Consistent cycles in $\frac{1}{2}$-arc-transitive graphs

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Abstract

A directed cycle C of a graph is called $\frac{1}{k}$-consistent if there exists an automorphism of the graph which acts as a k-step rotation of C. These cycles have previously been considered by several authors in the context of arc-transitive graphs. In this paper we extend these results to the case of graphs which are vertex-transitive, edge-transitive but not arc-transitive.

1 Introduction

A long neglected result of J. H. Conway [1, 2] states that an arc-transitive graph of valence d has exactly $d-1$ orbits of the so-called consistent directed cycles, where directed cycle is consistent whenever there exists an automorphism of a graph which acts on the cycle as a one step rotation. The original result of Conway first appeared in [1] and several rigorous proofs, together with some generalizations of the result, were recently provided in [5, 9, 10]. In this paper we extend the result of Conway to graphs Γ admitting a group of automorphisms G acting transitively on the vertices and the edges but intransitively on the directed edges of Γ. Moreover, we consider the so called $\frac{1}{k}$-consistent cycles of Γ, that is, cycles which may not allow a one-step rotation, but rather a k-step rotation for $k \geq 2$.

If such a graph Γ has girth at least $2k+1$, then we show that the number of G-orbits of directed $\frac{1}{k}$-consistent cycles in Γ is exactly

$$\frac{1}{k}\sum_{\ell | k} \varphi\big(\frac{k}{\ell}\big)\big((d-1)^{\ell}+1\big)$$

where φ denotes the Euler totient (and the summation is taken over all positive divisors of k); see Theorem 5.2.

In Sections 2 and 3 we set up notation and terminology. In Sections 4 and 5 we state and prove several results about the number of orbits of directed $\frac{1}{k}$-consistent cycles, including the above mentioned result. Finally, in Section 6 we illustrate the developed theory by considering the smallest $\frac{1}{2}$-arc-transitive graph, the Doyle-Holt graph.

2 Definitions

In this paper, we consider three closely-related structures in a connected graph Γ, each of which can be called a “cycle” in some context. First, a cycle is often defined as a sequence $\alpha = [v_0, v_1, v_2, \ldots, v_{r-1}]$, $r \geq 3$, of pairwise distinct vertices of Γ such that each vertex v_i is adjacent to v_{i+1}, the addition being mod r. We will use the word *cyclet* to mean such a sequence, and we will think of it as a *rooted, directed cycle.* To be explicit, a cyclet $\alpha = [v_0, v_1, v_2, \ldots, v_{r-1}]$ is distinct from its *shift* $\alpha^1 = [v_1, v_2, \ldots, v_{r-1}, v_0]$ and its *inverse* $\alpha^{-1} = [v_0, v_{r-1}, v_{r-2}, \ldots, v_2, v_1]$.

Second, consider the equivalence relation on the set of cyclets generated by the shift relationship. Equivalence classes under this relation will be called *directed cycles.* The directed cycle containing α will be called $\vec{\alpha}$; that is, the directed cycle $\vec{\alpha}$ is the set of all *t-shifts* $\alpha^t = [v_t, \ldots, v_{r-1}, v_0, \ldots, v_{t-1}]$ of α, for $t = 0, 1, \ldots, r-1$.

Finally, a *cycle* is an equivalence class of cyclets under the relation generated by shift and inverse relationships. This is equivalent to thinking of a cycle, as we often do, as a connected subgraph of Γ in which each vertex has valence 2. Thus, the cyclet α and its inverse cyclet α^{-1} induce the same cycle $\tilde{\alpha}$, but distinct directed cycles $\vec{\alpha}$ and $\vec{\alpha}^{-1}$, respectively.

An *s-arc* in Γ is a sequence $\alpha = [v_0, v_1, v_2, \ldots, v_s]$ of vertices of Γ in which every two consecutive vertices are adjacent, and every three consecutive vertices are distinct. A 1-arc is an *arc* and a 0-arc can be viewed as a vertex.

For a subgroup G of $\mathrm{Aut}(\Gamma)$, we say that G is $\frac{1}{2}$-*arc-transitive* or that Γ is $(G, \frac{1}{2})$-*arc-transitive* provided that G acts transitively on the set of vertices and the set of edges of Γ, but intransitively on the set of arcs of Γ.

