Subdivision yields Alexander duality on independence complexes

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Submitted: Dec 1, 2008; Accepted: Apr 24, 2009; Published: May 12, 2009
Mathematics Subject Classification: 55P10, 05C69, 05E25

Dedicated to Anders Björner on the occasion of his 60th birthday.

Abstract

We study how the homotopy type of the independence complex of a graph changes if we subdivide edges. We show that the independence complex becomes the Alexander dual if we place one new vertex on each edge of a graph. If we place two new vertices on each edge then the independence complex is the wedge of two spheres. Placing three new vertices on an edge yields the suspension of the independence complex.

1 Introduction

Independence complexes of various graph classes: e.g. trees, cycles, 2D grids were studied in numerous papers [2, 4, 5, 6, 9, 10, 11, 12]. We study how edge subdivision (definition 1) changes the homotopy type of the independence complex. This is motivated by the homology calculation [7] of $\mathrm{Ind}(G_3)$. Schoutens [15] observed and proved that $\tilde{\mathrm{H}}_i(\mathrm{Ind}(G), \mathbb{R}) \cong \tilde{\mathrm{H}}_{n-i-2}(\mathrm{Ind}(G_2), \mathbb{R})$ using the double complex and the tic-tac-toe lemma. This explains that the reduced Euler characteristic sometimes changes the sign if we place one new vertex on each edge of a graph: $\tilde{\chi}(\mathrm{Ind}(G)) = (-1)^{|V(G)|} \cdot \tilde{\chi}(\mathrm{Ind}(G_2))$. Alexander duality explains this on the homotopy level. $\mathrm{Ind}(G)$ is a subcomplex of a simplex with $n = |V(G)|$ vertices. If G is connected, then $\mathrm{Ind}(G)$ is a subcomplex of S^{n-2}, the boundary of a simplex with n vertices. We can consider this S^{n-2} as the equator of S^{n-1}. We will show that the complement of $\mathrm{Ind}(G)$, $S^{n-1} \setminus \mathrm{Ind}(G)$ is homotopy equivalent to $\mathrm{Ind}(G_2)$. In section 2 we review some definitions and collect the necessary tools for the proofs. In

*This research has been supported by DIAMANT (an NWO mathematics cluster).

section 3 we will show that $\mathrm{Ind}(G_2)$ is the suspension of the Alexander dual of $\mathrm{Ind}(G)$. In section 4 we prove that $\mathrm{Ind}(G_3)$ is a wedge of spheres unless G is a tree. We study how the homotopy type changes if we remove a vertex from G_3. In section 5 we deal with $\mathrm{Ind}(G_n)$ and show that $\mathrm{Ind}(G_{n+3}) \simeq \mathrm{susp}^e(\mathrm{Ind}(G_n))$. From this we get recursively the homotopy information of $\mathrm{Ind}(G_n)$.

2 Preliminaries

We assume that the reader is familiar with basic topological concepts and constructions (homotopy, wedge, suspension, join), the definition of graphs, simplicial complexes and their properties. Introductory chapters of books like [14, 3, 13] should provide a sufficient background. Here we only review a few things to fix the notation.

We assume that graphs $G = (V(G), E(G))$ are *simple*, i.e., without loops and parallel edges. A graph will be connected unless otherwise stated.

Definition 1 *Let G be a graph. The graph G_n is obtained from G by replacing each edge by a path of length n.*

For example $G_1 = G$. If P is just an edge, then P_n is the path with n edges. Let C be the loop. Now C_n is the cycle with n vertices. Clearly $(C_n)_3$ is C_{3n}. We will consider $V(G_2) = V(G) \cup E(G)$ and $V(G_3) \supset V(G)$.
A subset of the vertex set of a graph is *independent* if no two vertices in it are adjacent.

