Poset homology of Rees products, and q-Eulerian polynomials

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Dedicated to Anders Björner on the occasion of his 60th birthday

Abstract

The notion of Rees product of posets was introduced by Björner and Welker in [8], where they study connections between poset topology and commutative algebra. Björner and Welker conjectured and Jonsson [25] proved that the dimension of the top homology of the Rees product of the truncated Boolean algebra $B_n \setminus \{0\}$ and the n-chain C_n is equal to the number of derangements in the symmetric group $\mathfrak{S}_n$. Here we prove a refinement of this result, which involves the Eulerian numbers, and a q-analog of both the refinement and the original conjecture, which comes from replacing the Boolean algebra by the lattice of subspaces of the n-dimensional vector space over the q element field, and involves the $(\mathrm{maj}, \mathrm{exc})$-$q$-Eulerian polynomials studied in previous papers of the authors [32, 33]. Equivariant versions of the refinement and the original conjecture are also proved, as are type BC versions (in the sense of Coxeter groups) of the original conjecture and its q-analog.

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Contents

1 Introduction and statement of main results

In their study of connections between topology of order complexes and commutative algebra in [8], Björner and Welker introduced the notion of Rees product of posets, which is a combinatorial analog of the Rees construction for semigroup algebras. They stated a conjecture that the Möbius invariant of a certain family of Rees product posets is given by the derangement numbers. Our investigation of this conjecture led to a surprising new q-analog of the classical formula for the exponential generating function of the Eulerian polynomials, which we proved in [33] by establishing certain quasisymmetric function identities. In this paper, we return to the original conjecture (which was first proved by Jonsson [25]). We prove a refinement of the conjecture, which involves Eulerian polynomials, and we prove a q-analog and equivariant version of both the conjecture and its refinement, thereby connecting poset topology to the subjects studied in our earlier paper.

The terminology used in this paper is explained briefly here and more fully in Section 2. All posets are assumed to be finite.

Given ranked posets P, Q with respective rank functions r_P, r_Q, the *Rees product* $P * Q$ is the poset whose underlying set is

$$\{(p, q) \in P \times Q : r_P(p) \geq r_Q(q)\},$$

with order relation given by $(p_1, q_1) \leq (p_2, q_2)$ if and only if all of the conditions

- $p_1 \leq_P p_2$,
- $q_1 \leq_Q q_2$, and
- $r_P(p_1) - r_P(p_2) \geq r_Q(q_1) - r_Q(q_2)$

hold. In other words, (p_2, q_2) covers (p_1, q_1) in $P * Q$ if and only if p_2 covers p_1 in P and either $q_2 = q_1$ or q_2 covers q_1 in Q.

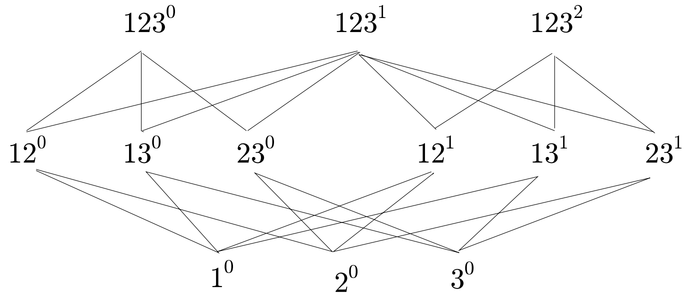

Figure 1. $(B_3 \setminus \{\emptyset\}) * C_3$

Let B_n be the Boolean algebra on the set $[n] := \{1, \ldots, n\}$ and C_n be the chain $\{0 < 1 < \ldots < n-1\}$. This paper is concerned with the Rees product $(B_n \setminus \{\emptyset\}) * C_n$ and various analogs. The Hasse diagram of $(B_3 \setminus \{\emptyset\}) * C_3$ is given in Figure 1 (the pair (S, j) is written as S^j with set brackets omitted).

Recall that for a poset P, the *order complex* ΔP is the abstract simplicial complex whose vertices are the elements of P and whose k-simplices are totally ordered subsets of size $k+1$ from P. The (reduced) homology of P is given by $\tilde{H}_k(P) := \tilde{H}_k(\Delta P; \mathbb{C})$. A poset P is said to be *Cohen-Macualay* if the homology of each open interval of $P \cup \{\hat{0}, \hat{1}\}$ is concentrated in its top dimension, where $\hat{0}$ and $\hat{1}$ are respective minimum and maximum elements appended to P. A poset is said to be *acyclic* if its homology is trivial in all dimensions. Björner and Welker [8, Corollary 2] prove that the Rees product of any Cohen-Macaulay poset with any acyclic Cohen-Macaualy poset is Cohen-Macaulay. Hence $(B_n \setminus \{\emptyset\}) * C_n$ is Cohen-Macaulay, since both $B_n \setminus \{\emptyset\}$ and C_n are Cohen-Macaulay and C_n is acyclic.

For any poset P with a minimum element $\hat{0}$, let P^- denote the truncated poset $P \setminus \{\hat{0}\}$. The theorem of Jonsson as conjectured by Björner and Welker in [8] is as follows.

Theorem 1.1 (Jonsson [25])**.** *We have*

$$\dim \tilde{H}_{n-1}(B_n^- * C_n) = d_n,$$

where d_n is the number of derangements (fixed-point-free elements) in the symmetric group $\mathfrak{S}_n$.

Our refinement of Theorem 1.1 is Theorem 1.2 below. Indeed, Theorem 1.1 follows immediately from Theorem 1.2, the Euler characteristic interpretation of the Mobius function, the recursive definition of the Mobius function, and the well-known formula

$$d_n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} (n-m)! \,. \tag{1.1}$$

Let P be a ranked and bounded poset of length n with minimum element $\hat{0}$ and maximum element $\hat{1}$. The maximal elements of $P^- * C_n$ are of the form $(\hat{1}, j)$, for $j = 0 \ldots, n-1$. Let $I_j(P)$ denote the open principal order ideal generated by $(\hat{1}, j)$. If P is Cohen-Macaulay then the homology of the order complex of $I_j(P)$ is concentrated in dimension $n-2$.

Theorem 1.2. *For all $j = 0, \dots, n-1$, we have*

$$\dim \tilde{H}_{n-2}(I_j(B_n)) = a_{n,j},$$

where $a_{n,j}$ is the Eulerian number indexed by n and j; that is $a_{n,j}$ is the number of permutations in $\mathfrak{S}_n$ with j descents, equivalently with j excedances.

We have obtained two different proofs of Theorem 1.2 both as applications of general results on Rees products that we derive. One of these proofs, which appears in [34], involves the theory of lexicographical shellability [3]. The other, which is given in Sections 3 and 4, is based on the recursive definition of the Möbius function applied to the Rees product of B_n with a poset whose Hasse diagram is a tree. This proof yields a *q-analog* (Theorem 1.3) of Theorem 1.2, in which the Boolean algebra B_n is replaced by its q-analog, $B_n(q)$, the lattice of subspaces of the n-dimensional vector space $\mathbb{F}_q^n$ over the q element field $\mathbb{F}_q$, and the Eulerian number $a_{n,j}$ is replaced by a q-Eulerian number. The proof also yields an $\mathfrak{S}_n$-equivariant version (Theorem 1.5) of Theorem 1.2. The proofs of these results also appear in Sections 3 and 4. A q-analog and equivariant version of Theorem 1.1 are derived as consequences in Section 5.

Recall that the *major index*, $\mathrm{maj}(\sigma)$, of a permutation $\sigma \in \mathfrak{S}_n$ is the sum of all the descents of σ, i.e.

$$\mathrm{maj}(\sigma) := \sum_{i:\sigma(i)>\sigma(i+1)} i,$$

and the *excedance number*, $\mathrm{exc}(\sigma)$, is the number of excedances of σ, i.e.,

$$\mathrm{exc}(\sigma) := |\{i \in [n-1] : \sigma(i) > i\}|.$$

Recall that the excedance number is equidistributed with the number of descents on $\mathfrak{S}_n$. The Eulerian polynomials are defined by

$$A_n(t) = \sum_{j=0}^{n-1} a_{n,j} t^j = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)},$$

for $n \geq 1$, and $A_0(t) = 1$. (Note that it is common in the literature to define the Eulerian polynomials to be $tA_n(t)$.) For $n \geq 1$, define the q-Eulerian polynomial

$$A_n^{\mathrm{maj,exc}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

and let $A_0^{\mathrm{maj,exc}}(q,t) = 1$. For example,

$$A_3^{\mathrm{maj,exc}}(q,t) := 1 + (2q + q^2 + q^3)t + q^2t^2.$$

For all j, the q-Eulerian number $a_{n,j}^{\mathrm{maj,exc}}(q)$ is the coefficient of t^j in $A_n^{\mathrm{maj,exc}}(q,t)$. The study of the q-Eulerian polynomials $A_n^{\mathrm{maj,exc}}(q,t)$ was initiated in our recent paper [32] and was subsequently further investigated in [33, 14, 15, 16]. There are various other q-analogs of the Eulerian polynomials that had been extensively studied in the literature prior to our paper; for a sample see [1, 2, 10, 12, 13, 17, 18, 20, 21, 22, 23, 24, 29, 30, 35, 37, 38, 42]. They involve different combinations of Mahonian and Eulerian permutation statistics, such as the major index and the descent number, the inversion index and the descent number, the inversion index and the excedance number.