Let G be a group of symmetries of the graph Γ. A cyclet $\alpha = [v_0, v_1, v_2, \ldots, v_{r-1}]$ is *G-consistent* provided there is $g \in G$ such that $v_i^g = v_{i+1}$ for all i, the addition being modulo r. Such a symmetry g is called a *shunt* for α. Observe that if a cyclet α is consistent, so are all of its shifts, and so are all of t heir inverses. If α is consistent, we will say the same for the directed cycle $\vec{\alpha}$ and the cycle $\tilde{\alpha}$.

More generally, a cyclet $\alpha = [v_0, v_1, v_2, \ldots, v_{r-1}]$ is *$(G, \frac{1}{k})$-consistent* provided there is $g \in G$ such that $v_i^g = v_{i+k}$ for all i, the addition again being modulo r. Such a symmetry g is called a *$\frac{1}{k}$-shunt* for α. Note that we do not require that k is a divisor of r. If $k' = \gcd(k, r)$, then α is $(G, \frac{1}{k})$-consistent if and only if it is $(G, \frac{1}{k'})$-consistent.

Observe that if a cyclet α is $(G,\frac{1}{k})$-consistent, so are all of its shifts, as well as the inverse. Hence we may call the cycle $\tilde{\alpha}$, and the directed cycle $\vec{\alpha}$, $(G,\frac{1}{k})$-consistent whenever α is $(G,\frac{1}{k})$-consistent. When G is clear from the context, the reference to G will be omitted.

Of course, a group $G \leq \mathrm{Aut}(\Gamma)$ acts upon the set of $(G,\frac{1}{k})$-consistent cyclets, as well as directed or undirected $(G,\frac{1}{k})$-consistent cycles.

3 Signature of k-arcs and overlap of cyclets

Let Γ be a graph and let $G \leq \mathrm{Aut}(\Gamma)$. Assume that Γ is $(G, \frac{1}{2})$-arc-transitive. In this case the valency d of Γ is an even number, as proved by Tutte [12]. Fix an edge of Γ and choose an orientation on this edge. The orbit of this oriented edge under G induces the orientation on the edges of Γ, and G acts transitively on the set of so obtained oriented edges. For every vertex v of Γ, exactly $d/2$ edges incident with v are oriented "into" v, while the other $d/2$ edges are oriented "out of" v.

Let $\alpha = [u_0, u_1, u_2, \ldots, u_k]$ be a k-arc of Γ. By the *signature* of α we mean the sequence $[\epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{k-1}]$, where $\epsilon_i = 1$ if the edge $\{u_i, u_{i+1}\}$ is oriented out of u_i, and 0 otherwise. The *signature* of a $\frac{1}{k}$-consistent cyclet $[u_0, u_1, \ldots, u_r]$ is defined as the signature of its initial k-arc $[u_0, u_1, \ldots, u_k]$.

Following [10] we define the *overlap* $m(\alpha, \beta)$ of two cyclets $\alpha = [u_0, u_1, \ldots, u_{r-1}]$ and $\beta = [v_0, v_1, \ldots, v_{s-1}]$ to be -1 if $u_0 \neq v_0$ and

$$m(\alpha, \beta) = \max\{t : u_i = v_i \text{ for } i = 0, 1, 2, \ldots, t\}$$

otherwise. Similarly if $\mathcal{A}$ and $\mathcal{B}$ are two sets of cyclets in Γ, then we define the *overlap* of $\mathcal{A}$ and $\mathcal{B}$ to be the maximal overlap between (distinct) representatives of $\mathcal{A}$ and $\mathcal{B}$:

$$m(\mathcal{A}, \mathcal{B}) = \max\{m(\alpha, \beta) : \alpha \in \mathcal{A}, \beta \in \mathcal{B}, \alpha \neq \beta\}.$$

If $\mathcal{F}$ is a family of orb its of cyclets, then a set F of representatives, one from each element of $\mathcal{F}$, will be called *compact* provided that $m(\alpha, \beta) = m(\alpha^G, \beta^G)$ for each pair of distinct elements $\alpha, \beta \in F$. The following two lemmas were proved in [10].