Definition 2 *Let G be a graph. The independence complex of G, denoted by $\mathrm{Ind}(G)$, is a simplicial complex with vertex set $V(G)$, and $\sigma \in \mathrm{Ind}(G)$ if σ is an independent set in G.*

We will consider the independence complex of connected graphs. If G is the disjoint union of H and J then $\mathrm{Ind}(G)$ is the join of $\mathrm{Ind}(H)$ and $\mathrm{Ind}(J)$. In a graph G, the *neighborhood* of a vertex v, $\mathrm{N}_G(v)$ is the set of vertices which are adjacent to v. If it is clear which G is meant, we just write $\mathrm{N}(v)$. We will use the following lemma from [6].

Lemma 3 (fold lemma) *Let G be a graph and $v, w \in V(G)$. If $\mathrm{N}(v) \subseteq \mathrm{N}(w)$ then $\mathrm{Ind}(G)$ is homotopy equivalent to $\mathrm{Ind}(G \setminus \{w\})$.*

Let X be a topological space, and let $X = \cup_{i \in I} X_i$ be a covering. The nerve of a covering is a simplicial complex, denoted $\mathcal{N}(X_I)$, whose set of vertices is given by I, and whose set of simplices is described as follows: the finite subset $S \subseteq I$ gives a simplex of $\mathcal{N}(X_I)$ if and only if the intersection $\cap_{i \in S} X_i$ is non-empty. We will need the nerve lemma [3, 13].

Lemma 4 (nerve lemma) *Let K be a simplicial complex, and let $K = \cup_{i=1}^n A_i$ be a covering of K by its subcomplexes, such that every non-empty intersection of the covering sets is contractible. Then K and $\mathcal{N}(A_I)$ are homotopy equivalent.*

Let K be a simplicial complex with the ground set V. The star of a vertex v of K is $\mathrm{star}_K(v) = \{\sigma \in K \colon \sigma \cup \{v\} \in K\}$. We define the combinatorial *Alexander dual* of K as a simplicial complex $K^* = \{A \subset V \colon V \setminus A \notin K\}$. If $|V| = n$ we can consider

$K \subset S^{n-2}$ unless K is the $n-1$-dimensional simplex. It is easy to see that K^* is homotopy equivalent to $S^{n-2} \setminus K$. The Alexander duality [1, 8] gives that the ith reduced homology group is isomorphic to the $n-i-3$rd reduced cohomology group of the complement: $\tilde{\mathrm{H}}_i(K) \cong \tilde{\mathrm{H}}^{n-i-3}(S^{n-2} \setminus K)$. In our combinatorial settings: $\tilde{\mathrm{H}}_i(K) \cong \tilde{\mathrm{H}}^{n-i-3}(K^*)$.

3 The independence complex of G_2

Theorem 5 *Let G be a graph with n vertices. The independence complex* $\mathrm{Ind}(G_2)$ *is homotopy equivalent to the Alexander dual of* $\mathrm{Ind}(G)$*. Here* $\mathrm{Ind}(G)$ *is considered as a simplicial complex on $n+1$ vertices such that actually no simplex contains the extra $(n+1)$st vertex.*

Proof. For $v \in V(G)$ let $K_v = \mathrm{star}_{\mathrm{Ind}(G_2)}(v)$. We define $K_\emptyset$ to be the induced subcomplex by the vertex set $V(G_2) \setminus V(G)$. This way we obtain a covering of $\mathrm{Ind}(G_2)$. $K_\emptyset$ is a simplex, K_v is a cone with apex v so they are contractible. The intersection $K_{v_1} \cap \cdots \cap K_{v_k}$ is again a cone with apex e.g. v_1, since $V(G)$ forms an independent set in G_2. So $K_{v_1} \cap \cdots \cap K_{v_k}$ is non-empty and contractible. The intersection $K_\emptyset \cap K_{v_1} \cap \cdots \cap K_{v_k}$ is empty if $V(G) \setminus \{v_1, \ldots, v_k\}$ is an independent set. If $V(G) \setminus \{v_1, \ldots, v_k\}$ is not an independent set, then there are edges $e_1, \ldots, e_l \in E(G)$ spanned by $V(G) \setminus \{v_1, \ldots, v_k\}$. Now this intersection is a simplex with vertex set $e_1, \ldots, e_l \in V(G_2)$.
We can apply the nerve lemma (lemma 4) and get that $\mathrm{Ind}(G_2)$ is homotopy equivalent to a simplicial complex on $n+1$ vertices. The extra $(n+1)$st vertex corresponds to $K_\emptyset$. The non-empty intersections correspond to complements of non-independent sets, exactly as in the Alexander duality, which completes the proof. ☐