Like $B_n^- * C_n$, the q-analog $B_n(q)^- * C_n$ is Cohen-Macaulay. Hence $I_j(B_n(q))$ has vanishing homology below its top dimension $n-2$. We prove the following q-analog of Theorem 1.2.

Theorem 1.3. *For all $j = 0, 1, \dots, n-1$,*

$$\dim \tilde{H}_{n-2}(I_j(B_n(q))) = q^{\binom{n}{2}+j}\, a_{n,j}^{\mathrm{maj,exc}}(q^{-1}). \tag{1.2}$$

As a consequence we obtain the following q-analog of Theorem 1.1.

Corollary 1.4. *For all $n \geq 0$, let $\mathcal{D}_n$ be the set of derangements in $\mathfrak{S}_n$. Then*

$$\dim \tilde{H}_{n-1}(B_n(q)^- * C_n) = \sum_{\sigma \in \mathcal{D}_n} q^{\binom{n}{2} - \mathrm{maj}(\sigma) + \mathrm{exc}(\sigma)}.$$

The symmetric group $\mathfrak{S}_n$ acts on B_n in an obvious way and this induces an action on $B_n^- * C_n$ and on each $I_j(B_n)$. From these actions, we obtain a representation of $\mathfrak{S}_n$ on $\tilde{H}_{n-1}(B_n^- * C_n)$ and on each $\tilde{H}_{n-2}(I_j(B_n))$. We show that these representations can be described in terms of the Eulerian quasisymmetric functions that we introduced in [32, 33].

The Eulerian quasisymmetric function $Q_{n,j}$ is defined as a sum of fundamental quasisymmetric functions associated with permutations in $\mathfrak{S}_n$ having j excedances. The fixed-point Eulerian quasisymmetric function $Q_{n,j,k}$ refines this; it is a sum of fundamental quasisymmetric functions associated with permutations in $\mathfrak{S}_n$ having j excedances and k fixed points. (The precise definitions are given in Section 2.1.) Although it's not apparent from their definition, the $Q_{n,j,k}$, and thus the $Q_{n,j}$, are actually symmetric functions. A key result of [33] is the following formula, which reduces to the classical formula for the exponential generating function for Eulerian polynomials,

$$\sum_{n,j,k \geq 0} Q_{n,j,k}(\mathbf{x}) t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt) - tH(z)}, \tag{1.3}$$

where $H(z) := \sum_{n \geq 0} h_n z^n$, and h_n denotes the nth complete homogeneous symmetric function.

Our equivariant version of Theorem 1.2 is as follows.

Theorem 1.5. *For all $j = 0, 1, \dots, n-1$,*

$$\mathrm{ch} \tilde{H}_{n-2}(I_j(B_n)) = \omega Q_{n,j}, \tag{1.4}$$

where ch *denotes the Frobenius characteristic and ω denotes the standard involution on the ring of symmetric functions.*

We derive the following equivariant version of Theorem 1.1 as a consequence.

Corollary 1.6. *For all $n \geq 1$,*

$$\mathrm{ch}\tilde{H}_{n-1}(B_n^- * C_n) = \sum_{j=0}^{n-1} \omega Q_{n,j,0}.$$

The expression on the right hand side of (1.3) has occurred several times in the literature (see [33, Sec. 7]), and these occurrences yield corollaries of Theorem 1.5 and Corollary 1.6. We discuss three of these corollaries in Section 5. One is a consequence of a formula of Procesi [28] and Stanley [39] on the representation of the symmetric group on the cohomology of the toric variety associated with the Coxeter complex of $\mathfrak{S}_n$. Another corollary is a consequence of a refinement of a result of Carlitz, Scoville and Vaughan [11] due to Stanley (cf. [33, Theorem 7.2]) on words with no adjecent repeats. The third is a consequence of MacMahon's formula [26, Sec. III, Ch.III] for multiset derangements.

In Section 6, we present type BC analogs (in the context of Coxeter groups) of both Theorem 1.1 and its q-analog, Corollary 1.4. In the type BC analog of Theorem 1.1, the Boolean algebra B_n is replaced by the poset of faces of the n-dimensional cross polytope (whose order complex is the Coxeter complex of type BC). The type BC derangements are the elements of the type BC Coxeter group that have no fixed points in their action on the vertices of the cross polytope. In the type BC analog of Corollary 1.4, the lattice of subspaces $B_n(q)$ is replaced by the poset of totally isotropic subspaces of $\mathbb{F}_q^{2n}$ (whose order complex is the building of type BC).

2 Preliminaries

2.1 Quasisymmetric functions and permutation statistics

In this section we review some of our work in [33].

A *permutation statistic* is a function $f : \bigcup_{n\geq 1} \mathfrak{S}_n \to \mathbb{N}$. (Here $\mathbb{N}$ is the set of nonnegative integers and $\mathbb{P}$ is the set of positive integers.) Two well studied permutation statistics are the *excedance statistic* exc and the *major index* maj. For $\sigma \in \mathfrak{S}_n$, $\mathrm{exc}(\sigma)$ is the number of excedances of σ and $\mathrm{maj}(\sigma)$ is the sum of all descents of σ, as described above. We also define the *fixed point statistic* $\mathrm{fix}(\sigma)$ to be the number of $i \in [n]$ satisfying $\sigma(i) = i$, and the *comajor index* comaj by

$$\mathrm{comaj}(\sigma) := \binom{n}{2} - \mathrm{maj}(\sigma).$$

Remark 2.1. Note that our definition of comaj is different from a commonly used definition in which the comajor index of $\sigma \in \mathfrak{S}_n$ is defined to be $n\,\mathrm{des}(\sigma) - \mathrm{maj}(\sigma)$, where $\mathrm{des}(\sigma)$ is the number of descents of σ.

For any collection $\mathrm{f}_1, \ldots, \mathrm{f}_r$ of permutation statistics, and any $n \in \mathbb{P}$, we define the *generating polynomial*

$$A_n^{\mathrm{f}_1,\ldots\mathrm{f}_r}(t_1, \ldots, t_r) := \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{r} t_i^{\mathrm{f}_i(\sigma)}.$$

A *symmetric function* is a power series of bounded degree (with coefficients in some given ring R) in countably many variables $x_1, x_2, \ldots$ that is invariant under any permutation of the variables. A *quasisymmetric function* is a power series f in these same variables such that for any $k \in \mathbb{P}$ and any three k-tuples $(i_1 > \ldots > i_k)$, $(j_1 > \ldots > j_k)$ and $(a_1, \ldots, a_k)$ from $\mathbb{P}^k$, the coefficients in f of $\prod_{s=1}^{k} x_{i_s}^{a_s}$ and $\prod_{s=1}^{k} x_{j_s}^{a_s}$ are equal. Every symmetric function is a quasisymmetric function. We write $f(\mathbf{x})$ for any power series $f(x_1, x_2, \ldots)$.

Recall that, for $n \in \mathbb{N}$, the complete homogeneous symmetric function $h_n(\mathbf{x})$ is the sum of all monomials of degree n in $x_1, x_2, \ldots$, and the elementary symmetric function $e_n(\mathbf{x})$ is the sum of all such monomials that are squarefree. The *Frobenius characteristic* map ch sends each virtual $\mathfrak{S}_n$-representation to a symmetric function (with integer coefficients) that is homogeneous of degree n. There is a unique involutory automorphism ω of the ring of symmetric functions that maps $h_n(\mathbf{x})$ to $e_n(\mathbf{x})$ for every $n \in \mathbb{N}$. For any representation V of S_n, we have

$$\omega(\mathrm{ch}(V)) = \mathrm{ch}(V \otimes \mathrm{sgn}), \tag{2.1}$$

where sgn is the sign representation of $\mathfrak{S}_n$.

For $n \in \mathbb{P}$ and $S \subseteq [n-1]$, define

$$F_{S,n} = F_{S,n}(\mathbf{x}) := \sum_{\substack{i_1 \geq \ldots \geq i_n \geq 1 \\ j \in S \Rightarrow i_j > i_{j+1}}} x_{i_1} \ldots x_{i_n}$$

and let $F_{\emptyset,0} = 1$. Each $F_{S,n}$ is a quasisymmetric function. The involution ω extends to an involution on the ring of quasisymmetric functions. In fact,

$$\omega(F_{S,n}) = F_{[n-1]\setminus S,n}.$$

For $n \in \mathbb{P}$, set $[\overline{n}] := \{\overline{1}, \ldots, \overline{n}\}$ and order $[n] \cup [\overline{n}]$ by

$$\overline{1} < \ldots < \overline{n} < 1 < \ldots < n. \tag{2.2}$$

For $\sigma = \sigma_1 \ldots \sigma_n \in \mathfrak{S}_n$, written in one line notation, we obtain $\overline{\sigma}$ by replacing σ_i with $\overline{\sigma_i}$ whenever i is an excedance of σ. We now define $\mathrm{DEX}(\sigma)$ to be the set of all $i \in [n-1]$ such that i is a descent of $\overline{\sigma}$, i.e. the element in position i of $\overline{\sigma}$ is larger, with respect to the order (2.2), than that in position $i+1$. For example, if $\sigma = 42153$, then $\overline{\sigma} = \overline{4}21\overline{5}3$ and $\mathrm{DEX}(\sigma) = \{2, 3\}$.