Lemma 3.1 [10, Lemma 3.5] *Let Γ be a graph and $G \leq \mathrm{Aut}(\Gamma)$. Every family of pairwise distinct G-orbits of cyclets in Γ has a compact set of representatives.*

Lemma 3.2 [10, Lemma 4.1] *Let k be a positive integer, let Γ be a graph of girth at least $2k+1$, let $[y_0, y_1, \ldots, y_k, y_{k+1}]$ be a $(k+1)$-arc in Γ, and let g be an automorphism of Γ which maps the arc (y_0, y_1) to the arc (y_k, y_{k+1}). Then there exists a unique cyclet starting in y_0 for which g is a $\frac{1}{k}$-shunt. Moreover, this cyclet is of the form $[y_0, \ldots, y_{k-1}, y_0^g, \ldots, y_{k-1}^g, y_0^{g^2}, \ldots]$.*

4 Orbits of consistent cyclets

To facilitate the formulation and proof of our main result, we introduce the following function δ.

Definition 4.1 *Let d and k be nonnegative integers, let $\sigma = [\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}]$ be a sequence such that $\epsilon_i \in \{0,1\}$ for $0 \le i \le k-1$, and let $z(\sigma) = \sum_{i=0}^{k-1} |\epsilon_{i+1} - \epsilon_i|$, addition in the subscripts being modulo k. Then we define*

$$\delta_d(\sigma) = \left(\frac{d}{2} - 1\right)^{z(\sigma)} \left(\frac{d}{2}\right)^{k-z(\sigma)}.$$

The following lemma is a crucial step towards our main result. Its statement and proof are analogues of Theorem 4.3 and its proof in [10]. In fact, the proof of Lemma 4.2 below can be obtained from the proof of [10, Theorem 4.3] simply by replacing the function $\delta_{(\Gamma,G)}$ in [10] by the function δ_d defined above, and some other minor and obvious modifications. In order to avoid repetition, we leave the details of the proof to the reader and only provide the main ideas.

Lemma 4.2 *Let k be a positive integer, let Γ be a graph of girth at least $2k+1$ and valency d, and let G be a $\frac{1}{2}$-arc-transitive subgroup of $\mathrm{Aut}(\Gamma)$. Let $\sigma = [\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}]$ be a sequence such that $\epsilon_i \in \{0,1\}$ for $0 \le i \le k-1$. Then there are exactly $\delta_d(\sigma)$ G-orbits of $\frac{1}{k}$-consistent cyclets with k-signature σ.*

Proof: Let $\{\mathcal{A}_1, \ldots, \mathcal{A}_t\}$ be the family of G-orbits of $\frac{1}{k}$-consistent cyclets having signature σ. By Lemma 3.1, we can choose $\alpha_i \in \mathcal{A}_i$ such that $m(\alpha_i, \alpha_j) = m(\mathcal{A}_i, \mathcal{A}_j)$ for $1 \le i < j \le t$. Clearly, by $\frac{1}{2}$-arc-transitivity of G, it follows that $m(\alpha_i, \alpha_j) \ge 1$ for $1 \le i, j \le t$. Let v_0 and v_1 be the first two vertices of α_1 (and thus of every α_i).

Consider the set $\mathcal{T}$ of $(k-1)$-arcs $[x_0, x_1, \ldots, x_{k-1}]$ such that $[x_0, \ldots, x_{k-1}, v_0, v_1]$ is a $(k+1)$-arc and such that the signature of $[x_0, \ldots, x_{k-1}, v_0]$ is σ. Since the girth of Γ is at least $2k+1$, it is easy to see that $|\mathcal{T}| = \delta_d(\sigma)$.

Following the fourth paragraph of the proof of [10, Theorem 4.3] word by word, with δ' replaced with $\delta_d(\sigma)$, we obtain that $t \le \delta_d(\sigma)$.

Suppose now that $t < \delta_d(\sigma)$. Then there is at least one $(k-1)$-arc $[x_0, \ldots, x_{k-1}]$ in $\mathcal{T}$ which is not a "tail" of α_i for any i, $1 \le i \le t$ (where by a "tail" of a cyclet we mean the k-tuple of the last k vertices in the cyclet). As in the fifth paragraph of the proof of [10, Theorem 4.3] we construct a $\frac{1}{k}$-consistent cyclet $[v_0, v_1, \ldots, x_0, \ldots, x_{k-1}]$. Note that the signature of the so constructed cyclet is also σ.