Theorem 6 *The independence complex* $\mathrm{Ind}(G_2)$ *is homotopy equivalent to the suspension of the Alexander dual of* $\mathrm{Ind}(G)$. $\mathrm{Ind}(G_2) \simeq \mathrm{susp}\left((\mathrm{Ind}(G))^*\right)$.

Proof. By theorem 5 we know that $\mathrm{Ind}(G_2) \simeq (\mathrm{Ind}(G) \subset \sigma^n)^*$. The later Alexander dual is the cone over $(\mathrm{Ind}(G))^*$ together with a simplex on $V(G)$. If we contract this simplex we get a homotopy equivalent CW complex. The suspension is the double cone over $(\mathrm{Ind}(G))^*$. A cone is contractible, so we might contract one to obtain a homotopy equivalent CW complex. Since these CW complexes are the same we have finished the proof. ☐

Remark. Let G be a graph with n vertices and e edges. Since $G_4 = (G_2)_2$ by the Alexander duality (theorem 5) we get that $\mathrm{Ind}(G_2) \simeq S^{n-1} \setminus \mathrm{Ind}(G)$, $\mathrm{Ind}(G) \simeq S^{n-1} \setminus \mathrm{Ind}(G_2)$ and $\mathrm{Ind}(G_4) \simeq S^{n+e-1} \setminus \mathrm{Ind}(G_2) = S^{n-1} * S^{e-1} \setminus \mathrm{Ind}(G_2) \simeq \mathrm{Ind}(G) * S^{e-1}$. The join with S^{e-1} is the same as the suspension iterated e times, so $\mathrm{Ind}(G_4) \simeq \mathrm{susp}^e(\mathrm{Ind}(G))$. A similar formula can be obtained for G_{2^k} by repeating this.

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Lemma 7 *Let T be a tree.* $\mathrm{Ind}(T_3)$ *is contractible.*

Proof. We proceed by induction on the number of edges of T. If T has only one edge, then T_3 is a path of length 3 and it is easy to check that $\mathrm{Ind}(T_3)$ is contractible. Lets assume that T has $e+1$ edges. Since T is a tree, there is a degree one vertex $x \in V(T)$. Let $y = \mathrm{N}_T(x)$ be its only neighbor. In T_3 there are two new vertices u, v between x and y. Since $\mathrm{N}_{T_3}(x) = \{u\} \subset \{u, y\} = \mathrm{N}_{T_3}(v)$ we get from lemma 3 that $\mathrm{Ind}(T_3) = \mathrm{Ind}(T_3 \setminus \{v\})$. $T_3 \setminus \{v\}$ is a disjoint union of an edge and H_3, where H is a tree with only e edges. So $\mathrm{Ind}(T_3)$ is the join of S^0 and $\mathrm{Ind}(H_3)$, which is contractible by the induction. □

Theorem 8 *Let G be a graph but not a tree with n vertices and e edges.* $\mathrm{Ind}(G_3)$ *is homotopy equivalent to a wedge of spheres $S^{e-1} \vee S^{n-1}$.*

Before the proof we remark that it is easy to find one of the spheres. $G_3 \setminus V(G)$ is the disjoint union of e edges, so $\mathrm{Ind}(G_3)$ contains as a subcomplex the corresponding cross-polytope boundary S^{e-1}.

