An Inequality Related to Vizing's Conjecture

W. Edwin Clark and Stephen Suen
Department of Mathematics, University of South Florida,
Tampa, FL 33620-5700, USA
eclark@math.usf.edu suen@math.usf.edu

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Abstract

Let $\gamma(G)$ denote the domination number of a graph G and let $G \,\square\, H$ denote the Cartesian product of graphs G and H. We prove that $\gamma(G)\gamma(H) \leq 2\gamma(G\square H)$ for all simple graphs G and H.

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We use $V(G)$, $E(G)$, $\gamma(G)$, respectively, to denote the vertex set, edge set and domination number of the (simple) graph G. For a pair of graphs G and H, the Cartesian product $G \,\square\, H$ of G and H is the graph with vertex set $V(G) \times V(H)$ and where two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. In 1963, V. G. Vizing [2] conjectured that for any graphs G and H,

$$\gamma(G)\gamma(H) \leq \gamma(G \,\square\, H). \tag{1}$$

The reader is referred to Hartnell and Rall [1] for a summary of recent progress on Vizing's conjecture. We note that there are graphs G and H for which equality holds in (1). However, it was previously unknown [1] whether there exists a constant c such that

$$\gamma(G)\gamma(H) \leq c\,\gamma(G \,\square\, H).$$

We shall show in this note that $\gamma(G)\gamma(H) \leq 2\,\gamma(G \,\square\, H)$.

For $S \subseteq V(G)$ we let $N_G[S]$ denote the set of vertices in $V(G)$ that are in S or adjacent to a vertex in S, *i.e.*, the set of vertices in $V(G)$ dominated by vertices in S.

Theorem 1 *For any graphs G and H,*

$$\gamma(G)\gamma(H) \le 2\gamma(G \,\square\, H).$$

Proof. Let D be a dominating set of $G \,\square\, H$. It is sufficient to show that

$$\gamma(G)\gamma(H) \le 2|D|. \tag{2}$$

Let $\{u_1, u_2, \ldots, u_{\gamma(G)}\}$ be a dominating set of G. Form a partition $\{\Pi_1, \Pi_2, \ldots, \Pi_{\gamma(G)}\}$ of $V(G)$ so that for all i: (i) $u_i \in \Pi_i$, and (ii) $u \in \Pi_i$ implies $u = u_i$ or u is adjacent to u_i. This partition of $V(G)$ induces a partition $\{D_1, D_2, \ldots, D_{\gamma(G)}\}$ of D where

$$D_i = (\Pi_i \times V(H)) \cap D.$$

Let P_i be the projection of D_i onto H. That is,

$$P_i = \{v \,|\, (u, v) \in D_i \text{ for some } u \in \Pi_i\}.$$

Observe that for any i, $P_i \cup (V(H) - N_H[P_i])$ is a dominating set of H, and hence the number of vertices in $V(H)$ not dominated by P_i satisfies the inequality

$$|V(H) - N_H[P_i]| \ge \gamma(H) - |P_i|. \tag{3}$$

For $v \in V(H)$, let

$$Q_v = D \cap (V(G) \times \{v\}) = \{(u, v) \in D \,|\, u \in V(G)\}.$$

and C be the subset of $\{1, 2, \ldots, \gamma(G)\} \times V(H)$ given by

$$C = \{\, (i, v) \,|\, \Pi_i \times \{v\} \subseteq N_{G\square H}[\, Q_v \,]\, \}.$$

Let $N = |C|$. By counting in two different ways we shall find upper and lower bounds for N. Let

$$\begin{aligned} L_i &= \{(i, v) \in C \,|\, v \in V(H)\}, \text{ and} \\ R_v &= \{(i, v) \in C \,|\, 1 \le i \le \gamma(G)\}. \end{aligned}$$

Clearly

$$N = \sum_{i=1}^{\gamma(G)} |L_i| = \sum_{v \in V(H)} |R_v|.$$

Note that if $v \in V(H) - N_H[P_i]$, then the vertices in $\Pi_i \times \{v\}$ must be dominated by vertices in Q_v and therefore $(i, v) \in L_i$. This implies that $|L_i| \ge |V(H) - N_H[P_i]|$. Hence

$$N \ge \sum_{i=1}^{\gamma(G)} |V(H) - N_H[P_i]|$$

and it follows from (3) that

$$\begin{aligned} N &\geq \gamma(G)\gamma(H) - \sum_{i=1}^{\gamma(G)} |P_i| \\ &\geq \gamma(G)\gamma(H) - \sum_{i=1}^{\gamma(G)} |D_i|. \end{aligned}$$

So we obtain the following lower bound for N.

$$N \geq \gamma(G)\gamma(H) - |D|. \tag{4}$$

For each $v \in V(H)$, $|R_v| \leq |Q_v|$. If not,

$$\{\, u \mid (u, v) \in Q_v \} \cup \{\, u_j \mid (j, v) \notin R_v \}$$

is a dominating set of G with cardinality

$$|Q_v| + (\gamma(G) - |R_v|) = \gamma(G) - (|R_v| - |Q_v|) < \gamma(G),$$

and we have a contradiction. This observation shows that

$$N = \sum_{v \in V(H)} |R_v| \leq \sum_{v \in V(H)} |Q_v| = |D|. \tag{5}$$

It follows from (4) and (5) that

$$\gamma(G)\gamma(H) - |D| \leq N \leq |D|,$$

and the desired inequality (2) follows. ■

References

[1] Bert Hartnell and Douglas F. Rall, Domination in Cartesian Products: Vizing's Conjecture, in *Domination in Graphs—Advanced Topics* edited by Haynes, *et al*, Marcel Dekker, Inc, New York, 1998, 163–189.

[2] V. G. Vizing, The cartesian product of graphs, *Vyčisl. Sistemy* **9**, 1963, 30–43.