Among all $(G,\frac{1}{k})$-consistent cyclets of the form $[v_0, v_1, \dots, x_0, \dots, x_{k-1}]$ and with signature σ choose one (say τ) which maximizes the sum $\sum_{i=1}^{t} m(\alpha_i, \tau)$. Let $g \in G$ be a $\frac{1}{k}$-shunt for τ. The cyclet τ belongs to exactly one of the G-orbits $\mathcal{A}_1, \dots, \mathcal{A}_t$. Without loss of generality we may assume that $\tau \in \mathcal{A}_1$. Let $h \in G$ be such that $\tau^h = \alpha_1$, and let $m = m(\alpha_1, \tau)$. Note that $m \geq 1$. Moreover, by our choice of $\alpha_i \in \mathcal{A}_i$, we see that $m(\alpha_i, \tau) \leq m(\alpha_i, \alpha_1)$ for every $i \neq 1$.

Next, consider the automorphism $gh \in G$. Since g is a $\frac{1}{k}$-shunt for τ, it maps x_0 to v_0 and x_1 to v_1. Since h fixes v_0 and v_1, it follows that $x_0^{gh} = v_0$ and $x_1^{gh} = v_1$. Therefore, in view of Lemma 3.2, there exists a $\frac{1}{k}$-consistent cyclet τ' of the form $[v_0, v_1, \dots, x_0, \dots, x_{k-1}]$ for which gh is a $\frac{1}{k}$-shunt.

As in the proof of [10, Theorem 4.3], we show that $m(\alpha_1, \tau') = m(\alpha_1, \tau) + k$ and $m(\alpha_i, \tau') \geq m(\alpha_i, \tau)$ for $2 \leq i \leq t$, contradicting the choice of τ. This shows that $t = \delta_d(\sigma)$ and completes the proof. ∎

We are now able to count the number of all orbits of $\frac{1}{k}$-consistent cyclets for a given $\frac{1}{2}$-arc-transitive group action.

Theorem 4.3 *Let k be a positive integer, let Γ be a graph of girth at least $2k+1$ and valency d, and let G be a $\frac{1}{2}$-arc-transitive subgroup of* $\mathrm{Aut}(\Gamma)$. *Then there are exactly $(d-1)^k + 1$ G-orbits of $\frac{1}{k}$-consistent cyclets in Γ.*

Proof: Let S be the set of all signatures of length k. In view of Lemma 4.2 we need to show that

$$\sum_{\sigma \in S} \left(\frac{d}{2} - 1\right)^{z(\sigma)} \left(\frac{d}{2}\right)^{k - z(\sigma)} = (d-1)^k + 1,$$

where $z(\sigma)$ for $\sigma \in S$ is as in Definition 4.1. Note that $z(\sigma)$, being the number of changes in the "cyclical interpretation" of the sequence σ, is an even integer in the range between 0 and k. Observe that for a fixed even z in that range the number of $\sigma \in S$ with $z(\sigma) = z$ equals $2\binom{k}{z}$. Hence

$$\sum_{\sigma \in S} \left(\frac{d}{2} - 1\right)^{z(\sigma)} \left(\frac{d}{2}\right)^{k - z(\sigma)} = 2 \sum_{\substack{0 \leq z \leq k \\ z \text{ even}}} \binom{k}{z} \left(\frac{d}{2} - 1\right)^{z} \left(\frac{d}{2}\right)^{k - z} =$$

$$= \left(\frac{d}{2} + \left(\frac{d}{2} - 1\right)\right)^k + \left(\frac{d}{2} - \left(\frac{d}{2} - 1\right)\right)^k = (d-1)^k + 1,$$

as required. ∎

5 Precisely $\frac{1}{k}$-consistent cyclets and orbits of directed cycles

A $\frac{1}{k}$-consistent cyclet is called *precisely $\frac{1}{k}$-consistent* (relative to a group G) if it is not $\frac{1}{\ell}$-consistent for any $\ell < k$. Note that if a cyclet is $\frac{1}{k}$-consistent and precisely $\frac{1}{\ell}$-consistent, then $\ell \mid k$.

We shall first count the number of orbits of precisely $\frac{1}{k}$-consistent cyclets. Let k be a positive integer, let Γ be a graph of girth at least $2k+1$ and valency d, and let G be a $\frac{1}{2}$-arc-transitive subgroup of $\mathrm{Aut}(\Gamma)$. Let $f(k)$ denote the number of G-orbits of precisely $\frac{1}{k}$-consistent cyclets in Γ. It follows from Theorem 4.3 that

$$\sum_{\ell | k} f(\ell) = (d-1)^k + 1.$$

The number of orbits of precisely $\frac{1}{k}$-consistent cyclets now follows directly from the Möbius Inversion Formula (see [6, Theorem 10.4]).