Proof. For $x \in V(G)$ let $K_x = \mathrm{star}_{\mathrm{Ind}(G_3)}(x)$. We define $K_\emptyset$ to be the induced subcomplex by the vertex set $V(G_3) \setminus V(G)$. This way we obtain a covering of $\mathrm{Ind}(G_3)$. As we observed before $K_\emptyset$ is a cross-polytope boundary so it is S^{e-1}. K_x is a cone with apex x so it is contractible. The intersection $K_{x_1} \cap \cdots \cap K_{x_k}$ is again a cone with apex e.g. x_1, since $V(G)$ is an independent set in G_3, so $K_{x_1} \cap \cdots \cap K_{x_k}$ is non-empty and contractible. The intersection $K_\emptyset \cap K_{x_1} \cap \cdots \cap K_{x_k}$ is empty if $V(G) = \{x_1, \ldots, x_k\}$. If $V(G) \neq \{x_1, \ldots, x_k\}$ let $y \in V(G) \setminus \{x_1, \ldots, x_k\}$ such that y has a neighbor x_i in G. y exists since G is connected. In G_3 there are two new vertices u, v between x_i and y, let $v \in \mathrm{N}_{G_3}(y)$. It is easy to see that the intersection $K_\emptyset \cap K_{x_1} \cap \cdots \cap K_{x_k}$ is a cone with apex v, so it is contractible. We are ready to understand the nerve of this covering. We covered $\mathrm{Ind}(G_3)$ with $n+1$ sets, and only the intersection of all sets was empty, so the nerve is the boundary of a simplex which is S^{n-1}.

Observe that $K_\emptyset$ is the only non-contractible subcomplex so we can not apply the nerve lemma (lemma 4) yet. We show that there is a maximal simplex of $\sigma \in K_\emptyset (= S^{e-1})$ such that the interior of σ does not intersect any other K_x. We choose a spanning tree T in G. Since G was not a tree, there is an edge $vw \in E(G)$, $vw \notin E(T)$. We assign to each vertex of $x \in G$ an edge e_x such that the edge contains the vertex, and different vertices have different assigned edges. If we pick a vertex $x \in G$, then there is a unique path in T which starts in x and ends in v. We assign the first edge of this path to x. To finish this we assign vw to v. Now in G_3 we choose $v_x \in \mathrm{N}_{G_3}(x)$ such that v_x is a vertex of the path of length 3 introduced instead of e_x during the construction of G_3. Because of the construction, these chosen vertices v_x form a maximal simplex σ in $\mathrm{Ind}(G_3)$ and $K_\emptyset$ as well.

Now in the interior of σ we choose an $(e-1)$-dimensional simplex τ. τ does not intersect K_x $(x \in V(G))$, because of the construction of σ. We modify $K_\emptyset$ by removing the interior of τ. Since $K_\emptyset$ was the boundary of the cross-polytope, after the modification it will be contractible, it is homeomorphic to the disc. To obtain a covering of $\mathrm{Ind}(G_3)$ we cover τ by e $(e-1)$-dimensional simplices corresponding to the cone over the boundary of τ.

The nerve of this new covering will be the previously described S^{n-1}; and the covering of τ together with the modified $K_\emptyset$ provides the boundary of a simplex with e vertices.

S^{n-1} and this new simplex boundary have only the vertex corresponding to the modified $K_\emptyset$ in common, which completes the proof. $\square$

Remark. Let G be a graph with n vertices and e edges. Since $G_6 = (G_2)_3$, from theorem 8 and lemma 7 we get that $\mathrm{Ind}(G_6)$ is homotopy equivalent to $S^{2e-1} \vee S^{e+n-1}$ unless G is a tree, when it is contractible. Similarly $\mathrm{Ind}(G_{3k})$ is homotopy equivalent to $S^{k\cdot e-1} \vee S^{(k-1)\cdot e+n-1}$ or contractible.

In physics independent sets correspond to configurations of electrons. It is interesting to know what happens if a cosmic ray hits one possible place of the electron. This corresponds to deleting a vertex in the graph.