For $n \in \mathbb{P}$, $0 \leq j < n-1$ and $0 \leq k \leq n$, we introduced in [33] the *fixed point Eulerian quasisymmetric functions*

$$Q_{n,j,k} = Q_{n,j,k}(\mathbf{x}) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j \\ \mathrm{fix}(\sigma) = k}} F_{\mathrm{DEX}(\sigma),n}(\mathbf{x}),$$

and the *Eulerian quasisymmetric functions*

$$Q_{n,j} := \sum_{k=0}^{n} Q_{n,j,k}.$$

We also set $Q_{0,0} = Q_{0,0,0} = 1$. It turns out that the fixed point Eulerian quasisymmetric functions (and therefore the Eulerian quasisymmetric functions) are symmetric.

We define two power series in the variable z with coefficients in the ring of symmetric functions,

$$H(z) := \sum_{n\geq 0} h_n(\mathbf{x})z^n,$$

and

$$E(z) := \sum_{n\geq 0} e_n(\mathbf{x})z^n.$$

The key result in [33] is as follows.

Theorem 2.2 ([33], Theorem 1.2)**.** *We have*

$$\sum_{n,j,k\geq 0} Q_{n,j,k}(\mathbf{x})t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)} \tag{2.3}$$

$$= \frac{H(rz)}{1-\sum_{n\geq 2} t[n-1]_t h_n z^n}, \tag{2.4}$$

where $[n]_t = 1 + t + \cdots + t^{n-1}$.

It is shown in [33] that the stable principal specialization (that is, substitution of q^{i-1} for each variable x_i) of $F_{\mathrm{DEX}(\sigma),n}$ is given by

$$F_{\mathrm{DEX}(\sigma),n}(1, q, q^2, \dots) = (q;q)_n^{-1} q^{maj(\sigma)-\mathrm{exc}(s)},$$

where $(p;q)_n := \prod_{i=1}^{n}(1-pq^{i-1})$. Hence

$$\sum_{j,k\geq 0} Q_{n,j,k}(1, q, \dots)t^j r^k := (q;q)_n^{-1} A_n^{\mathrm{maj,exc,fix}}(q; q^{-1}t, r).$$

Using the stable principal specialization we obtained from Theorem 2.2 a formula for $A_n^{\mathrm{maj,exc,fix}}$. From that formula, we derived the two following results. Before stating them, we recall the following q-analogs: for $0 \leq k \leq n$,

$$[n]_q := 1 + q + \cdots + q^{n-1},$$

$$[n]_q! := \prod_{j=1}^{n} [j]_q, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!},$$

$$\mathrm{Exp}_q(z) := \sum_{n\geq 0} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}, \quad \exp_q(z) := \sum_{n\geq 0} \frac{z^n}{[n]_q!}.$$

Corollary 2.3 ([33], Corollary 4.5)**.** *We have*

$$\sum_{n\geq 0} A_n^{\text{comaj,exc,fix}}(q,t,r)\frac{z^n}{[n]_q!} = \frac{(1-tq^{-1})\text{Exp}_q(rz)}{\text{Exp}_q(ztq^{-1})-(tq^{-1})\text{Exp}_q(z)}. \tag{2.5}$$

Corollary 2.4 ([33], Corollary 4.6)**.** *For all $n \geq 0$, we have*

$$\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{fix}(\sigma)=k}} q^{\text{comaj}(\sigma)}t^{\text{exc}(\sigma)} = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\sigma\in\mathcal{D}_{n-k}} q^{\text{comaj}(\sigma)}t^{\text{exc}(\sigma)}.$$

Consequently,

$$\sum_{\sigma\in\mathcal{D}_n} q^{\text{comaj}(\sigma)}t^{\text{exc}(\sigma)} = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k}^{\text{comaj,exc}}(q,t).$$

2.2 Homology of posets

We say that a poset P is *bounded* if it has a minimum element $\hat{0}_P$ and a maximum element $\hat{1}_P$. For any poset P, let $\widehat{P}$ be the bounded poset obtained from P by adding a minimum element and a maximum element and let P^+ be the poset obtained from P by adding only a maximum element. For a poset P with minimum element $\hat{0}_P$, let $P^- = P \setminus \{\hat{0}_P\}$. For $x \leq y$ in P, let (x,y) denote the open interval $\{z \in P : x < z < y\}$ and $[x,y]$ denote the closed interval $\{z \in P : x \leq z \leq y\}$. A subset I of a poset P is said to be a *lower order ideal* of P if for all $x < y \in P$, we have $y \in I$ implies $x \in I$. For $y \in P$, by *closed principal lower order ideal generated by* y, we mean the subposet $\{x \in P : x \leq y\}$. Similarly the *open principal lower order ideal generated by* y is the subposet $\{x \in P : x < y\}$. Upper order ideals are defined similarly. A *chain* of length n in P is an $n+1$ element subposet of P for which the induced order relation is a total order.

A poset P is said to be *ranked* (or pure) if all its maximal chains are of the same length. The *length* of a ranked poset P is the common length of its maximal chains. If P is a ranked poset, the rank $r_P(y)$ of an element $y \in P$ is the length of the closed principal lower order ideal generated by y.

A poset P is said to be *homotopy Cohen-Macaulay* if each open interval (x,y) of $\hat{P}$ has the homotopy type of a wedge of $(l([x,y])-2)$-spheres. Clearly homotopy Cohen-Macaulay is a stronger property than Cohen-Macaulay. We will make use of the following tool for establishing homotopy Cohen-Macaulayness.

Definition 2.5 ([6, 7])**.** A bounded poset P is said to *admit a recursive atom ordering* if its length $l(P)$ is 1, or if $l(P) > 1$ and there is an ordering $a_1, a_2, \ldots, a_t$ of the atoms of P that satisfies:

(i) For all $j = 1, 2, \ldots, t$ the interval $[a_j, \hat{1}_P]$ admits a recursive atom ordering in which the atoms of $[a_j, \hat{1}_P]$ that belong to $[a_i, \hat{1}_P]$ for some $i < j$ come first.

(ii) For all $i < j$, if $a_i, a_j < y$ then there is a $k < j$ and an atom z of $[a_j, \hat{1}_P]$ such that $a_k < z \leq y$.

Björner and Wachs [6] prove that every bounded ranked poset that admits a recursive atom ordering is homotopy Cohen-Macaulay (see also [43, Section 4.2]).

The *Möbius invariant* of a bounded poset P is given by

$$\mu(P) := \mu_P(\hat{0}_P, \hat{1}_P),$$

where μ_P is the Möbius function on P. It follows from a well known result of P. Hall (see [40, Proposition 3.8.5]) and the Euler-Poincaré formula that if poset P has length n then

$$\mu(\hat{P}) = \sum_{i=0}^{n} (-1)^i \dim \tilde{H}_i(P). \tag{2.6}$$

Hence if P is Cohen-Macaulay then for all $x \leq y$ in $\hat{P}$

$$\mu_P(x, y) = (-1)^r \dim \tilde{H}_r((x, y)), \tag{2.7}$$

where $r = r_P(y) - r_P(x) - 2$, and if $y = x$ or y covers x we set $\tilde{H}_r((x, y)) = \mathbb{C}$.

Suppose a group G acts on a poset P by order preserving bijections (we say that P is a G-poset). The group G acts simplicially on ΔP and thus arises a linear representation of G on each homology group of P. Now suppose P is ranked of length n. The given action also determines an action of G on $P * X$ for any length n ranked poset X defined by $g(a, x) = (ga, x)$ for all $a \in P$, $x \in X$ and $g \in G$. For a ranked G-poset P of length n with a minimum element $\hat{0}$, the action of G on P restricts to an action on P^-, which gives an action of G on $P^- * C_n$. This action restricts to an action of G on each subposet $I_j(P)$.

We will need the following result of Sundaram [41] (see [43, Theorem 4.4.1]): If G acts on a bounded poset P of length n then we have the virtual G-module isomorphism,

$$\bigoplus_{r=0}^{n} (-1)^r \bigoplus_{x \in P/G} \tilde{H}_{r-2}((\hat{0}, x)) \uparrow_{G_x}^{G} \cong_G 0, \tag{2.8}$$

where P/G denotes a complete set of orbit representatives, G_x denotes the stabilizer of x, and $\uparrow_{G_x}^{G}$ denotes the induction of the G_x module from G_x to G. Here $H_{r-2}((\hat{0}, x))$ is the trivial representation of G_x if $x = \hat{0}$ or x covers $\hat{0}$.

3 Rees products with trees

We prove the results stated in the introduction by working with the Rees product of the (nontruncated) Boolean algebra B_n with a tree and its q-analog, the Rees product of the (nontruncated) subspace lattice $B_n(q)$ with a tree. Theorems 4.1 and 4.5 will then be used to relate these Rees products to the ones considered in the introduction.

For $n, t \in \mathbb{P}$, let $T_{t,n}$ be the poset whose Hasse diagram is a complete t-ary tree of height n, with the root at the bottom. By *complete* we mean that every nonleaf node has exactly t children and that all the leaves are distance n from the root.

Since B_n and $B_n(q)$ are homotopy Cohen-Macaulay, it is an immediate consequence of the following result that $B_n * T_{t,n}$ and $B_n(q) * T_{t,n}$ are also homotopy Cohen-Macaulay.