Proposition 5.1 *With the above notation and assumptions, the number of orbits of precisely $\frac{1}{k}$-consistent cyclets equals*

$$f(k) = \sum_{\ell | k} \mu\big(\frac{k}{\ell}\big)\big((d-1)^\ell + 1\big),$$

where μ denotes the Möbius function.

The second result of this section gives the number of orbits of $\frac{1}{k}$-consistent directed cycles for a $\frac{1}{2}$-arc-transitive group $G \le \mathrm{Aut}(\Gamma)$.

Recall that a directed cycle $\vec{\alpha}$ is $\frac{1}{k}$-consistent (relative to a group $G \le \mathrm{Aut}(\Gamma)$) if an underlying cyclet α is $\frac{1}{k}$-consistent. The group G acts on the set of all $\frac{1}{k}$-consistent directed cycles in a natural way. Suppose that $\vec{\mathcal{A}}$ is an orbit of $\frac{1}{k}$-consistent directed cycles, and let $\mathcal{A}$ be the set of all cyclets which underlie a member of $\vec{\mathcal{A}}$. Then $\mathcal{A}$ is a union of G-orbits of $\frac{1}{k}$-consistent cyclets. Suppose that $\alpha \in \mathcal{A}$ is precisely $\frac{1}{\ell}$-consistent for some divisor ℓ of k. Then it is easy to see that every element of $\mathcal{A}$ is precisely $\frac{1}{\ell}$-consistent, and that the shifts $\alpha, \alpha^1, \alpha^2, \ldots, \alpha^{\ell-1}$ form a complete set of representatives of G-orbits of cyclets contained in $\mathcal{A}$. In particular, $\mathcal{A}$ is a union of ℓ distinct G-orbits of $\frac{1}{k}$-consistent cyclets (each being a "shift" of a fixed one).

We can now state and prove the main result of the paper, mentioned already in Introduction.

Theorem 5.2 *Let k be a positive integer, let Γ be a graph of valency d and of girth at least $2k+1$, and let G be a $\frac{1}{2}$-arc-transitive subgroup of* $\mathrm{Aut}(\Gamma)$. *Then the number of G-orbits of directed $\frac{1}{k}$-consistent cycles in Γ is exactly*

$$\frac{1}{k} \sum_{\ell | k} \varphi\big(\frac{k}{\ell}\big)\big((d-1)^\ell + 1\big)$$

where φ denotes the Euler totient (and the summation is taken over all positive divisors of k).

In particular, the number of G-orbits of consistent directed cycles in Γ is d, and if the girth of Γ is at least 5, then the number of G-orbits of $\frac{1}{2}$-consistent directed cycles in Γ is $\frac{d^2-d+2}{2}$, among which d of them are also consistent.

The proof of the above theorem is a word by word translation of the proof of [10, Theorem 5.1] with $\delta_{(\Gamma,G)}(\ell)$ replaced with $(d-1)^\ell+1$.

A $\frac{1}{k}$-consistent directed cycle is said to be *symmetric* provided that there exists an automorphism in G sending it to the inverse of one of its shifts, and it is said to be *chiral* otherwise. Note that orbits of chiral directed cycles always come in pairs (with a directed cycle and its inverse in different members of this pair). Hence the number of orbits of chiral directed cycles is always even. Note that every G-consistent directed cycle (with respect to a $\frac{1}{2}$-arc-transitive group G) is always chiral. If one defines a $\frac{1}{k}$-consistent (undirected) cycle as pair of the two underlying $\frac{1}{k}$-consistent directed cycles, then the following result follows immediately from the above discussion.

Proposition 5.3 *If Γ is a d-valent graph admitting a $\frac{1}{2}$-arc-transitive group of automorphisms G, then it has precisely $\frac{d}{2}$ G-orbits of consistent undirected cycles.*

6 Examples and applications

In the study of $\frac{1}{2}$-arc-transitive groups of automorphisms, the notion of alternating cycles plays a very important role. A $\frac{1}{2}$-consistent (un)directed cycle is alternating if and only if it consists of cyclets with signature $[0,1]$ and $[1,0]$. The number of orbits of these can be deduced easily from Lemma 4.2.