Lemma 9 *Let G be a graph with e edges and $x \in V(G)$ a vertex. $\mathrm{Ind}(G_3 \setminus \{x\})$ is homotopy equivalent to S^{e-1}.*

Proof. Let y be the neighbor of x in G. In G_3 there are two new vertices u, v between x and y. Since x was deleted $\mathrm{N}_{G_3}(u) = \{v\} \subseteq \mathrm{N}_{G_3}(y)$, so $\mathrm{Ind}(G_3 \setminus \{x\})$ is homotopy equivalent to $\mathrm{Ind}(G_3 \setminus \{x, y\})$. By continuing along the edges of G we get that $\mathrm{Ind}(G_3 \setminus \{x\})$ is homotopy equivalent to $\mathrm{Ind}(G_3 \setminus \{V(G)\})$ (G was connected). $G_3 \setminus \{V(G)\}$ is a graph containing e disjoint edges, so $\mathrm{Ind}(G_3 \setminus \{x\})$ is homotopy equivalent to the join of edge many S^0, which is S^{e-1}; the boundary of the cross-polytope. $\square$

Lemma 10 *Let G be a graph with n vertices and e edges. Let $u \in V(G_3)$, $u \notin V(G)$ be a vertex. $\mathrm{Ind}(G_3 \setminus \{u\})$ is homotopy equivalent to S^{n-1} or $S^{m-1} \vee S^{n-1}$ or it is contractible, where $n \le m \le e$.*

Proof. Let x and y be neighbors in G such that $u, v \in V(G_3)$ are between them.

Case 1. Assume that $G_3 \setminus \{u\}$ is connected. We define a new graph $\tilde{G}$ from G by removing the edge between x and y, and adding a new vertex $\tilde{x}$ connected to y. $\tilde{G}$ is connected since $G_3 \setminus \{u\}$ was connected. We choose a spanning tree T in $\tilde{G}$. Since $\tilde{x}$ has degree 1 the edge between $\tilde{x}$ and y is in T. Let $z \neq x$ be another neighbor (in G) of y such that the edge zy is in T. In G_3 there are two vertices u_1, v_1 between y and z. Now $\mathrm{N}_{G_3 \setminus \{u\}}(v) = \{y\} \subset \{v_1, y\} = \mathrm{N}_{G_3 \setminus \{u\}}(u_1)$, so from lemma 3 we get that $\mathrm{Ind}(G_3 \setminus \{u\})$ is homotopy equivalent to $\mathrm{Ind}(G_3 \setminus \{u, u_1\})$. We can recursively repeat this procedure on the edges of T. In each step we choose the closest edge to $\tilde{x}$ where we have not performed this step yet. The procedure allows us to delete one vertex from the corresponding path in G_3, without changing the homotopy type of the independence complex. Let H be the graph obtained this way from $G_3 \setminus \{u\}$. Let ab be an edge in G but not an edge of T. In H there are two vertices c, d between a and b. In T there is a unique path from a to $\tilde{x}$. Following this path in $H \subset G_3$ we denote the neighbor of a by v_a. We define v_b similarly. $\mathrm{N}_H(v_a) = \{a\} \subset \{a, d\} = \mathrm{N}_H(c)$ so by lemma 3 $\mathrm{Ind}(H)$ is homotopy equivalent to $\mathrm{Ind}(H \setminus \{c\})$. Now $\mathrm{N}_{H \setminus \{c\}}(v_b) = \{b\} = \mathrm{N}_{H \setminus \{c\}}(d)$ so by lemma 3 $\mathrm{Ind}(H \setminus \{c\})$ is homotopy equivalent to $\mathrm{Ind}(H \setminus \{c, d\})$. Repeatedly we can remove the middle vertices of each edge corresponding to edge of $E(G) \setminus E(T)$. At the end we get a graph consisting of n disjoint edges resulting in S^{n-1} for the independence complex.

Case 2. Now $G_3 \setminus \{u\}$ is not connected, it has then two components. One of the component is H_3 for an appropriate graph H. If H is a tree then $\mathrm{Ind}(H_3)$ is contractible by lemma 7, $\mathrm{Ind}(G_3 \setminus \{u\})$ is contractible as well. If H is not a tree with n_H vertices and e_H edges, then by theorem 8 $\mathrm{Ind}(H_3)$ is homotopy equivalent to $S^{e_H-1} \vee S^{n_H-1}$. Now the other connected component can be considered as F_3 with an extra vertex and edge for some graph F. Similar to Case 1 we get that $\mathrm{Ind}(F_3)$ is homotopy equivalent to S^{n_F-1}, where F has n_F vertices. $\mathrm{Ind}(G_3 \setminus \{u\})$ is the join of the independence complexes of its two components, so it is homotopy equivalent to $(S^{e_H-1} \vee S^{n_H-1}) * S^{n_F-1} \cong S^{e_H+n_F-1} \vee S^{n_H+n_F-1} = S^{m-1} \vee S^{n-1}$. It is easy to see that $e_H + n_F - 1 \leq e_H + e_F < e$ and $e_H + n_F - 1 \geq n_H + n_F - 1 = n - 1$, since a tree has vertex-1 edges. $\square$