Theorem 3.1. *Let P be a ranked poset of length n. If P is (homotopy) Cohen-Macaulay then so is $P * T_{t,n}$.*

Proof. Given a ranked poset Q of length l and a set $S \subseteq \{0, \dots, l\}$, the rank selected subposet Q_S is defined to be the induced partial order on the subset $\{q \in Q : r_Q(q) \in S\}$. By Lemma 11 of [8],

$$P * T_{t,n} = P \circ (T_{t,n} \times C_{n+1})_{\{0,\dots,n\}},$$

where $\circ$ is the Segre product introduced in [8]. Björner and Welker [8] prove that the Segre product of (homotopy) Cohen-Macaulay posets is (homotopy) Cohen-Macaulay. Hence to prove the theorem we need only show that $(T_{t,n} \times C_{n+1})_{\{0,\dots,n\}}$ is homotopy Cohen-Macaulay. We do this by showing that $(T_{t,n} \times C_{n+1})^+_{\{0,\dots,n\}}$ admits a recursive atom ordering.

In order to describe the recursive atom ordering, we first describe a natural bijection $x \mapsto w_x$ from $T_{t,n}$ to $\{w \in [t]^* : l(w) \leq n\}$, where $[t]^*$ denotes the set of all words over the alphabet $[t]$ and $l(w)$ denotes the length of w. First let w_x be the empty word if x is the root of $T_{t,n}$. Then assuming the word $w_x \in [t]^*$ has already been assigned to the parent x of the node y, we let $w_y = w_x i$, where y is the ith child of x (under some fixed ordering of the children of each node). Next we define a partial order relation $\leq_W$ on

$$W_n := \{0^k w : w \in [t]^*, 0 \leq k + l(w) \leq n\}$$

by

$$0^k u \leq_W 0^j v$$

if $k \leq j$ and $u \in [t]^*$ is a prefix of $v \in [t]^*$, that is, $v = uw$ for some $w \in [t]^*$. The map $\varphi : (T_{t,n} \times C_{n+1})_{\{0,\dots,n\}} \to W_n$ defined by

$$\varphi(x, k) = 0^k w_x,$$

is clearly a poset isomorphism.

We now describe a recursive atom ordering of W_n^+. The atoms are the words of length 1,

$$0, 1, 2, \dots, t.$$

For each atom j, the interval $[j, \hat{1}]$ is isomorphic to W_{n-1}^+ with element $0^k ju$ of $[j, \hat{1}]$ corresponding to element $0^k u$ of W_{n-1}. We claim that the increasing order $0 < 1 < \cdots < t$ on the atoms of W_n^+ is a recursive atom ordering. Indeed, the atoms of $[j, \hat{1}]$ are $0j, j1, j2, \dots, jt$ and the only atom that can belong to some $[i, \hat{1}]$ where $i < j$ is $0j$. By induction we can assume that $0j, j1, j2, \dots, jt$ is a recursive atom ordering of $[j, \hat{1}]$, since this atom ordering corresponds to the atom ordering $0, 1, 2, \dots, t$ of W_{n-1}. Hence condition (i) of Definition 2.5 holds. For condition (ii) we note that if y is greater than atoms $i < j$ of W_n then $y \geq_W 0j$, which is an atom of both $[0, \hat{1}]$ and $[j, \hat{1}]$. $\square$

The following result, which is interesting in its own right, will be used to prove the results stated in the introduction.

Theorem 3.2. *For all $n, t \geq 1$ we have*

$$\begin{aligned}\dim \tilde{H}_{n-2}((B_n * T_{t,n})^-) &= tA_n(t) &(3.1)\\ \dim \tilde{H}_{n-2}((B_n(q) * T_{t,n})^-) &= tA_n^{\mathrm{comaj,exc}}(q, qt) &(3.2)\\ \mathrm{ch}\tilde{H}_{n-2}((B_n * T_{t,n})^-) &= t\sum_{j=0}^{n-1} \omega Q_{n,j} t^j. &(3.3)\end{aligned}$$

Corollary 3.3. *For all $n \geq 1$ we have*

$$\begin{aligned}\dim \tilde{H}_{n-2}((B_n * C_{n+1})^-) &= n!\\ \dim \tilde{H}_{n-2}((B_n(q) * C_{n+1})^-) &= \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{comaj}(\sigma)+\mathrm{exc}(\sigma)}\\ \mathrm{ch}\tilde{H}_{n-2}((B_n * C_{n+1})^-) &= \sum_{j=0}^{n-1} \omega Q_{n,j}.\end{aligned}$$

To prove (3.1) and (3.2), we make use of two easy Rees product results. A bounded ranked poset P is said to be *uniform* if $[x, \hat{1}_P] \cong [y, \hat{1}_P]$ whenever $r_P(x) = r_P(y)$ (see [40, Exercise 3.50]). We will say that a sequence of posets $(P_0, P_1, \ldots, P_n)$ is uniform if for each $k = 0, 1, \ldots, n$, the poset P_k is uniform of length k and

$$P_k \cong [x, \hat{1}_{P_n}]$$

for each $x \in P_n$ of rank $n-k$. The sequences $(B_0, \ldots, B_n)$ and $(B_0(q), \ldots, B_n(q))$ are examples of uniform sequences as are the sequence of set partition lattices $(\Pi_0, \ldots, \Pi_n)$ and the sequence of face lattices of cross polytopes $(\widehat{PCP}_0, \ldots, \widehat{PCP}_n)$.

The following result is easy to verify.

Proposition 3.4. *Suppose P is a uniform poset of length n. Then for all $t \in \mathbb{P}$, the poset $R := (P * T_{t,n})^+$ is uniform of length $n+1$. Moreover, if $x \in P$ and $y \in R$ with $r_P(x) = r_R(y) = k$ then*

$$[y, \hat{1}_R] \cong ([x, \hat{1}_P] * T_{t,n-k})^+.$$

Proposition 3.5. *Let $(P_0, P_1, \ldots, P_n)$ be a uniform sequence of posets. Then for all $t \in \mathbb{P}$,*

$$1 + \sum_{k=0}^{n} W_k(P_n)[k+1]_t \mu((P_{n-k} * T_{t,n-k})^+) = 0, \qquad (3.4)$$

where $W_k(P)$ is the number of elements of rank k in P.

Proof. Let $R := (P_n * T_{t,n})^+$ and let y have rank k in R. By Proposition 3.4,

$$\mu_R(y, \hat{1}_R) = \mu((P_{n-k} * T_{t,n-k})^+).$$

Clearly

$$W_k(R) = W_k(P_n)[k+1]_t$$

for all $0 \leq k \leq n$. Hence (3.4) is just the recursive definition of the Möbius function applied to the dual of R. □

To prove (3.1) either take dimension in (3.3) or set $q = 1$ in the proof of (3.2) below.

Proof of (3.2). We apply Proposition 3.5 to the uniform sequence $(B_0(q), B_1(q), \ldots, B_n(q))$. The number of k-dimensional subspaces of $\mathbb{F}_q^n$ is given by

$$W_k(B_n(q)) = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Write $\mu_n(q,t)$ for $\mu((B_n(q) * T_{t,n})^+)$. Hence by Proposition 3.5,

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q [k+1]_t \mu_{n-k}(q,t) = -1. \tag{3.5}$$

Setting

$$F_{q,t}(z) := \sum_{j \geq 0} \mu_j(q,t) \frac{z^j}{[j]_q!}$$

and

$$G_{q,t}(z) := \sum_{k \geq 0} [k+1]_t \frac{z^k}{[k]_q!},$$

we derive from (3.5) that

$$F_{q,t}(z) = -\exp_q(z) G_{q,t}(z)^{-1}. \tag{3.6}$$

If we assume $t > 1$ we have

$$\begin{aligned} G_{q,t}(z) &= \frac{1}{1-t} \sum_{k \geq 0} (1 - t^{k+1}) \frac{z^k}{[k]_q!} \\ &= \frac{\exp_q(z) - t \exp_q(tz)}{1-t}. \end{aligned}$$

We calculate that

$$F_{q,t}(-z) = -(1-t) - t \frac{(1-t)\exp_q(-tz)}{\exp_q(-z) - t\exp_q(-tz)}.$$

Using the fact that $\exp_q(-z)\mathrm{Exp}_q(z) = 1$, we have

$$F_{q,t}(-z) = -(1-t) - t\frac{(1-t)\mathrm{Exp}_q(z)}{\mathrm{Exp}_q(tz) - t\mathrm{Exp}_q(z)}.$$

It now follows from Corollary 2.3 that for all $n \geq 1$ and $t > 1$,

$$\mu_n(q,t) = (-1)^{n-1} t \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{comaj}(\sigma)+\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}. \tag{3.7}$$

One can see from (3.5) and induction that $\mu_n(q,t)$ is a polynomial in t. Hence since (3.7) holds for infinitely many integers t, it holds as an identity of polynomials, which implies that it holds for $t = 1$.