Proposition 6.1 *If Γ is a d-valent graph of girth at least 5 admitting a $\frac{1}{2}$-arc-transitive group of automorphisms G, then the number of G-orbits of alternating $\frac{1}{2}$-consistent directed cycles is precisely $(\frac{d}{2}-1)^2$. In particular, if d is divisible by 4, then there exists at least one symmetric $\frac{1}{2}$-consistent directed cycle in Γ.*

Proof: It follows by Lemma 4.2 that the number of $\frac{1}{2}$-consistent cyclets of signature $[0,1]$ is $(\frac{d}{2}-1)^2$, and the same of signature $[1,0]$. Note that all such cyclets are precisely $\frac{1}{2}$-consistent, and so a cyclet and its shift belong to different orbits of G. Therefore the number of G-orbits of $\frac{1}{2}$-consistent directed cycles is $\frac{1}{2}((\frac{d}{2}-1)^2+(\frac{d}{2}-1)^2)=(\frac{d}{2}-1)^2$. If d is divisible by 4, then this number is odd, and so at least one orbit consists of symmetric directed cycles. ■

6.1 $\frac{1}{2}$-arc-transitive group actions on tetravalent graphs

Among all graphs admitting a $\frac{1}{2}$-arc-transitive groups action, those of valency 4 have received by far the most attention (see for example [7, 8, 11, 13]). Let us summarize our results for the case of tetravalent graphs Γ admitting a $\frac{1}{2}$-arc-transitive group of automorphisms G.

By Theorems 4.3 and 5.2 there exist precisely four orbits of consistent cyclets in Γ and the same number of orbits of consistent directed cycles. Since G is $\frac{1}{2}$-arc-transitive, none of these orbits is symmetric. Hence there are precisely two orbits of G-consistent (undirected) cycles in Γ.

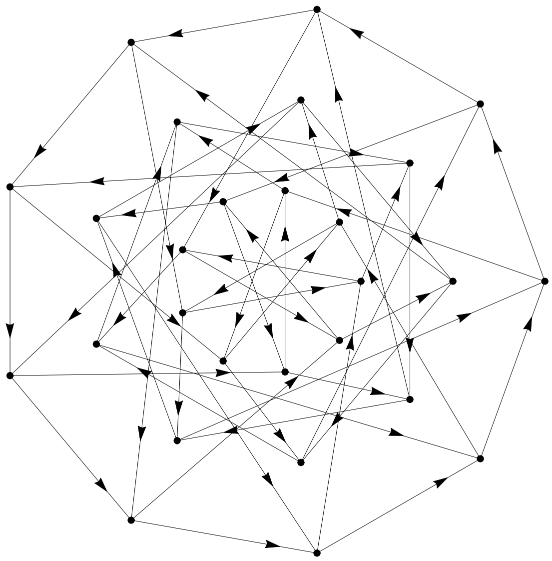

Figure 1: Consistent cycles of length 9 in the Doyle-Holt graph.

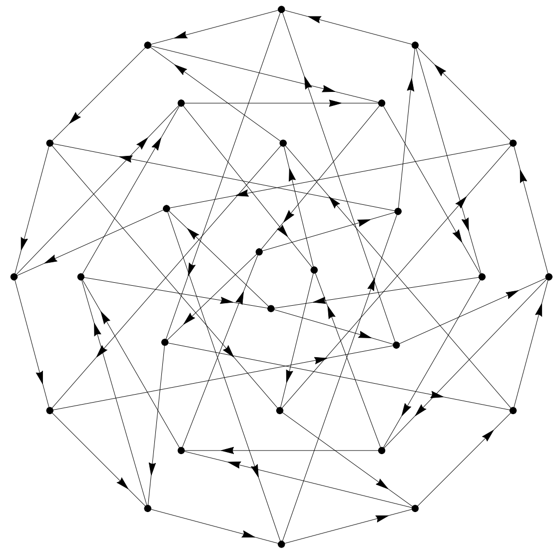

Figure 2: Consistent cycles of length 6 and $\frac{1}{2}$-consistent cycles of length 12.

Suppose now that the girth of Γ is at least 5. Then by Proposition 5.1 there exist precisely six orbits of precisely $\frac{1}{2}$-consistent cyclets, two of them of signature $[0,0]$, two of signature $[1,1]$, one of signature $[0,1]$ and one of signature $[1,0]$. One of the two orbits of signature $[0,0]$ is clearly a shift of the other of the same signature. The same holds true for the two orbits of signature $[1,1]$. The two thus obtained directed cycles with signatures $[0,0]$ and $[1,1]$ are chiral, and one is the inverse of the other, resulting in a single orbit of $\frac{1}{2}$-consistent undirected cycles.