5 The independence complex of G_n

The following theorem will explain the homotopy type of the independence complex of G_n (for $n \geq 4$). In [12] this was proved for the special case when G is a path or a cycle.

Theorem 11 *Let G be a graph and $uv \in E(G)$ an edge. Let $\tilde{G}$ be a graph obtained from G by replacing the edge uv by a path of length 4. Now $\mathrm{Ind}(\tilde{G})$ is homotopy equivalent to the suspension of $\mathrm{Ind}(G)$. $\mathrm{Ind}(\tilde{G}) \simeq \mathrm{susp}(\mathrm{Ind}(G))$.*

Proof. Let $V(\tilde{G}) = V(G) \cup \{1, 2, 3\}$, 2 is the middle vertex of this edge subdivision. Let $A = \mathrm{star}_{\mathrm{Ind}(\tilde{G})}(2)$ and $B = \mathrm{star}_{\mathrm{Ind}(\tilde{G})}(1) \cup \mathrm{star}_{\mathrm{Ind}(\tilde{G})}(3)$. A is a cone with apex 2, so it is contractible. Since there is no edge between 1 and 3 we get that $\mathrm{star}_{\mathrm{Ind}(\tilde{G})}(1) \cap \mathrm{star}_{\mathrm{Ind}(\tilde{G})}(3)$ is a cone with apex 1. By lemma 4 we get that B is contractible. It is easy to see that $B \cap A = \mathrm{Ind}(G)$, so by [3, Lemma 10.4(ii)], $\mathrm{Ind}(\tilde{G}) \simeq \mathrm{susp}(\mathrm{Ind}(G))$. $\square$

Let G be a graph with n vertices and e edges. By theorem 11 we get that $\mathrm{Ind}(G_{n+3}) \simeq \mathrm{susp}^e(\mathrm{Ind}(G_n))$. This gives that $\mathrm{Ind}(G_{3k+1}) \simeq \mathrm{susp}^{e \cdot k}(\mathrm{Ind}(G))$. Using theorem 6 we have that $\mathrm{Ind}(G_{3k+2}) \simeq \mathrm{susp}^{e \cdot k}(\mathrm{Ind}(G_2)) \simeq \mathrm{susp}^{e \cdot k+1}(\mathrm{Ind}(G)^*)$. In other words $S^{e \cdot k+n-1} \setminus \mathrm{Ind}(G)$ is homotopy equivalent to $\mathrm{Ind}(G_{3k+2})$. From theorem 8 and lemma 7 we obtain that $\mathrm{Ind}(G_{3k+3}) \simeq \mathrm{susp}^{e \cdot k}(\mathrm{Ind}(G_3)) \simeq \mathrm{susp}^{e \cdot k}(S^{e-1} \vee S^{n-1}) \simeq S^{(k+1) \cdot e-1} \vee S^{k \cdot e+n-1}$ unless G is a tree, when it is contractible.

In G_n we subdivide each edge of G into n pieces. It is not necessary to subdivide each edge into the same number of pieces. As long as the number of pieces mod 3 is the same for each edge, we can keep track the homotopy changes using theorem 11 and the previous sections.

Acknowledgments

I would like to thank Liza Huijse and Kareljan Schoutens for suggesting to study G_2, G_3 and for explaining the physics relevance of the independence complex. I would like to thank Gunther Cornelissen and Jan Draisma for organizing the DIAMANT meets GQT workshop in the Lorentz Center where this research was initiated.

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