Since by Theorem 3.1, the poset $(B_n(q) * T_{t,n})^-$ is Cohen-Macaulay, equation (3.2) holds. □

We say that a bounded ranked G-poset P is G-uniform if the following holds,

- P is uniform
- $G_x \cong G_y$ for all $x, y \in P$ such that $r_P(x) = r_P(y)$
- there is an isomorphism between $[x, \hat{1}_P]$ and $[y, \hat{1}_P]$ that intertwines the actions of G_x and G_y for all $x, y \in P$ such that $r_P(x) = r_P(y)$. We will write

$$[x, \hat{1}_P] \cong_{G_x, G_y} [y, \hat{1}_P].$$

Given a sequence of groups $G = (G_0, G_1, \dots, G_n)$, we say that a sequence of posets $(P_0, P_1, \dots, P_n)$ is G-uniform if

- P_k is G_k-uniform of length k for each k
- $G_k \cong (G_n)_x$ and $P_k \cong_{G_k, (G_n)_x} [x, \hat{1}_{P_n}]$ whenever $r_{P_n}(x) = n - k$.

For example, the sequence $(B_0, B_1, \dots, B_n)$ is $(\mathfrak{S}_0 \times \mathfrak{S}_n, \mathfrak{S}_1 \times \mathfrak{S}_{n-1}, \dots, \mathfrak{S}_n \times \mathfrak{S}_0)$-uniform, where the action of $\mathfrak{S}_i \times \mathfrak{S}_{n-i}$ on B_i is given by

$$(\sigma, \tau)\{a_1, \dots, a_s\} = \{\sigma(a_1), \dots, \sigma(a_s)\}$$

for $\sigma \in \mathfrak{S}_i$, $\tau \in \mathfrak{S}_{n-i}$ and $\{a_1, \dots, a_s\} \in B_i$. In other words $\mathfrak{S}_i$ acts on subsets of $[i]$ in the usual way and $\mathfrak{S}_{n-i}$ acts trivially.

The following proposition is easy to verify.

Proposition 3.6 (Equivariant version of Proposition 3.4)**.** *Suppose P is a G-uniform poset of length n. Then for all $t \in \mathbb{P}$, the G-poset $R := (P * T_{t,n})^+$ is G-uniform of length $n+1$. Moreover, if $x \in P$ and $y \in R$ with $r_P(x) = r_R(y) = k$ then*

$$[y, \hat{1}_R] \cong_{G_y, G_x} ([x, \hat{1}_P] * T_{t,n-k})^+.$$

If $(P_0, P_1, \ldots, P_n)$ is a $(G_0, G_1, \ldots, G_n)$-uniform sequence of posets, we can view G_k as a subgroup of G_n for each $k = 0, \ldots, n$. For G-uniform poset P, let $W_k(P; G)$ be the number of G-orbits of the rank k elements of P. The Lefschetz character of a G-poset P of length $n \geq 0$ is defined to be the virtual representation

$$L(P; G) := \bigoplus_{j=0}^{n} (-1)^j \tilde{H}_j(P).$$

Note that by (2.6) the dimension of the Lefschetz character $L(P; G)$ is precisely $\mu(\hat{P})$.

Proposition 3.7 (Equivariant version of Proposition 3.5)**.** *Let $(P_0, P_1, \ldots, P_n)$ be a $(G_0, G_1, \ldots, G_n)$-uniform sequence of posets. Then for all $t \in \mathbb{P}$,*

$$1_{G_n} \oplus \bigoplus_{k=0}^{n} W_k(P_n; G_n)[k+1]_t L((P_{n-k} * T_{t,n-k})^-; G_{n-k}) \uparrow_{G_{n-k}}^{G_n} = 0. \tag{3.8}$$

Proof. Sundaram's equation (2.8) applied to the dual of a G-poset P is equivalent to the following equivariant version of the recursive definition of the Möbius function:

$$\bigoplus_{y \in P/G} L((y, \hat{1}_P); G_y) \uparrow_{G_y}^{G} = 0, \tag{3.9}$$

where $L((y, \hat{1}_P); G_y)$ is the trivial representation if $y = \hat{1}_P$ and is the negative of the trivial representation if y is covered by $\hat{1}_P$. We apply (3.9) to the G_n-uniform poset $R := (P_n * T_{t,n})^+$. Let y have rank k in R. It follows from Proposition 3.6 that

$$L((y, \hat{1}_R); (G_n)_y) \uparrow_{(G_n)_y}^{G_n} \cong L((P_{n-k} * T_{t,n-k})^-; G_{n-k}) \uparrow_{G_{n-k}}^{G_n}.$$

Clearly,

$$W_k(R; G_n) = W_k(P_n; G_n)[k+1]_t$$

for all k. Thus (3.8) follows from (3.9). $\square$

Proof of (3.3). Now we apply Proposition 3.7 to the $(\mathfrak{S}_0 \times \mathfrak{S}_n, \mathfrak{S}_1 \times \mathfrak{S}_{n-1}, \ldots, \mathfrak{S}_n \times \mathfrak{S}_0)$-uniform sequence $(B_0, B_1, \ldots, B_n)$. Let

$$L_n(t) := \operatorname{ch} L((B_n * T_{t,n})^-; \mathfrak{S}_n).$$

Clearly $W_k(B_n; \mathfrak{S}_n) = 1$ for all $k = 0, \ldots, n$. Therefore by Proposition 3.7,

$$\sum_{k=0}^{n} [k+1]_t h_k L_{n-k}(t) = -h_n. \tag{3.10}$$

Setting

$$F_t(z) := \sum_{j \geq 0} L_j(t) z^j$$

and

$$G_t(z) := \sum_{k\geq 0}[k+1]_t h_k z^k,$$

we derive from (3.10) that

$$F_t(z)G_t(z) = -H(z). \tag{3.11}$$

Now if $t > 1$,

$$\begin{aligned} G_t(z) &= \frac{1}{1-t}\sum_{k\geq 0}(1-t^{k+1})h_k z^k \\ &= \frac{H(z) - tH(tz)}{1-t}, \end{aligned}$$

and we thus have

$$F_t(z) = -\frac{(1-t)H(z)}{H(z) - tH(tz)}. \tag{3.12}$$

We calculate that

$$F_t(-z) = -(1-t) - t\frac{(1-t)H(-tz)}{H(-z) - tH(-tz)}. \tag{3.13}$$

Using the fact that $H(-z)E(z) = 1$ we have

$$F_t(-z) = -(1-t) - t\frac{(1-t)E(z)}{E(tz) - tE(z)}.$$

By applying the standard symmetric function involution ω, we obtain

$$\omega F_t(-z) = -(1-t) - t\frac{(1-t)H(z)}{H(tz) - tH(z)}.$$

It follows from this and Theorem 2.2 that for all $n \geq 1$ and $t > 1$,

$$\omega L_n(t) = (-1)^{n-1} t \sum_{j=0}^{n-1} Q_{n,j} t^j. \tag{3.14}$$

By (3.10) and induction, $L_n(t)$ is a polynomial in t. Hence (3.14) holds for $t = 1$ as well. Since $(B_n * T_{t,n})^-$ is Cohen-Macaulay we are done. □

4 The tree lemma

The following result and Theorem 3.2 are all that is needed to prove Theorems 1.2 and 1.3, since B_n and $B_n(q)$ are self-dual and Cohen-Macaulay.

Theorem 4.1 (Tree Lemma)**.** *Let P be a bounded, ranked poset of length n. Then for all $t \in \mathbb{P}$,*

$$\sum_{j=1}^{n} \mu(\widehat{I_{j-1}(P)})t^j = -\mu((P^* * T_{t,n})^+), \tag{4.1}$$

where P^* is the dual of P.

Before we can prove Theorem 4.1, we need a few lemmas. Set

$$R(P) := P * \{x_0 < x_1 < \ldots < x_n\}$$

and for $i \in [n]$, let $R_i(P)$ be the closed principal lower order ideal in $R(P)$ generated by $(\hat{1}_P, x_i)$. Set

$$R_i^+(P) := \{(a, x_j) \in R_i(P) : j > 0\}$$

and

$$R_i^-(P) := R_i(P) \setminus R_i^+(P).$$

Lemma 4.2. *The posets $R_i^+(P)$ and $I_{i-1}(P)^+$ are isomorphic.*

Proof. The map that sends (a, x_j) to $(a, j-1)$ is an isomorphism. □

An *antiisomorphism* from poset X to a poset Y is an isomorphism ψ from X to Y^*. In other words, ψ is an order reversing bijection from X to Y with order reversing inverse.

Lemma 4.3. *For $0 \leq i \leq n$, the map $\psi_i : R_i(P) \to R_i(P^*)$ given by $\psi_i((a, x_j)) = (a, x_{i-j})$ is an antiisomorphism.*

Proof. We show first that ψ_i is well-defined, that is, if $(a, x_j) \in R_i(P)$ then $(a, x_{i-j}) \in R_i(P^*)$. For $a \in P$ and $j \in \{0, \ldots, n\}$ we have $(a, x_j) \in R_i(P)$ if and only if the three conditions

(1) $0 \leq j \leq i$

(2) $r_P(a) \geq j$

(3) $n - r_P(a) \geq i - j$

hold. If (1), (2), (3) hold then so do all of

(1′) $0 \leq i - j \leq i$

(2′) $r_{P^*}(a) = n - r_P(a) \geq i - j$

(3′) $n - r_{P^*}(a) = r_P(a) \geq j = i - (i - j)$,

and (1′), (2′), (3′) together imply that $(a, x_{i-j}) \in R_i(P^*)$. The map $\psi_i^* : R_i(P^*) \to R_i(P)$ given by $\psi_i^*((a, x_j)) = (a, x_{i-j})$ is also well-defined by the argument just given, and $\psi_i^* = \psi_i^{-1}$, so ψ_i is a bijection.

