\cdot p} \\ &= A\frac{(p^{s-1}rd-1)(p^{s-1}rd-2)\cdots(p^{s-1}(r-1)d+1)}{(p^{s-1}d-1)!} = A\binom{p^{s-1}rd-1}{p^{s-1}d-1} \end{aligned}$$

Here, $A$ is a product of fractions of the form $\frac{p^s rd-a}{p^s d-a}$ for all $a$ relatively prime to $p$. Since the numerator and denominator of each of these fractions are units in $\mathbb{Z}/(p^s)$, all the fractions are equivalent to 1 mod $p^s$. Thus $A \equiv 1 \pmod{p^s}$. This proves that

$$\binom{p^s rd-1}{p^s d-1} - \binom{p^{s-1}rd-1}{p^{s-1}d-1} \equiv 0 \pmod{p^s},$$

and therefore $na^n_{n,k} \equiv 0 \pmod{p^s}$ for each prime divisor $p$ of $n$. Therefore, by the Chinese Remainder Theorem, $na^n_{n,k} \equiv 0 \pmod{n}$, and this proves the lemma. □

**Lemma 20.** *For $n \geq 2$, $a^n_{n,0} = 0$, and for $k \geq 1$, $a^n_{n,k} > 0$.*

**Proof.** We know the case $k = 0$. Using [13, Lemma 7.2], $a^1_{n,0} = 1$ and $a^d_{n,0} = 0$ for $d > 1$. For $k \geq 1$, let $r = k+1$. Given a prime divisor $p$ of $n$, let $q_p = n/p$. It is well-known that summing over prime divisors of $n$, $\sum_{p|n} \frac{1}{p^2} < 1$ (This inequality also holds when summed over all primes). So we will prove that

$$\binom{nr-1}{n-1} > p^2 \binom{q_p r-1}{q_p-1}. \tag{22}$$

If we let $(a)_b = b!\binom{a}{b}$ for integers $a$ and $b$, then this is equivalent to

$$(nr-1)_{n-1} > p^2(n-1)_{n-q_p}(q_p r-1)_{q_p-1}.$$

Since $pq_p = n$, it follows that $p^2(q_p r-1)(q_p r-2) < (nr-1)(nr-2)$. Thus it is enough to show that

$$(nr-3)_{n-3} > (n-1)_{n-q_p}(q_p r-3)_{q_p-3}.$$

The left side is equal to $(nr-3)_{q_p-3}(nr-q_p)_{n-q_p}$. It is clear that $nr-3 > q_p r-3$ and $nr-q_p > n-1$ (since $r > 1$). Thus (22) is true, so for each prime $p|n$, $\frac{1}{p^2}\binom{nr-1}{n-1} > \binom{q_p r-1}{q_p-1}$. Therefore,

$$\binom{nr-1}{n-1} > \left(\sum_{\substack{p|n \\ p \text{ prime}}} \frac{1}{p^2}\right)\binom{nr-1}{n-1} > \sum_{\substack{p|n \\ p \text{ prime}}} \binom{q_p r-1}{q_p-1}$$

This inequality holds for each integer $n$. In (21), if one term $\mathfrak{H}_{k,d}$ (cf. proof of Theorem 13) is subtracted, then $\mathfrak{H}_{k,pd}$ is added for some prime $p$, which is a lot larger than $\mathfrak{H}_{k,d}$ by (22). Also, the coefficient of $\mathfrak{H}_{k,n}$ is always 1. Therefore, $a^n_{n,k} > 0$. □

Thus we have proved the following, which by Theorem 13 proves that $\Psi_{n,k}$ is a character of $\mathfrak{S}_n$

**Theorem 21.** *$a^n_{n,k}$ is a non-negative integer for all $n \geq 2$, and it is zero only if $k = 0$.*

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