On the other hand, the orbit of $\frac{1}{2}$-consistent cyclet of signature $[0,1]$ is a shift of the orbit of those of signature $[1,0]$, giving rise to a single orbit of directed cycles, the latter being necessarily symmetric.

To summarize, a tetravalent graph Γ admitting a $\frac{1}{2}$-arc-transitive group of automorphisms G of girth at least 5 has precisely two orbits of consistent (undirected) cycles (both being chiral) and two orbits of precisely $\frac{1}{2}$-consistent (undirected) cycles, one being symmetric and the other chiral. Note that the symmetric orbit of $\frac{1}{2}$-consistent cycles corresponds to the set of alternating cycles, as introduced and thoroughly studied in [7].

To illustrate this situation we consider the smallest $\frac{1}{2}$-arc-transitive, called the Doyle-Holt graph [3, 4], denoted by $\mathcal{H}$, together with its full automorphism group $G = \mathrm{Aut}(\mathcal{H})$, of order 54. Usually this graph on 27 vertices is presented as a tri-circulant with three sets of vertices of size 9, u_i, v_i and w_i, $i \in \mathbb{Z}_9$, and edges of the form $u_i v_{i\pm1}$, $v_i w_{i\pm2}$, $w_i u_{i\pm4}$. Three drawings of $\mathcal{H}$ are shown in Figures 1, 2 and 3. The orientation of edges induced by the $\frac{1}{2}$-arc-transitive action of G is represented by the arrows on the edges.

As discussed above, $\mathcal{H}$ has precisely 2 orbits of consistent undirected cycles, both being chiral. The lengths of these cycles are 9 and 6, respectively. Since the stabiliser of an edge in $\mathcal{H}$ in trivial, each of these two orbits gives rise to an edge decomposition of $\mathcal{H}$ into cycles.

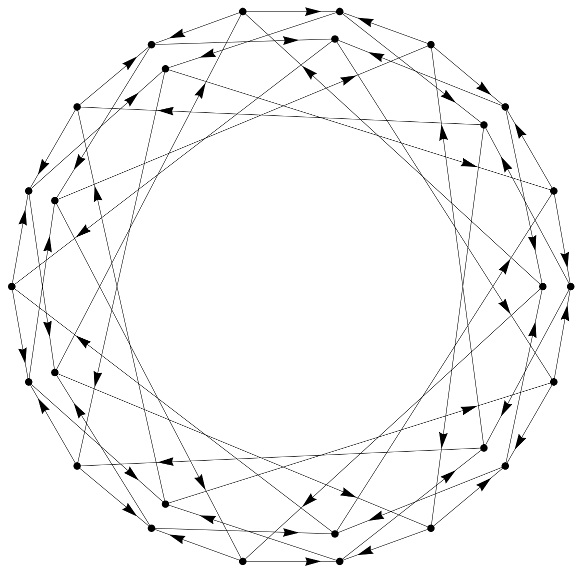

Figure 3: $\frac{1}{2}$-consistent cycles of length 18.

The consistent cycles of length 9 can be easily seen in Figure 1. The vertices in this figure are arranged in three concentric rings, each inducing one of the six consistent 9-cycles in the orbit. These three consistent cycles share the same shunt, being the rotation of the figure by $2\pi/9$ about the center. The edges going between the rings induce the other three consistent cycles in the orbit.

Now consider the drawing of the Doyle-Holt graph in Figure 2. Here the vertices are arranged in four concentric rings, consisting of 3, 6, 6, and 12 vertices, respectively. Observe that the simultaneous rotation of the outer three rings by $\pi/3$ and the inner ring by $2\pi/3$ about the center of the drawing induces an automorphism of the graph, which is clearly a shunt for the 6-cycle induced by one of the two rings of size 6. The same automorphism is also a $\frac{1}{2}$-shunt for the outer 12 cycle. Since the girth of $\mathcal{H}$ is 5, our theory implies that there are precisely two orbits of $\frac{1}{2}$-consistent cycles, one of them containing the outer cycle of length 12 in Figure 2.

The other orbit of precisely $\frac{1}{2}$-consistent cycles in $\mathcal{H}$ is visible in Figure 3. It consists of three alternating cycles of length 18, one of them being the outer cycle in the figure, and the other two jumping between the outer and the inner ring.

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