Now for (a, x_j) and (b, x_k) in $R_i(P)$, we have $(a, x_j) < (b, x_k)$ if and only if the three conditions

(4) $a \leq_P b$

(5) $j \leq k$

(6) $r_P(b) - r_P(a) \geq k - j$

hold. If (4), (5), (6) hold then so do all of

(4′) $b \leq_{P^*} a$

(5′) $i-k \leq i-j$

(6′) $r_{P^*}(a) - r_{P^*}(b) = r_P(b) - r_P(a) \geq k-j = (i-j)-(i-k)$,

and (4′), (5′), (6′) together imply that in $R_i(P^*)$ we have $(b, x_{i-k}) \leq (a, x_{i-j})$. Therefore, ψ_i is order reversing, and the same argument shows that ψ_i^* is order reversing. $\square$

Corollary 4.4. *For $1 \leq i \leq n$ we have*

$$\mu(\widehat{I_{i-1}(P)}) = \sum_{(a,x_i)\in R_i(P^*)} \mu_{R_i(P^*)}((\hat{1}_P, x_0), (a, x_i)). \tag{4.2}$$

In case the notation has confused the reader, we remark before proving Corollary 4.4 that the sum on the right side of equality (4.2) is taken over all pairs (a, x_i) such that $a \in P$ with $r_P(a) \leq n-i$ (so $r_{P^*}(a) \geq i$), and that $\hat{1}_P$, being the maximum element of P, is the minimum element of P^* (so $(\hat{1}_P, x_0)$ is the minimum element of $R_i(P^*)$).

Proof. We have

$$\begin{aligned}
\mu(\widehat{I_{i-1}(P)}) &= -\sum_{\alpha\in I_{i-1}(P)^+} \mu_{\widehat{I_{i-1}(P)}}(\alpha, (\hat{1}_P, i-1)) \\
&= -\sum_{\beta\in R_i^+(P)} \mu_{R_i^+(P)}(\beta, (\hat{1}_P, x_i)) \\
&= \sum_{\gamma=(a,x_0)\in R_i^-(P)} \mu_{R_i(P)}(\gamma, (\hat{1}_P, x_i)) \\
&= \sum_{\gamma=(a,x_0)\in R_i^-(P)} \mu_{R_i(P^*)}(\psi_i((\hat{1}_P, x_i)), \psi_i(\gamma)) \\
&= \sum_{\gamma=(a,x_0)\in R_i^-(P)} \mu_{R_i(P^*)}((\hat{1}_P, x_0), (a, x_i)) \\
&= \sum_{(a,x_i)\in R_i(P^*)} \mu_{R_i(P^*)}((\hat{1}_P, x_0), (a, x_i)).
\end{aligned}$$

Indeed, the first equality follows from the definition of the Möbius function; the second follows from Lemma 4.2; the third follows from the definition of the Möbius function and the fact that $\mu_{R_i^+(P)}$ is the restriction of $\mu_{R_i(P)}$ to $R_i^+(P) \times R_i^+(P)$ (as $R_i^+(P)$ is an upper order ideal in $R_i(P)$); the fourth follows from Lemma 4.3 and the last two follow from the definition of ψ_i. $\square$

Proof of Tree Lemma (Theorem 4.1). The poset $T_{t,n}$ has exactly t^j elements of rank j for each $j = 0, \dots, n$. Let r_T be the rank function of $T_{t,n}$ and let $\hat{0}_T$ be the minimum element of $T_{t,n}$.

We have

$$\begin{aligned}\mu((P^* * T_{t,n})^+) &= -\sum_{\alpha \in P^* * T_{t,n}} \mu_{P^* * T_{t,n}}((\hat{1}_P, \hat{0}_T), \alpha) \\ &= -\sum_{j=0}^{n} \sum_{\alpha \in P^*_{n,t,j}} \mu_{P^* * T_{t,n}}((\hat{1}_P, \hat{0}_T), \alpha),\end{aligned}$$

where

$$P^*_{n,t,j} := \{(a, w) \in P^* * T_{t,n} : r_T(w) = j\}.$$

We have

$$\begin{aligned}\sum_{\alpha \in P^*_{n,t,0}} \mu_{P^* * T_{t,n}}((\hat{1}_P, \hat{0}_T), \alpha) &= \sum_{a \in P^*} \mu_{P^* * T_{t,n}}((\hat{1}_P, \hat{0}_T), (a, \hat{0}_T)) \\ &= \sum_{a \in P^*} \mu_{P^*}(\hat{1}_P, a) \\ &= 0.\end{aligned}$$

Now fix $j \in [n]$. For any $w \in T_{t,n}$ with $r_T(w) = j$, the interval $[\hat{0}_T, w]$ in $T_{t,n}$ is a chain of length j. Therefore, for any $(a, w) \in P^*_{n,t,j}$, the interval $[(\hat{1}_P, \hat{0}_T), (a, w)]$ in $P^* * T_{t,n}$ is isomorphic with the interval $[(\hat{1}_P, x_0), (a, x_j)]$ in $R_j(P^*)$. For any $a \in P^*$, the four conditions

- $r_{P^*}(a) \geq j$,
- $(a, w) \in P^*_{n,t,j}$ for some $w \in T_{t,n}$,
- $(a, v) \in P^*_{n,t,j}$ for every $v \in T_{t,n}$ satisfying $r_T(v) = j$,
- $(a, x_j) \in R_j(P^*)$

are all equivalent. There are exactly t^j elements $v \in T_{t,n}$ of rank j. It follows that

$$\sum_{\alpha \in P^*_{n,t,j}} \mu_{P^* * T_{t,n}}((\hat{1}_P, \hat{0}_T), \alpha) = t^j \sum_{(a, x_j) \in R_j(P^*)} \mu_{R_j(P^*)}((\hat{1}_P, x_0), (a, x_j)),$$

and the Tree Lemma now follows from Corollary 4.4. □

Since B_n is Cohen-Macaulay and self-dual, the following result shows that Theorem 1.5 is equivalent to (3.3).

Theorem 4.5 (Equivariant Tree Lemma)**.** *Let P be a bounded, ranked G-poset of length n. Then for all $t \in \mathbb{P}$,*

$$\bigoplus_{j=1}^{n} t^j L(I_{j-1}(P); G) \cong_G -L((P^* * T_{t,n})^-; G). \tag{4.3}$$

Consequently, if P is Cohen-Macaulay then for all $t \in \mathbb{P}$,

$$\bigoplus_{j=1}^{n} t^j \tilde{H}_{n-2}(I_{j-1}(P)) \cong_G \tilde{H}_{n-1}((P^* * T_{t,n})^-).$$

Proof. The proof is an equivariant version of the proof of the Tree Lemma. In particular, the isomorphism of Lemma 4.2 is G-equivariant, as is the antiisomorphism of Lemma 4.3.

The equivariant version of (4.2) is

$$L(I_{i-1}(P); G) = \bigoplus_{(a,x_i) \in R_i(P^*)/G} L(((\hat{1}_P, x_0), (a, x_i)); G_a) \uparrow_{G_a}^{G} . \tag{4.4}$$

To prove (4.4) we let (3.9) play the role of the recursive definition of Möbius function in the proof of (4.2).

To prove (4.3) we follow the proof of the Tree Lemma again letting (3.9) play the role of the recursive definition of Möbius function, and in the last step applying (4.4) instead of (4.2). ☐

5 Corollaries

In this section we restate and prove Corollaries 1.4 and 1.6 and discuss some other corollaries that were mentioned in the introduction.

Corollary 5.1 (to Theorem 1.3)**.** *For all $n \geq 0$, let $\mathcal{D}_n$ be the set of derangements in $\mathfrak{S}_n$. Then*

$$\dim \tilde{H}_{n-1}(B_n(q)^- * C_n) = \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{comaj}(\sigma)+\mathrm{exc}(\sigma)}.$$

Proof. Since $B_n(q)^- * C_n$ is Cohen-Macaulay and the number of m-dimensional subspaces of $\mathbb{F}_q^n$ is $\left[\begin{array}{c} n \\ m \end{array} \right]_q$, the Möbius function recurrence for $(B_n(q)^- * C_n) \cup \{\hat{0}, \hat{1}\}$ is equivalent to

$$\dim \tilde{H}_{n-1}(B_n(q)^- * C_n) = \sum_{m=0}^{n} \left[\begin{array}{c} n \\ m \end{array} \right]_q (-1)^{n-m} \sum_{j=0}^{m-1} \dim \tilde{H}_{m-2}(I_j(B_m(q))).$$

It therefore follows from Theorem 1.3 that

$$\dim \tilde{H}_{n-1}(B_n(q)^- * C_n) = \sum_{m=0}^{n} \left[\begin{array}{c} n \\ m \end{array} \right]_q (-1)^{n-m} \sum_{\sigma \in \mathfrak{S}_m} q^{\mathrm{comaj}(\sigma)+\mathrm{exc}(\sigma)}.$$

The result thus follows from Corollary 2.4. ☐

Corollary 5.2 (to Theorem 1.5)**.** *We have*

$$\sum_{n\geq 0}\mathrm{ch}\tilde{H}_{n-1}(B_n^- * C_n)z^n = \frac{1}{1-\sum_{i\geq 2}(i-1)e_i z^i}. \tag{5.1}$$

Equivalently,

$$\mathrm{ch}\tilde{H}_{n-1}(B_n^- * C_n) = \sum_{j=0}^{n-1}\omega Q_{n,j,0}. \tag{5.2}$$

Proof. Applying (2.8) to the Cohen-Macaulay $\mathfrak{S}_n$-poset $\widehat{B_n^- * C_n}$, we have

$$\tilde{H}_{n-1}(B_n^- * C_n) \cong_{\mathfrak{S}_n} \bigoplus_{m=0}^{n}(-1)^{n-m}\bigoplus_{j=0}^{m-1}\left(\tilde{H}_{m-2}(I_j(B_m))\otimes 1_{\mathfrak{S}_{n-m}}\right)\uparrow_{\mathfrak{S}_m\times\mathfrak{S}_{n-m}}^{\mathfrak{S}_n},$$

where 1_G denotes the trivial representation of a group G. From this we obtain

$$\mathrm{ch}\tilde{H}_{n-1}(B_n^- * C_n) = \sum_{m=0}^{n}(-1)^{n-m}\sum_{j=0}^{m-1}\mathrm{ch}\tilde{H}_{m-2}(I_j(B_m))h_{n-m}. \tag{5.3}$$

Hence

$$\sum_{n\geq 0}\mathrm{ch}\tilde{H}_{n-1}(B_n^- * C_n)\,z^n \;=\; H(-z)\sum_{n\geq 0}z^n\sum_{j=0}^{n-1}\mathrm{ch}\tilde{H}_{n-2}(I_j(B_n)).$$

It follows from Theorem 1.5 and (2.4) that

$$\sum_{n\geq 0}z^n\sum_{j=0}^{n-1}\mathrm{ch}\tilde{H}_{n-2}(I_j(B_n))t^j = \frac{E(z)}{1-\sum_{n\geq 2}t[n-1]_t e_n z^n}.$$

By setting $t=1$ and using the fact that $E(z)H(-z)=1$, we obtain (5.1). Equation (5.2) follows from (5.1) and (2.4). ☐

We now present some additional corollaries of Theorem 1.5 and Corollary 1.6, which follow from the occurrence of the right hand side of (1.3) in various results in the literature.

Let X_n be the toric variety naturally associated to the Coxeter complex Δ_n for the reflection group $\mathfrak{S}_n$. (See, for example, [9] for a discussion of Coxeter complexes and [19] for an explanation of how toric varieties are associated to polytopes.) The action of $\mathfrak{S}_n$ on Δ_n induces an action on X_n and thus a representation on each cohomology group of X_n. Now X_n can have nontrivial cohomology only in dimensions $2j$, for $0\leq j\leq n-1$. (See for example [19, Section 4.5].) Using work of Procesi [28], Stanley shows in [39] that

$$\sum_{n\geq 0}\sum_{j=0}^{n-1}\mathrm{ch}H^{2j}(X_n)\,t^j z^n = \frac{(1-t)H(z)}{H(zt)-tH(z)}.$$

Combining this with Theorem 1.5 and equations (1.3) and (2.1), we obtain the following result.

Corollary 5.3 (to Theorem 1.5)**.** *For all $j = 0, \ldots, n-1$, we have the following isomorphism of $\mathfrak{S}_n$-modules*

$$\tilde{H}_{n-2}(I_j(B_n)) \cong_{\mathfrak{S}_n} H^{2j}(X_n) \otimes \mathrm{sgn}.$$

It would be interesting to find a topological explanation for this isomorphism, in particular one that extends the isomorphism to other Coxeter groups.

Another corollary is an immediate consequence of a refinement of a result of Carlitz, Scoville and Vaughan [11] due to Stanley (cf. [33, Theorem 7.2]).

Corollary 5.4. *For all $j = 0, \ldots, n-1$, let $W_{n,j}$ be the set of all words of length n over the alphabet of positive integers with the properties that no adjacent letters are equal and there are exactly j descents. Then*

$$\mathrm{ch}\tilde{H}_{n-2}(I_j(B_n)) = \sum_{w:=w_1\cdots w_n \in W_{n,j}} x_{w_1} x_{w_2} \cdots x_{w_n}.$$

The following equivariant version of Theorem 1.1 is an immediate consequence of Corollary 5.2 and MacMahon's formula [26, Sec. III, Ch.III] for multiset derangements. A *multiset derangement* of order n is a $2 \times n$ matrix $D = (d_{i,j})$ of positive integers such that

- $d_{1,j} \leq d_{1,j+1}$ for all $j \in [n-1]$,
- the multisets $\{d_{1,j} : j \in [n]\}$ and $\{d_{2,j} : j \in [n]\}$ are equal, and
- $d_{1,j} \neq d_{2,j}$ for all $j \in [n]$.

Given a multiset derangement D, we write x^D for $\prod_{j=1}^{n} x_{d_{1,j}}$.

Corollary 5.5 (to Corollary 5.2)**.** *For all $n \geq 1$, we have*

$$\mathrm{ch}\tilde{H}_{n-1}(B_n^- * C_n) = \sum_{D \in \mathcal{MD}_n} \mathrm{x}^D, \tag{5.4}$$

where $\mathcal{MD}_n$, is the set of all multiset derangements of order n.

6 Type BC-analogs

In this section we present type BC analogs (in the context of Coxeter groups) of both the Björner-Welker-Jonsson derangement result (Theorem 1.1) and its q-analog (Corollary 1.4).

A poset P with a $\hat{0}_P$ is said to be a *simplicial poset* if $[\hat{0}_P, x]$ is a Boolean algebra for all $x \in P$. The prototypical example of a simplicial poset is the poset of faces of a simplicial complex. In fact, every simplicial poset is isomorphic to the face poset of some regular CW complex (see [4]). The next result follows immediately from Theorem 1.2 and the definition of the Möbius function. For a ranked poset P of length n and $r \in \{0, 1, \ldots, n\}$, let $W_r(P)$ be the *rth* Whitney number of the second kind of P, that is, the number of elements of rank r in P.

Corollary 6.1 (of Theorem 1.2)**.** *Let P be a ranked simplicial poset of length n. Then*

$$\mu(\widehat{P^- * C_n}) = \sum_{r=0}^{n} (-1)^{r-1} W_r(P) r!.$$

We think of B_n as the poset of faces of an $(n-1)$-simplex whose barycentric subdivision is the Coxeter complex of type A. Then d_n is the number of derangements in the action of the associated Coxeter group $\mathfrak{S}_n$ on the vertices of the simplex. Let PCP_n be the poset of *simplicial* (that is, proper) faces of the n-dimensional crosspolytope CP_n (see for example [5, Section 2.3]), whose barycentric subdivision is the Coxeter complex of type BC. The associated Weyl group, which is isomorphic to the wreath product $\mathfrak{S}_n[\mathbb{Z}_2]$, acts by reflections on CP_n and therefore on its vertex set. Let d_n^{BC} be the number of derangements in this action on vertices.

Theorem 6.2. *For all n, we have*

$$\dim \tilde{H}_{n-1}(PCP_n^- * C_n) = d_n^{BC}.$$

Proof. It is well known and straightforward to prove by induction on n that, for $0 \leq r \leq n$, the number of $(r-1)$-dimensional faces of CP_n is $2^r\binom{n}{r}$. Corollary 6.1 gives

$$\mu(\widehat{PCP_n^- * C_n}) = \sum_{r=0}^{n} (-1)^{r-1} 2^r \binom{n}{r} r!.$$

Hence since PCP_n^- is Cohen-Macaulay, we have,

$$\dim \tilde{H}_{n-1}(PCP_n^- * C_n) = \sum_{r=0}^{n} (-1)^{n-r} 2^r \binom{n}{r} r!.$$

On the other hand, we may identify the vertices of CP_n with elements of $[n] \cup [\overline{n}]$, where $[\overline{n}] = \{\bar{1}, \ldots, \bar{n}\}$, so that the action of the Weyl group $W \cong \mathfrak{S}_n[\mathbb{Z}_2]$ is determined by the following facts.

- Each element $w \in W$ can be written uniquely as $w = (\sigma, v)$ with $\sigma \in \mathfrak{S}_n$ and $v \in \mathbb{Z}_2^n$.
- Any element of the form $(\sigma, 0)$ maps $i \in [n]$ to $\sigma(i)$ and $\overline{i} \in [\overline{n}]$ to $\overline{\sigma(i)}$.
- Any element of the form $(1, e_i)$, where e_i is the i^{th} standard basis vector in $\mathbb{Z}_2^n$, exchanges i and $\overline{i}$, and fixes all other vertices.

It follows that for each $S \subseteq [n]$, the pointwise stabilizer of S in W is exactly the pointwise stabilizer of $\overline{S} := \{\overline{i} : i \in S\}$ and is isomorphic to $\mathfrak{S}_{n-|S|}[\mathbb{Z}_2]$. Using inclusion-exclusion as is done to calculate d_n, we get

$$d_n^{BC} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} 2^{n-j} (n-j)!.$$

$\square$

Muldoon and Readdy [27] have recently obtained a dual version of Theorem 6.2 in which the Rees product of the dual of PCP_n with the chain is considered.

Next we consider a poset that can be viewed as both a q-analog of PCP_n and a type BC analog of $B_n(q)$. Let $\langle \cdot, \cdot \rangle$ be a nondegenerate, alternating bilinear form on the vector space $\mathbb{F}_q^{2n}$. A subspace U of $\mathbb{F}_q^{2n}$ is said to be *totally isotropic* if $\langle u, v \rangle = 0$ for all $u, v \in U$. Let $PCP_n(q)$ be the poset of totally isotropic subspaces of $\mathbb{F}_q^{2n}$. The order complex of $PCP_n(q)$ is the building of type BC, naturally associated to a finite group of Lie type B or C (see for example [9, Chapter V], [31, Appendix 6]). Thus we have both a q-analog of PCP_n and a type BC analog of $B_n(q)$ (since the order complex of $B_n(q)$ is the building of type A).

Clearly $PCP_n(q)$ is a lower order ideal of $B_{2n}(q)$.

Proposition 6.3. *The maximal elements of $PCP_n(q)$ all have dimension n. For $r = 0, \ldots, n$, the number of r-dimensional isotropic subspaces of $\mathbb{F}_q^{2n}$ is given by*

$$W_r(PCP_n(q)) = \left[\begin{array}{c} n \\ r \end{array} \right]_q (q^n + 1)(q^{n-1} + 1) \cdots (q^{n-r+1} + 1).$$

Proof. The first claim of the proposition is a well known fact (see for example [31, Chapter 1]). The second claim is also a known fact; we sketch a proof here. The number of ordered bases for any k-dimensional subspace of $\mathbb{F}_q^{2n}$ is

$$\prod_{j=0}^{k-1} (q^k - q^j).$$

On the other hand, we can produce an ordered basis for a k-dimensional totally isotropic subspace of $\mathbb{F}_q^{2n}$ in k steps, at each step i choosing

$$v_i \in \langle v_1, \ldots, v_{i-1} \rangle^\perp \setminus \langle v_1, \ldots, v_{i-1} \rangle.$$

The number of ways to do this is

$$\prod_{j=0}^{k-1} (q^{2n-j} - q^j),$$

and the proof is completed by division and manipulation. □

It was shown by Solomon [36] that $PCP_n(q)$ is Cohen-Macaulay. Hence so is the Rees product $PCP_n(q)^- * C_n$. We will show that the dimension of $\tilde{H}_{n-1}(PCP_n(q)^- * C_n)$ is a polynomial in q with nonnegative integral coefficients and give a combinatorial interpretation of the coefficients. We first need the following q-analog of Corollary 6.1. We say that a poset P with $\hat{0}_P$ is q-simplicial if each interval $[\hat{0}_P, x]$ is isomorphic to $B_j(q)$ for some j.

Corollary 6.4 (of Theorem 1.3)**.** *Let P be a ranked q-simplicial poset of length n. Then*

$$\mu(\widehat{P^- * C_n}) = \sum_{r=0}^{n} (-1)^{r-1} W_r(P) \sum_{\sigma \in \mathfrak{S}_r} q^{\mathrm{comaj}(\sigma)+\mathrm{exc}(\sigma)}.$$

Theorem 6.5. *For all $n \geq 0$, let $d_n(q) := \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{comaj}(\sigma)+\mathrm{exc}(\sigma)}$. Then*

$$\dim \tilde{H}_{n-1}(PCP_n(q)^- * C_n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} \prod_{i=k+1}^{n} (1+q^i)\ d_{n-k}(q). \tag{6.1}$$

Consequently, $\dim \tilde{H}_{n-1}(PCP_n(q)^- * C_n)$ *is a polynomial in q with nonnegative integer coefficients.*

Proof. We have by Proposition 6.3, Corollary 6.4, and the fact that $PCP_n(q)^- * C_n$ is Cohen-Macaulay,

$$\dim \tilde{H}_{n-1}(PCP_n(q)^- * C_n) = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_q \prod_{i=j+1}^{n} (1+q^i)\ a_{n-j}(q),$$

where $a_n(q) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{comaj}(\sigma)+\mathrm{exc}(\sigma)}$. On the other hand by Corollary 2.4, the right hand side of (6.1) equals

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} \prod_{i=k+1}^{n} (1+q^i) \sum_{m=0}^{n-k} (-1)^m \begin{bmatrix} n-k \\ m \end{bmatrix}_q a_{n-k-m}(q)$$

$$\begin{aligned} &= \sum_{j \geq 0} a_{n-j}(q) \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} \prod_{i=k+1}^{n} (1+q^i)(-1)^{j-k} \begin{bmatrix} n-k \\ j-k \end{bmatrix}_q \\ &= \sum_{j \geq 0} a_{n-j}(q) \begin{bmatrix} n \\ j \end{bmatrix}_q \sum_{k \geq 0} \begin{bmatrix} j \\ k \end{bmatrix}_q q^{k^2} \prod_{i=k+1}^{n} (1+q^i)(-1)^{j-k}. \end{aligned}$$

Thus to prove (6.1) we need only show that

$$\prod_{i=j+1}^{n} (1+q^i) = \sum_{k \geq 0} \begin{bmatrix} j \\ k \end{bmatrix}_q q^{k^2} \prod_{i=k+1}^{n} (1+q^i)(-1)^k,$$

holds for all n and j. By Gaussian inversion this is equivalent to,

$$q^{j^2}(-1)^j \prod_{i=j+1}^{n} (1+q^i) = \sum_{k \geq 0} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^{j-k} q^{\binom{j-k}{2}} \prod_{i=k+1}^{n} (1+q^i),$$

which is in turn equivalent to,

$$q^{j^2}(-1)^j = \sum_{k \geq 0} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^{j-k} q^{\binom{j-k}{2}} \prod_{i=k+1}^{j} (1+q^i). \tag{6.2}$$

To prove (6.2) we use the q-binomial formula,

$$\prod_{i=0}^{n-1}(x+yq^i) = \sum_{k\geq 0} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\binom{k}{2}} x^{n-k} y^k.$$

Set $y = 1$ and use Gaussian inversion to obtain

$$x^n = \sum_{k\geq 0} \left[\begin{array}{c} n \\ k \end{array} \right]_q (-1)^{n-k} \prod_{i=0}^{k-1}(x+q^i).$$

Now set $x = q^n$ to obtain

$$\begin{aligned} q^{n^2} &= \sum_{k\geq 0} \left[\begin{array}{c} n \\ k \end{array} \right]_q (-1)^{n-k} \prod_{i=0}^{k-1}(q^n+q^i) \\ &= \sum_{k\geq 0} \left[\begin{array}{c} n \\ k \end{array} \right]_q (-1)^{n-k} q^{\binom{k}{2}} \prod_{i=0}^{k-1}(q^{n-i}+1). \end{aligned}$$

□

Using the standard identification of elements of $\mathfrak{S}_n[\mathbb{Z}_2]$ with barred permutations (i.e., permutations written in one line notation with some subset of the letters barred), the derangements of Theorem 6.2 are the barred permutations $\sigma = \sigma_1 \cdots \sigma_n$ for which $\sigma_i \neq i$ for all $i \in [n]$. Let $\mathcal{D}_n^{BC}$ be the set of such barred permutations. If σ is a barred permutation, let $|\sigma|$ be the ordinary permutation obtained by removing the bars from σ. For $\sigma \in \mathcal{D}_n^{BC}$, let $\tilde{\sigma}$ be the word obtained by rearranging the letters of σ so that the fixed points of $|\sigma|$, which are all barred in σ, come first in increasing order with bars intact, followed by subword of nonfixed points of $|\sigma|$ also with bars intact. Now let S be the set of positions in which bars appear in $\tilde{\sigma}$. Define the *bar index*, $\operatorname{bnd}(\sigma)$ of σ to be $\sum_{i\in S} i$. For example if $\sigma = \bar{3}\bar{2}5\bar{4}\bar{6}1\bar{7}$ then $\tilde{\sigma} = \bar{2}\bar{4}\bar{7}\bar{3}5\bar{6}1$ and so $\operatorname{bnd}(\sigma) = 1+2+3+4+6$.

Corollary 6.6.

$$\dim \tilde{H}_{n-1}(PCP_n(q)^- * C_n) = \sum_{\sigma\in\mathcal{D}_n^{BC}} q^{\operatorname{comaj}(|\sigma|)+\operatorname{exc}(|\sigma|)+\operatorname{bnd}(\sigma)}.$$

Proof. By Corollary 2.4 we have,

$$\begin{aligned} &\sum_{\sigma\in\mathcal{D}_n^{BC}} q^{\operatorname{comaj}(|\sigma|)+\operatorname{exc}(|\sigma|)} p^{\operatorname{bnd}(\sigma)} \\ &= \sum_{k=0}^{n} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{fix}(\sigma)=k}} q^{\operatorname{comaj}(\sigma)+\operatorname{exc}(\sigma)} p^{\binom{k+1}{2}} \prod_{i=k+1}^{n}(1+p^i) \\ &= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\binom{k}{2}} d_{n-k}(q) p^{\binom{k+1}{2}} \prod_{i=k+1}^{n}(1+p^i). \end{aligned}$$

Now set $p = q$ and apply Theorem 6.5. □

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